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HENSTOCK INTEGRAL AND DINI-RIEMANN THEOREM

GIUSEPPE RAO - FRANCESCO TULONE

In [5] an analogue of the classical Dini-Riemann theorem related to non-absolutely convergent series of real number is obtained for the Lebesgue improper integral. Here we are extending it to the case of the Henstock integral.

1. Introduction

The classical Dini-Riemann theorem (see [2]) stating that if a series of real numbers is non-absolutely convergent, then it can be rearranged so that the new series converges to any arbitrary assigned value, was extended in [5] for the Lebesgue improper integral, using a measure preserving mapping instead of permutation.

In the same paper we have noticed that this fact is not true for some non-absolute integrals. An example is the Kolmogorov A-integral (see [1] and [7]) which being non-absolute is known to be invariant under measure preserving mapping.

In this paper we extend the previous result to the case of Henstock integral. Once again we present a direct construction of measure preserving mapping that changes the value of the integral.

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2. Notations and results

All the functions we are considering here are real valued and defined in $[0,1]$ and μ, μ^* are understood as the Lebesgue measure and outer Lebesgue measure respectively.

We remind that a map ϕ is called measure preserving if the image $\phi(A)$ of any measurable set A is measurable and $\mu(\phi(A)) = \mu(A)$

The definition of the Henstock integral can be found for example in [3]. The only property of Henstock integral we need is the following theorem.

Theorem 2.1. *If a function $f : [0,1] \to I\!R$ is Lebesgue improper integrable, then it is Henstock integrable on $[0,1]$ with the same integral value.*

Proof. This theorem is a special case of [3, Theorem 2.8.3] having in mind that each Lebesgue integrable function is Henstock integrable with the same value. □

The result of this paper is the following theorem.

Theorem 2.2. *For any Henstock integrable function $f : [0,1] \to I\!R$ which is not Lebesgue integrable and for any $\alpha \in I\!R$ there exists a measure preserving mapping $\psi_\alpha : [0,1] \to [0,1]$ one-to-one up to a set of measure zero such that $f(\psi_\alpha(x))$ is also Henstock integrable function with integral value equal to α.*

Proof. We start with a modification of a construction given in [4].

Consider the measurable sets $A_n = \{x \in [0,1] : n-1 \le f(x) < n\}$ and $B_n = \{x \in [0,1] : -n \le f(x) < -n+1\}$ for $n = 1,2,\dots$.

Putting

$$A_n^k = A_n \bigcup \left[\frac{k-1}{n^2}, \frac{k}{n^2}\right] \quad \text{and} \quad B_n^k = B_n \bigcup \left[\frac{k-1}{n^2}, \frac{k}{n^2}\right]$$

we have $A_n = \bigcup_{k=1}^{n^2} A_n^k$ and $B_n = \bigcup_{k=1}^{n^2} B_n^k$. It is clear that

$$(\bigcup_{k,n} A_n^k) \cup (\bigcup_{k,n} B_n^k) = [0,1]. \tag{1}$$

By the definition of the above sets for each n and k we get

$$0 \le \int_{A_n^k} f \le n \cdot \frac{1}{n^2} = \frac{1}{n} \quad \text{and} \quad 0 \ge \int_{B_n^k} f \ge -n \cdot \frac{1}{n^2} = -\frac{1}{n},$$

so that

$$\lim_{n\to\infty} \int_{A_n^k} f = \lim_{n\to\infty} \int_{B_n^k} f = 0,$$

independently of k.

We note that, since the function f is not Lebesgue integrable but Henstock integrable, then (see [4])

$$\sum_n \sum_k \int_{A_n^k} f = +\infty \quad \text{and} \quad \sum_n \sum_k \int_{B_n^k} f = -\infty.$$

As in the proof of classical Dini-Riemann theorem we can introduce a linear numeration of the sequence

$$\left\{ \int_{A_n^k} f \ , \ \int_{B_m^h} f \right\}_{n,k,m,h}$$

denoting it as $\{c_i\}_{i=1}^{\infty}$ in such a way that $\sum_{i=1}^{\infty} c_i = \alpha$.

We denote by C_i the set A_n^k or B_m^h for which $\int_{A_n^k} f$ or $\int_{B_m^h} f$ is equal to c_i. We note that on each C_i the function f keeps the sign.

By (1) we have $\cup_{i=1}^{\infty} C_i = [0,1]$ and since C_i are non-overlapping the equality $\sum_{i=1}^{\infty} \mu(C_i) = 1$ holds.

Let $D_i \subset C_i$ be the subset of all density points of C_i that belong to C_i. We take into account only those C_i for which D_i is nonempty. The sets D_i are mutually disjoint. We still have $\sum_{i=1}^{\infty} \mu(D_i) = 1$ and $\mu([0,1] \setminus (\cup_i D_i)) = 0$.

We put $t_0 := 0$ and $t_j := \sum_{i=1}^{j} \mu(D_i)$ for $j \geq 1$. Now we define the function $\varphi : \cup_i D_i \to [0,1]$ so that

$$\varphi(x) = \sum_{i=1}^{j-1} \mu(D_i) + \mu(D_j \cap [0,x]) = t_{j-1} + \mu(D_j \cap [0,x]) \quad \text{for} \quad x \in D_j. \tag{2}$$

This function is strictly increasing on D_j for each fixed j. Indeed if we take two points x_1 and x_2 of the same D_j, $x_1 < x_2$, then

$$\varphi(x_2) - \varphi(x_1) = \mu(D_j \cap [0,x_2]) - \mu(D_j \cap [0,x_1]) = \mu(D_j \cap (x_1,x_2]) > