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A CHARACTERIZATION OF ACM 0-DIMENSIONAL SCHEMES IN Q

LUCIA MARINO

Let $\mathbb{X} \subset Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a reduced 0-dimensional subscheme of the quadric Q and let $P \in \mathbb{X}$ be any point. Using the separating degree of P for $\mathbb{X}$ we give a sufficient condition so that $\mathbb{X}$ is ACM. This result, together with the previous ones (see [9]) gives a new characterization of ACM 0-dimensional schemes of Q by using separators.

1. Introduction

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be the smooth (abstract) quadric and let $\mathbb{X} \subset Q$ be a reduced 0-dimensional scheme. A form f is a *separator* for $P \in \mathbb{X}$ if $f(P) \neq 0$ and $f(Q) = 0$ for all $Q \in \mathbb{X} \setminus \{P\}$. The *set of minimal bi-degrees of separators for P* is called the *set of separating degrees of P in* $\mathbb{X}$; we denote it by

$$\text{s.deg}_{\mathbb{X}} P.$$

The ordering we are using is the natural partial order on $\mathbb{N}^2$, i.e., $(a,b) \leq (c,d)$ if and only if $a \leq c$ and $b \leq d$. Compared with the conductor degree of a point of a 0-dimensional scheme in the projective space $\mathbb{P}^r$, ([1], [2], [6], [7], [8], [9]) the separating degrees of a point of the reduced 0-dimensional schemes on the smooth quadric Q is a new investigation that justifies the use of this name.

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Presently, characterizations of ACM 0-dimensional schemes of multi projective spaces $\mathbb{P}^{n_1} \times ... \times \mathbb{P}^{n_k}$ are not known. Precisely, a geometrical classification of ACM 0-dimensional schemes of Q using the Hilbert Function of the considered scheme is given in [5], and combinatorial classifications are presented by [11] and [6].

In this paper we give a new characterization of ACM 0-dimensional schemes of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ using the separating degree $\text{s.deg}_{\mathbb{X}} P$, for $P \in \mathbb{X}$. In particular,

$$\mathbb{X} \quad \text{is ACM} \Leftrightarrow |\,\text{s.deg}_{\mathbb{X}} P| = 1 \quad \text{for all P} \in \mathbb{X}.$$

The necessary condition ($\Rightarrow$) was proved by the author in a previous paper (see [9])and shortly presented in Section 3; the sufficient condition ($\Leftarrow$) is proved in Section 6, Proposition 6.7.

Results on ACM reduced 0-dimensional schemes of $\mathbb{P}^{n_1} \times ... \times \mathbb{P}^{n_k}$ can be found in [11]. Probably, the main result of this paper could be generalized in order to give a characterization of ACM reduced 0-dimensional schemes of multi-projective spaces.

2. Preliminaries and notation

Here we collect some terminology (see [5] for details). Let $\mathbb{X} \subset Q = \mathbb{P}^1 \times \mathbb{P}^1$ be the quadric and let $\mathscr{O}_Q$ be its structure sheaf.

If $D \subset Q$ is any divisor of type (a,b) we denote by $\mathscr{O}_Q(a,b)$ the associated sheaf. We use the ring $S = \bigoplus_{a,b} H^0\mathscr{O}_Q(a,b)$. S is, in a natural way, a k-algebra using product of sections. It is easy to check that S is generated, as a bi-graded k-algebra, by $H^0\mathscr{O}_Q(1,0)$ and $H^0\mathscr{O}_Q(0,1)$ (both vector spaces of dimension 2) since for every $a,b \geq 0$ the map $H^0\mathscr{O}_Q(a,b) \otimes H^0\mathscr{O}_Q(1,0) \otimes H^0\mathscr{O}_Q(0,1) \to H^0\mathscr{O}_Q(a+1,b+1)$ given by the product, is surjective. Let u,u' and v,v' be bases for $H^0\mathscr{O}_Q(1,0)$ and $H^0\mathscr{O}_Q(0,1)$; then we have a bi-graded ring isomorphism $S \cong k[u,u';v,v']$. We use the above isomorphism to identify elements of S and elements of $k[u,u';v,v']$; of course we deal only with bi-homogeneous ideals of S.

When $s \in H^0\mathscr{O}_Q(a,b)$ its zero locus $(s)_0$ will be called a curve of type (a,b). We mention as lines of type $(1,0)$ or $(1,0)$-lines, and lines of type $(0,1)$ or $(0,1)$-lines respectively, $L = (l)_0$ and $L' = (l')_0$, with $l \in H^0\mathscr{O}_Q(1,0)$ and $l' \in H^0\mathscr{O}_Q(0,1)$. Every point $P \in Q$ is the intersection of two lines $l \in H^0\mathscr{O}_Q(1,0)$, $l' \in H^0\mathscr{O}_Q(0,1)$. If l and l' have equations $a'u - au' = 0$, $b'v - bv' = 0$ respectively, then the 4-tuple $(a,a';b,b')$ gives the coordinates of P.

When no confusion can arise we will not distinguish between curves and their defining forms. A saturated ideal of S of height 2 is a complete intersection iff it is generated by 2 elements of type $h(u,u') \otimes 1$, $1 \otimes h'(v,v')$, where h and

h' are any forms. From now on we shall mean by complete intersection on Q (c.i. for short) a subscheme whose saturated ideal has just 2 generators. (more details see in [5].)

Thus a zero-dimensional scheme $\mathbb{X} \subset Q$ is a complete intersection on Q only when $I_{\mathbb{X}}$ is generated by a curve of the type $(a,0)$ and a curve of type $(0,b)$. We can associate to $\mathbb{X}$ the bi-graded S-algebra $S_{\mathbb{X}} = S/I_{\mathbb{X}}$, where $I_{\mathbb{X}}$ is the homogeneous saturated ideal of $\mathbb{X}$ in S and $\mathscr{I}_X \subset \mathscr{O}_Q$ its ideal sheaf.

By analogy with the definition of Hilbert functions for graded modules, we can define the function $M_{\mathbb{X}} : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{N}$ by $M_{\mathbb{X}}(i,j) = \dim_k(S)_{(i,j)} - \dim_k(I_{\mathbb{X}})_{(i,j)} = \dim_k S_{\mathbb{X}}(i,j)$ where for every bi-graded S-module N we denote by $(N)_{(i,j)}$ the component of N of degree (i,j). The function $M_{\mathbb{X}}$ is the bigraded Hilbert function of $\mathbb{X}$. The function $M_{\mathbb{X}}$ can be represented as a matrix with infinitive integer entries,

$$M_{\mathbb{X}} = (M_{\mathbb{X}}(i,j)) = (m_{ij})$$

which will be called the *Hilbert matrix of* $\mathbb{X}$. Note that $M_{\mathbb{X}}(i,j) = 0$ for $i < 0$ or $j < 0$. So, from now on we restrict ourselves to the range $i \geq 0, j \geq 0$.

It is well known that not every 0-dimensional scheme $\mathbb{X} \subset Q$ is ACM; see for instance two "non-collinear" points on Q. The *ACM* 0-dimensional schemes of Q were classified in terms of their Hilbert function in [5].

3. The set of separating degrees: the ACM case.

We recall the following definition, given in [9].

Definition 3.1. Let $\mathbb{X} \subset Q$ be a reduced 0-dimensional scheme. We say that a form $f \in S$ is a *separator* for $P \in \mathbb{X}$ if $f(P) \neq 0$ and $f(Q) = 0$ for all $Q \in \mathbb{X} \setminus \{P\}$. The *set of minimal bi-degrees of separators for P* is called the *set of separating degrees of P in* $\mathbb{X}$; we denote it by

$$\text{s.deg}_{\mathbb{X}} P.$$

We observe that the cardinality of this set is not necessarily one for any point $P \in \mathbb{X}$. This is a very great difference with the conductor degree of P in a reduced 0-dimensional scheme of $\mathbb{P}^n$ (for more details on the the conductor degree of P see [9]).

Example 3.2. Let $\mathbb{X} = \{P_1, P_2\}$ two non-collinear points on Q. Each point of $\mathbb{X}$ has $\text{s.deg}_{\mathbb{X}} P = \{(0,1),(1,0)\}$

Theorem 3.3. *(Cayley-Bacharach on Q). Let*

$$\mathbb{Y} = C.I.((a,0),(0,b)) \subset Q$$

be a complete intersection on Q, let $P \in \mathbb{Y}$ and $\mathbb{Y}' = \mathbb{Y} \setminus \{P\}$. Then $M_{\mathbb{Y}'}(i,j) =$
$\begin{cases} M_{\mathbb{Y}}(i,j) - 1 & \forall (i,j) \geq (a-1,b-1) \\ M_{\mathbb{Y}}(i,j) & \text{otherwise} \end{cases}$

Proof. The proof is trivial. □

Corollary 3.4. *Let*

$$\mathbb{Y} = C.I.((a,0),(0,b)) \subset Q$$

be a complete intersection on Q and let P be a point of $\mathbb{Y}$. *Then* $\text{s.deg}_{\mathbb{Y}} P = \{(a-1,b-1)\}$.

Proof. The proof is a direct consequence of Theorem 3.3 □

We need some terminology to give a geometrical description of an ACM 0-dimensional scheme of Q (see [5], in [6] and [11]).

Definition 3.5. Let $a_1 < a_2 < ... < a_n$ and $b_1 > b_2 > ... > b_n$ be integers. The set $\mathscr{A}' = \{(a_1,b_1),\dots,(a_n,b_n)\}$ is said a *set of vertices.*

For any 0-dimensional ACM scheme $\mathbb{X} \subseteq Q$ the set of vertices of $\Delta M_{\mathbb{X}}$ (see [5], section 4) can be assumed to be of this type and conversely, given a set $\mathscr{A}'$ there exists an ACM 0-dimensional scheme of Q whose vertices are the elements of $\mathscr{A}'$. From now on we suppose that the points of any 0-dimensional scheme $\mathbb{X} \subseteq Q$ have integers coordinates.

For any couple $(a_q,b_q) \in \mathscr{A}'$ we denote by Δ_q the *rectangle* the set

$$\Delta_q = \{(r,s) \in \mathbb{Z}^2 |\ (1,1) \leq (r,s) \leq (a_q,b_q)\}.$$

For a fixed $\mathscr{A}'$ we set

$$\mathscr{A} = \cup_{1 \leq q \leq |\mathscr{A}'|} \Delta_q$$

For any couple $(i,1) \in \mathscr{A}$ we call R_{i0} the $(1,0)-$line of equation $u - iu' = 0$ and for any $(1,j) \in \mathscr{A}$ we call R_{0j} the line of equation $jv - v' = 0$.

Setting $P_{ij} = R_{i0} \cap R_{0j}$ the set

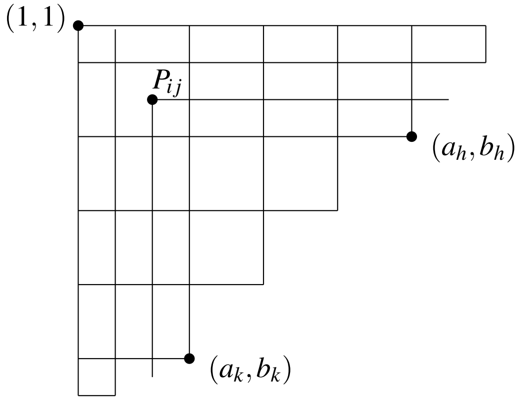
$$\mathbb{X} = \{P_{ij} |\ (i,j) \in \mathscr{A}\}$$

is an ACM 0-dimensional subscheme of Q whose vertices are the couples of $\mathscr{A}'$. We say that $\mathbb{X}$ has support in $\mathscr{A}$.

It is known, by [4], [6] and [11], that every ACM 0-dimensional scheme of Q can be described, after a suitable permutation of lines, as a ACM 0-dimensional subcheme with support on $\mathscr{A}$. This construction is equivalent to the Ferrer's diagram approach.

Remark 3.6. Every couple $(i,j) \in \mathscr{A}$ will determine two elements (a_h,b_h), $(a_k,b_k) \in \mathscr{A}'$ such that $a_{h-1} < i \leq a_h$ and $b_k \geq j > b_{k+1}$. Considering the corresponding $P_{ij} \in \mathbb{X}$, we can translate this notation with the existence of two rectangles Δ_h and Δ_k, where Δ_h is the "higher"rectangle containing the point P_{ij} and Δ_k is the "lowest"rectangle containing the point P_{ij}. Thus the couple $(i,j) \in \mathscr{A}$ determines two couples

$$(a_h,b_h),\ (a_k,b_k).$$



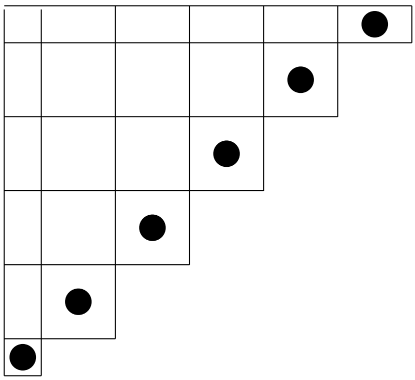
With the above notation, note that $h \leq k$.
Let

$$D_{ij} = \{(m,n) \mid (m,n) \geq (i,j)\}.$$

It has the property that $|D_{ij} \cap \mathscr{A}'| \geq 2$ (i.e. it contains at least two vertices) if and only if the point P_{ij} belongs to at least two rectangles Δ_q defined in 3.5.

Lemma 3.7. *With the above notation, if* $|D_{ij} \cap \mathscr{A}'| = 1$ *then* $\mathbb{X} \setminus \{P_{ij}\}$ *is ACM .*

Proof. By hypothesis, $\mathbb{X} \subset Q$, is a reduced 0-dimensional scheme, the point P_{ij} in one of parts marked in the following figure:



□

We recall the following result of [9]:

Proposition 3.8. *Let $\mathbb{X} \subset Q$ be an ACM reduced 0-dimensional scheme. Then for all $(i, j) \in \mathscr{A}$, the set of separating degrees of the point P_{ij} is given by*

$$\text{s.deg}_{\mathbb{X}} P_{ij} = \{(a_k - 1, b_h - 1)\}.$$

Corollary 3.9. *Let $\mathbb{X} \subset Q$ be an ACM 0-dimensional scheme then for each point $P \in \mathbb{X}$, $|\,\text{s.deg}_{\mathbb{X}} P| = 1$.*

This shows that the ACM 0-dimensional subschemes of Q have the same behaviour on the 0-dimensional subschemes of a projective space $\mathbb{P}^r$ with respect to the set of separating degrees.

Let us proceed to the general analysis of the unicity of the separator degrees of the point P belonging to a scheme $\mathbb{X} \subset Q$.

Proposition 3.10. *Let $\mathbb{X} \subset Q$ be a reduced 0-dimensional scheme. Let F be a separator of minimal bi-degree (α, β) for the point $P \in \mathbb{X}$. Then F is unique modulo $I_{\mathbb{X}}(\alpha, \beta)$ up to a scalar.*

Proof. Let $\mathbb{X}' = \mathbb{X} \setminus \{P\}$. If $h^0(\mathscr{I}_{\mathbb{X}'}(\alpha, \beta)) = p > 1$ then imposing to pass through the point P, we have a family of dimension $p - 1$. Since these are elements of $I_{\mathbb{X}}$ we have $\dim_k(I_{\mathbb{X}'}(\alpha, \beta)/I_{\mathbb{X}}(\alpha, \beta)) = 1$. □

Lemma 3.11. *Let $\mathbb{X} \subset Q$ be a reduced 0-dimensional scheme. Given $P \in \mathbb{X}$ with $(\alpha, \beta) \in \text{s.deg}_{\mathbb{X}} P$, let $\mathbb{X}' = \mathbb{X} \setminus \{P\}$. If $h^0(\mathscr{I}_{\mathbb{X}}(\alpha, \beta)) = r$ then $h^0(\mathscr{I}_{\mathbb{X}'}(\alpha, \beta)) = r + 1$.*

Proof. It is sufficient to prove that two indipendent separators for the point $P \in \mathbb{X}$, $f, g \in H^0(\mathscr{I}_{\mathbb{X}'}(\alpha,\beta))$, differ for an element of $H^0(\mathscr{I}_{\mathbb{X}}(\alpha,\beta))$. In the pencil $\lambda f + \mu g$ there exists an element passing through the point, thus

$$\exists\, h \in H^0(\mathscr{I}_{\mathbb{X}}(\alpha,\beta)) \mid h = \overline{\lambda} f + \overline{\mu} g \rightarrow g = kf + h'$$

with $h' \in H^0(\mathscr{I}_{\mathbb{X}}(\alpha,\beta))$.

It follows that if $H^0(\mathscr{I}_{\mathbb{X}}(\alpha,\beta)) =< f_1, f_2, ..., f_r >$ and $g \in H^0(\mathscr{I}_{\mathbb{X}'}(\alpha,\beta))$ is a separator for the point $P \in \mathbb{X}$, then $H^0(\mathscr{I}_{\mathbb{X}'}(\alpha,\beta)) =< g, g+f_1, g+f_2, ..., g+f_r >$. In particular, if $r = 0$ in degree (α,β) then there exists just one separator for the point P in $\mathbb{X}$. □

Let $\mathbb{X} \subset Q$ be a reduced ACM 0-dimensional scheme. Using the notation introduced in this section, it is known by [5] that the generators of $I_{\mathbb{X}}$ are forms having minimal degrees corresponding to the *corners* of $\mathscr{A}$:

$$\{(1,b+1),(a_1+1,b_2+1),...,(a_i+1,b_{i+1}+1),...,(a_{n-1},b_n+1),(a_n+1,1)\}.$$

Let $\mathscr{A}'$ be the set of vertices of $\mathbb{X}$, $(a_i,b_i) \in \mathscr{A}'$ and let P_i be the corresponding point. By [9],

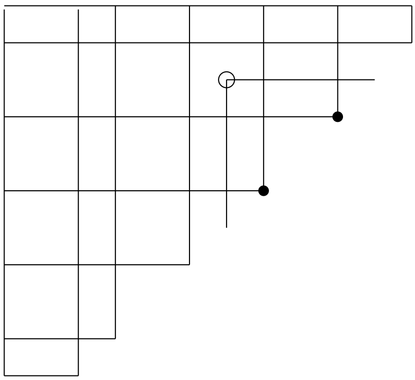
$$\text{s.deg}_{\mathbb{X}} P_i = \{(a_i-1,b_i-1)\}$$

Note that the couple (a_i-1,b_i-1) is a minimal degree of a generator of $\mathbb{X}' = \mathbb{X} \setminus \{P\}$ but it is not a minimal degree of a generator of $\mathbb{X}$, thus $I_{\mathbb{X}}(a_i-1,b_i-1) = 0 \rightarrow r = 0$. Hence we have the corollary

Corollary 3.12. *Let $\mathbb{X} \subset Q$ be a ACM reduced 0-dimensional scheme. Given $P \in \mathbb{X}$ a vertex, there exists only one separator for the point P in $\mathbb{X}$.*

4. Gaps

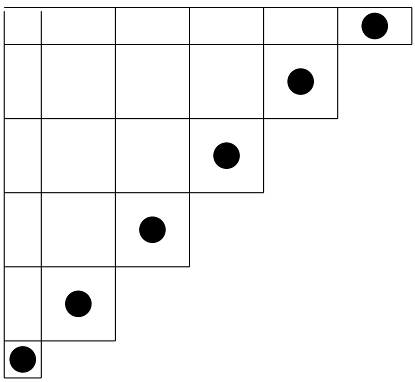
In this section we show that a reduced 0-dimensional scheme $\mathbb{X} \subset Q$ can be seen as a reduced ACM 0-dimensional scheme minus some points which lie in at least two rectangles Δ_q, (see Definition 3.5) as we see in the following figure



For any reduced 0-dimensional scheme $\mathbb{X}$ we consider a reduced ACM 0-dimensional scheme containing $\mathbb{X}$ having minimal degree. Such scheme always exists.

Remark 4.1. Let $\mathbb{X} \subset Q$ be a reduced 0-dimensional scheme and let $Y \supseteq \mathbb{X}$ a minimal 0-dimensional reduced ACM scheme. Let $T = Y \setminus \mathbb{X}$. If $\mathbb{X}$ is ACM then $T = \emptyset$, otherwise for every $P \in T$, P belongs to two different rectangles of the type Δ_q.

It is enough to note that erasing from Y a point P contained in just a rectangle (marked in black in the following picture) $Y \setminus \{P\}$ is still ACM, against the minimality of Y.



Moreover if $\mathbb{X} = Y \setminus \cup\{P_{ij}\}$ where each P_{ij} belongs to two different rectangles of type Δ_q it is easy to see that if we permute in all the possible ways the lines of type (0,1), and/or (1,0), we never obtain an ACM scheme according with the previous arguments (See Section 3).

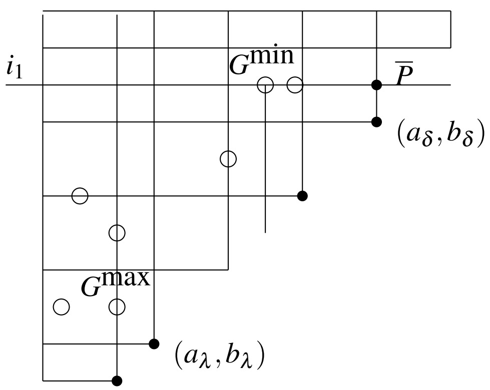
Definition 4.2. Let $Z = Y \setminus \mathbb{X}$. Z is said the set of gaps of $\mathbb{X}$.

Definition 4.3. Let $\mathbb{X} = Y \setminus \cup\{P_{ij}\}$. Let $(i_1, j_1) \in \mathscr{A} = \cup\Delta_q$ with $i_1 = \min\{i \mid P_{ij}$ is a gap of $\mathbb{X}\}$ and, fixed i_1, let $j_1 = \min\{j \mid P_{i_1 j}$ is a gap of $\mathbb{X}\}$. We define *Min Gap* $G^{\min}$ the point $P_{i_1,j_1} \in Y$. (i.e. among the higher gaps the Min Gap is the one on the left side). Let $(i_m, j_m) \in \mathscr{A}$, with $i_m = \max\{i \mid P_{ij}\ is\ a\ gap\ \}$ and, $j_m = \max\{j \mid P_{i_1 j}\ is\ a\ gap\ \}$. We define *Max Gap* $G^{\max}$ the point $P_{i_m,j_m} \in Y$. (i.e. among the lower gaps the Max Gap is the one on the right side).

Notation. As it has been seen in remark 3.6, each of the gaps $G^{\min}$ and $G^{\max}$ are associated to at least two couples of $\mathscr{A}'$, the set of the vertices. In particular to the gap $G^{\min}$ of $\mathbb{X}$ we associate the couple (a_δ, b_δ) where $a_\delta = \min\{a_p\ |\Delta_p$is a rectangle containing
$G^{\min}\}$; in similar way we have the couple (a_λ, b_λ) associated to the gap $G^{\max}$ of $\mathbb{X}$.

Definition 4.4. The couples $(a_\delta, b_\delta), (a_\lambda, b_\lambda) \in \mathscr{A}'$ related to the gaps $G^{\min}$ and $G^{\max}$ of $\mathbb{X}$ are called *special couples* respectively associated to the gaps $G^{\min}$ and $G^{\max}$. Moreover the point $\overline{P} = R_{i_1 0} \cap R_{0 b_\delta}$ of $\mathbb{X}$ with $R_{i_1 0}$ is the (1,0)-line of equation $u - i_1 u' = 0$ and $R_{0 b_\delta}$ the (0,1)-line of equation $b_\delta v - v' = 0$ is called *special point* of the scheme $\mathbb{X}$.

See the following figure.



5. Some results

Let $\mathbb{X} \subset Q$ be a 0-dimensional scheme and let Y be a minimal ACM scheme containing $\mathbb{X}$ described by $\mathscr{A} = \bigcup \Delta_q$.

Definition 5.1. The element (α, β) of s.deg$_{\mathbb{X}} P$ maximal with respect to the first component is called max $-$ *couple* for $\mathbb{X}$ of the point P. In similar way, the smallest element (α', β') of s.deg$_{\mathbb{X}} P$ with respect to the second component is called min $-$ *couple* for $\mathbb{X}$ of the point P.

Remark 5.2. If the scheme $\mathbb{X} \subset Q$ is ACM, then $(\alpha, \beta) = (\alpha', \beta')$.

Remark 5.3. Given $\widetilde{\mathbb{X}} \subset \mathbb{X}$ a reduced subscheme let $P \in \widetilde{\mathbb{X}}$. Note that a separator of P in $\mathbb{X}$ is also a separator of P in $\widetilde{\mathbb{X}}$. Then each couple of the set s.deg$_{\mathbb{X}} P$ not belonging to s.deg$_{\widetilde{\mathbb{X}}} P$ must be comparable with at least one couple of the set s.deg$_{\widetilde{\mathbb{X}}} P$.
It follows that if $(\widetilde{\alpha}, \widetilde{\beta}) \in$ s.deg$_{\mathbb{X}} P$ then it is comparable with at least one couple in s.deg$_{\widetilde{\mathbb{X}}} P$: let (γ, δ) be such couple. Hence

$$(1) \begin{cases} \widetilde{\alpha} \geq \gamma \\ \widetilde{\beta} \geq \delta \end{cases}$$

Lemma 5.4. *If the max-couple for $\widetilde{\mathbb{X}}$ of the point P, (α, β), doesn't belong to* s.deg$_{\mathbb{X}} P$ *and if there exists a separator for $P \in \mathbb{X}$ having bi-degree $(\overline{\alpha}, \beta)$ with $\overline{\alpha} > \alpha$ then,*

$$\exists\, \widetilde{\alpha} \mid (\widetilde{\alpha}, \beta) \in \text{s.deg}_{\mathbb{X}} P, \textit{ with } \widetilde{\alpha} \leq \overline{\alpha}$$

(Similarly, if $(\alpha', \beta') \notin$ s.deg$_{\mathbb{X}} P$ *and if there exists a separator for $P \in \mathbb{X}$ having bi-degree $(\alpha', \overline{\beta'})$ with $\overline{\beta'} > \beta'$ then,*

$$\exists\, \widetilde{\beta}' \mid (\alpha', \widetilde{\beta}') \in \text{s.deg}_{\mathbb{X}} P, \textit{ with } \widetilde{\beta}' \leq \overline{\beta'})$$

Proof. If $(\overline{\alpha}, \beta)$ is a minimal bi-degree of separator for P then $\widetilde{\alpha} = \overline{\alpha}$. If $(\overline{\alpha}, \beta)$ is not a minimal bi-degree of separator then there exists a minimal bi-degree $(\widetilde{\alpha}, \widetilde{\beta}) \in$ s.deg$_{\mathbb{X}} P$ where either

$$case\ (i) \begin{cases} \widetilde{\alpha} < \overline{\alpha} \\ \widetilde{\beta} \leq \beta \end{cases} \qquad \text{or} \quad case\ (ii) \begin{cases} \widetilde{\alpha} \leq \overline{\alpha} \\ \widetilde{\beta} < \beta \end{cases}$$

If we prove $\widetilde{\beta} = \beta$, the case (ii) is impossible.
Let (γ, δ) be the couple of the set s.deg$_{\widetilde{\mathbb{X}}} P$ comparable with the couple $(\widetilde{\alpha}, \widetilde{\beta}) \in$ s.deg$_{\mathbb{X}} P$. Now since (α, β) and (γ, δ) are in s.deg$_{\widetilde{\mathbb{X}}} P$ these couples must be either equals or not comparable. If these couples are equals, i.e. $\gamma = \alpha$, $\delta = \beta$,

it follows that $\beta = \delta \leq \widetilde{\beta}$; moreover, if the case (i) is true we have $\widetilde{\beta} \leq \beta$. Thus $\widetilde{\beta} = \beta$.
Conversely, if the case (ii) is true, that is $\widetilde{\beta} < \beta$, with $\gamma = \alpha$, $\delta = \beta$ is impossible because $\beta = \delta \leq \widetilde{\beta}$.
If the couples (α, β) and (γ, δ) are not comparable, since by hypothesis (α, β) is the max-couple for P in $\widetilde{\mathbb{X}}$ then $\alpha \geq \gamma$. Consequently, $\beta < \delta$. By (1), $\delta \leq \widetilde{\beta}$. For both cases (i) and (ii) we have $\beta < \beta$. It is impossible. The proof of the second result is similar to the proof given above and it is left to the reader. □

Corollary 5.5. *Let $(\alpha, \beta), (\alpha', \beta')$ be the max-couple and the min-couple for $\widetilde{\mathbb{X}}$ of the point P non belonging to* $\mathrm{s.deg}_{\mathbb{X}} P$. *If there exists a separator for $P \in \mathbb{X}$ having bi-degree $(\overline{\alpha}, \beta) \in \mathrm{s.deg}_{\mathbb{X}} P$ with $\overline{\alpha} > \alpha$ and if there exists a separator for $P \in \mathbb{X}$ having bi-degree $(\alpha', \overline{\beta'}) \in \mathrm{s.deg}_{\mathbb{X}} P$ with $\overline{\beta'} > \beta'$ then,* $|\mathrm{s.deg}_{\mathbb{X}} P| \geq 2$.

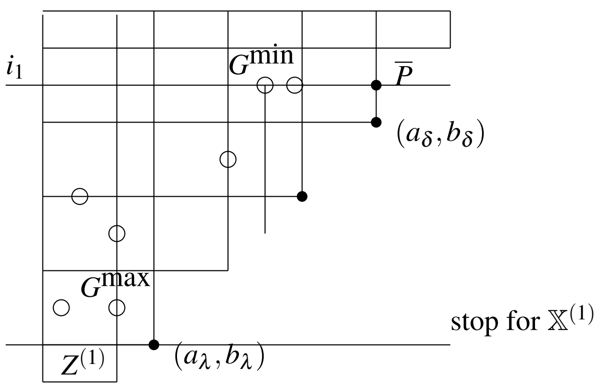
Proof. It is a obvious consequence of Lemma 5.4 □

6. The set of separating degrees for points in a 0-dimensional scheme on Q

Let $\mathbb{X} \subset Q$ be a 0-dimensional scheme; with the above notation $\mathbb{X} = Y \setminus \cup\{P_{ij}\}$ where Y is a ACM scheme of minimal degree containing $\mathbb{X}$.

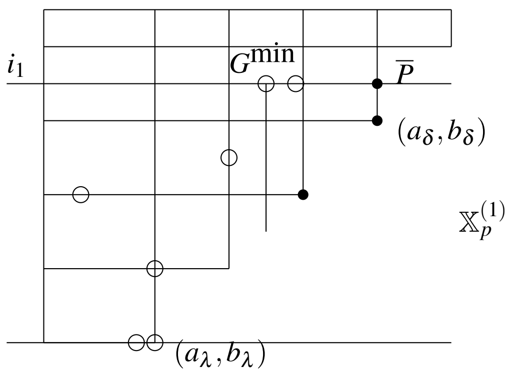
Definition 6.1. Let $\mathbb{X} \subset Q$ be a 0-dimensional scheme, and let (a_λ, b_λ) be the special couple associated to the gap $G^{\max}$(see definition 4.4). Let $\mathscr{A}'$ be the set of vertices of $\mathbb{X}$ and let $\mathscr{A}'^{(1)} = \{(a_i, b_i), i = 1, \ldots, \lambda\} \subset \mathscr{A}'$. The subscheme $\mathbb{X}^{(1)} \subset \mathbb{X}$, having $\mathscr{A}'^{(1)}$ as set of its vertices is said subscheme *of order one*.

Let $Z^{(1)}$ be the ACM subscheme of $\mathbb{X}$ such that $Z^{(1)} = \mathbb{X} \setminus \mathbb{X}^{(1)}$



The number of gaps of $\mathbb{X}^{(1)}$ is less than the number of gaps of $\mathbb{X}$. In fact in $\mathbb{X}^{(1)}$ permuting the (0,1)-line containing $G^{\max}$ with the (0,1)-line containing P_{a_λ,b_λ} and if it necessary permuting the (1,0)-line containing $G^{\max}$ with the (1,0)-line containing P_{a_λ,b_λ} we obtain a new configuration of $\mathbb{X}^{(1)}$ having G^{m} in the position (a_λ, b_λ) i.e. the gap $G^{\max}$ of $\mathbb{X}$ has been eliminated in $\mathbb{X}^{(1)}$. Moreover if permuting the lines we obtain a scheme where all gaps have been eliminated, then $\mathbb{X}^{(1)}$ is a ACM 0-dimensional subscheme of $\mathbb{X}$, otherwise $\mathbb{X}^{(1)}$ is not-ACM.

Definition 6.2. Let $\mathbb{X}^{(1)} \subset Q$ be the 0-dimensional subscheme of $\mathbb{X}$ of order one. We say *permutation scheme* $\mathbb{X}_p^{(1)}$ *related to* $\mathbb{X}^{(1)}$ a particular description of $\mathbb{X}^{(1)}$ with one gap of $\mathbb{X}$ in (a_λ, b_λ).



Remark 6.3. The subschemes $\mathbb{X}^{(1)}$ and $\mathbb{X}_p^{(1)}$ of $\mathbb{X}$ don't have the same behaviour. In fact $\mathbb{X}_p^{(1)} \cup Z^{(1)}$ is not equal to $\mathbb{X}$ and $\mathbb{X}^{(1)} \cup Z^{(1)} = \mathbb{X}$. Observe that to reconstruct the scheme $\mathbb{X}$ it is necessary to make inverse permutations on $\mathbb{X}_p^{(1)}$ to obtain $\mathbb{X}^{(1)}$ and $\mathbb{X}^{(1)} \cup Z^{(1)} = \mathbb{X}$. Thus at first we reconstruct $\mathbb{X}^{(1)}$ then we add $Z^{(1)}$ to obtain $\mathbb{X}$.

If $\mathbb{X}^{(1)} \subset \mathbb{X}$ is the 0-dimensional subscheme of order one, then we permute the lines to obtain $\mathbb{X}_p^{(1)}$. Going on it is possible to define the 0-dimensional subscheme of $\mathbb{X}^{(1)}$ of *order one*, called $\mathbb{X}^{(2)} \subset Q$, which will be of order 2 for $\mathbb{X}$, etc. In conclusion, there exists a 0-dimensional subscheme of $\mathbb{X}$ of maximal order n for some $n \in \mathbb{N}$ which will be a ACM 0-dimensional subscheme of $\mathbb{X}$, called $\mathbb{X}^{(n)}$. We assume $\mathbb{X}^{(0)} = \mathbb{X}$.

Note that if $\mathbb{X} \subset Q$ is a ACM 0-dimensional scheme then the derived scheme from $\mathbb{X}$ doesn't exists, because $\mathbb{X}$ has no gaps.

Proposition 6.4. *Let $\mathbb{X} \subset Q$ be a 0-dimensional scheme and $\mathbb{X}^{(1)}$ be the 0-dimensional subscheme of $\mathbb{X}$ of order one . Then if the special point $\overline{P}$ of the subscheme $\mathbb{X}^{(1)}$ has $|\,\mathrm{s.deg}_{\mathbb{X}^{(1)}}\overline{P}| \geq 2$ then $|\,\mathrm{s.deg}_{\mathbb{X}}\overline{P}| \geq 2$.*

Proof. Consider the special point $\overline{P} \in \mathbb{X}^{(1)}$, having $|\,\mathrm{s.deg}_{\mathbb{X}^{(1)}}\overline{P}| \geq 2$. (see Definition 4.4). Let (α,β) be the max-couple for the point $\overline{P}$ in $\mathbb{X}^{(1)}$ and (α',β') be the min-couple for the point $\overline{P}$ in $\mathbb{X}^{(1)}$.

It is obvious that if the max and min-couple for $\overline{P}$ in $\mathbb{X}^{(1)}$ belong to the set $\mathrm{s.deg}_{\mathbb{X}}\overline{P}$ then the proof is ended.

If $(\alpha,\beta),(\alpha',\beta') \notin \mathrm{s.deg}_{\mathbb{X}}\overline{P}$ we note that there always exist two separators for the special point $\overline{P}$ in $\mathbb{X}$ of bi-degrees $(\overline{\alpha},\beta)$, $(\alpha',\overline{\beta}')$ with $\overline{\alpha} > \alpha$, $\overline{\beta}' > \beta'$: it is sufficient to consider the form of bi-degree (α,β) and to add to it (1,0)- lines until to cover the ACM 0-dimensional sbscheme $Z^{(1)}$ of $\mathbb{X}$ so that we obtain the bi-degree $(\overline{\alpha},\beta)$ of a separator for $\overline{P}$ in $\mathbb{X}$. (In similar way we consider the form of bi-degree (α',β') and adding (0,1)-lines until to cover $Z^{(1)}$ we have $(\alpha',\overline{\beta}')$). Thus applying 5.4 there exist at least two couples of bi-degrees belonging to the set $\mathrm{s.deg}_{\mathbb{X}}\overline{P}$, i.e. $(\widetilde{\alpha},\beta),(\alpha',\widetilde{\beta}') \in \mathrm{s.deg}_{\mathbb{X}}P$, with $\alpha < \widetilde{\alpha} \leq \overline{\alpha}, \beta' < \widetilde{\beta}' \leq \overline{\beta}'$.

If $(\alpha,\beta) \in \mathrm{s.deg}_{\mathbb{X}}\overline{P},(\alpha',\beta') \notin \mathrm{s.deg}_{\mathbb{X}}\overline{P}$ then similarly we obtain two not comparable couples $(\alpha,\beta),(\alpha',\widetilde{\beta}') \in \mathrm{s.deg}_{\mathbb{X}}P$, with $\beta' < \widetilde{\beta}' \leq \overline{\beta}'$. (The proof is similar if $(\alpha,\beta) \notin \mathrm{s.deg}_{\mathbb{X}}\overline{P},(\alpha',\beta') \in \mathrm{s.deg}_{\mathbb{X}}\overline{P}$). □

The above proposition can be generalized

Proposition 6.5. *Let $\mathbb{X}^{(r-1)} \subset Q$ be a 0-dimensional subscheme of order $r-1$ of $\mathbb{X}$ and $\mathbb{X}^{(r)}$ the subcheme 0-dimensional of $\mathbb{X}$ of order r . Then if the special point $\overline{P}$ of the subscheme $\mathbb{X}^{(r)}$ has $|\,\mathrm{s.deg}_{\mathbb{X}^{(r)}}\overline{P}| \geq 2$ then $|\,\mathrm{s.deg}_{\mathbb{X}^{(r-1)}}\overline{P}| \geq 2, \forall\, r = 1,\ldots,n$, where n is the max order of $\mathbb{X}$.*

Proof. Use the same argument as in Proposition 6.4. □

Corollary 6.6. *With the above notation,*
if $|\,\mathrm{s.deg}_{\mathbb{X}^{(r)}}\overline{P}| \geq 2$ then $|\,\mathrm{s.deg}_{\mathbb{X}}\overline{P}| \geq 2$.

Proof. By the Proposition 6.5 if $|\,\mathrm{s.deg}_{\mathbb{X}^{(r)}}\overline{P}| \geq 2 \Rightarrow |\,\mathrm{s.deg}_{\mathbb{X}^{(r-1)}}\overline{P}| \geq 2 \Rightarrow \ldots \Rightarrow |\,\mathrm{s.deg}_{\mathbb{X}'}\overline{P}| \geq 2 \Rightarrow |\,\mathrm{s.deg}_{\mathbb{X}}\overline{P}| \geq 2$. □

Theorem 6.7. *Let $\mathbb{X} \subset Q$ be a 0-dimensional scheme.*

$$\textit{If } |\,\mathrm{s.deg}_{\mathbb{X}}P| = 1 \quad \textit{for all } P \in \mathbb{X} \textit{ then } \mathbb{X} \quad \textit{is ACM}.$$

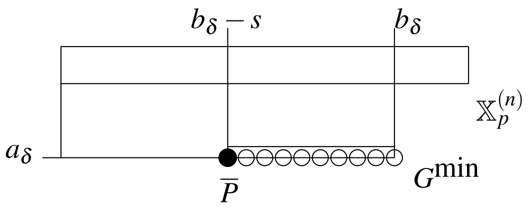
Proof. Let $\mathbb{X} \subset Q$ be a 0-dimensional scheme not ACM and $\mathbb{X}^{(n)} \subset \mathbb{X}$ be the subscheme of *order max n* so that

$$\mathbb{X} \supset \mathbb{X}^{(1)} \supset \mathbb{X}^{(2)} \supset \ldots \supset \mathbb{X}^{(n)}.$$

Consider the special point $\overline{P} = R_{i_1 0} \cap R_{0 b_\delta}$ of $\mathbb{X}$, defined in 4.4.

Call $Z^{(r)}$ the ACM 0-dimensional scheme such that $\mathbb{X}^{(r)} \cup Z^{(r)} = \mathbb{X}_p^{(r-1)}, r = 1, \ldots, n$.

By the above construction $\mathbb{X}_p^{(n)}$ is as in the following figure



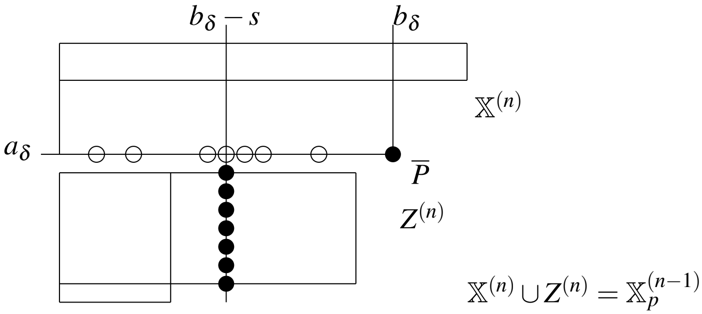
Now, $\mathbb{X}_p^{(n)}$ is evidently ACM. By Proposition 3.8 it is known that the set of separating degree of the point $\overline{P}$ is exactly

$$\text{s.deg}_{\mathbb{X}_p^{(n)}} \overline{P} = \{(a_\delta - 1, b_\delta - s - 1)\}$$

where s is the number of the gaps in the (1,0)-line of $G^{\min}$.
Since $\mathbb{X}^{(n)}$ is ACM (by Corollary 3.11) there exists one and one only separator F for the special point $\overline{P}$ in $\mathbb{X}^{(n)}$ having bi-degree $(a_\delta - 1, b_\delta - s - 1)$

$$F : R_{1,0} \cdot R_{2,0} \cdot \ldots \cdot R_{a_\delta - 1,0} \cdot R_{0,1} \cdot R_{0,2} \cdot \ldots \cdot R_{0,b_\delta - s - 1}$$

By Remark 6.3 to reconstruct the subscheme $\mathbb{X}^{(n-1)}$ it is necessary to make inverse permutations on $\mathbb{X}_p^{(n)}$ to obtain $\mathbb{X}^{(n)}$ and $\mathbb{X}^{(n)} \cup Z^{(n)} = \mathbb{X}_p^{(n-1)}$ (see the following figure in which the gaps on the right of $\overline{P}$ have moved in the left).



Observe that each of s lines of type (0,1) containing the gaps of $\mathbb{X}^{(n)}$ contain points of Z^n and moreover these points are not contained in F. Some of these points have been black marked in the above figure . It follows that F for the special point $\overline{P}$ is the separator in $\mathbb{X}^{(n)}$ but it is not a separator for $\overline{P}$ in $\mathbb{X}^{(n-1)}$. By Corollary 3.12, then we have

$$(a_\delta - 1, b_\delta - s - 1) \notin \text{s.deg}_{\mathbb{X}^{(n-1)}}\overline{P}$$

but easily we can construct a separator for $\overline{P}$ in $\mathbb{X}^{(n-1)}$ having bi-degree $(c, b_\delta - s - 1)$ with $c > a_\delta - 1$ (to consider the form F and to add to it (1,0)-lines until to cover the entire scheme).

Thus by Lemma 5.4

$$(\overline{c}, b_\delta - s - 1) \in \text{s.deg}_{\mathbb{X}^{(n-1)}}\overline{P}$$

where $a_\delta - 1 < \overline{c} \leq c$.
(Similarly, there exists the couple $(a_\delta - 1, \overline{d}) \in \text{s.deg}_{\mathbb{X}^{(n-1)}}\overline{P}$ where $b_\delta - s - 1 < \overline{d} \leq d$.) Then

$$|\,\text{s.deg}_{\mathbb{X}^{(n-1)}}\overline{P}| \geq 2.$$

Then by Corollary 6.6 $|\,\text{s.deg}_{\mathbb{X}}\overline{P}| \geq 2$.

□

This result gives a new characterization of ACM sets of points in the quadric Q. Precisely,

$$\mathbb{X} \quad \text{is ACM} \Leftrightarrow |\,\text{s.deg}_{\mathbb{X}}P| = 1 \quad \text{for all P} \in \mathbb{X}.$$

This result also provides another perspective on the problem of classifying ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, and perhaps this result will provide insight into more general problem of classifying ACM sets of points in multi-projective spaces.

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LUCIA MARINO
Dipartimento di Matematica e Informatica
Viale A. Doria, 6 - 95100 - Catania, Italy
e-mail: lmarino@dmi.unict.it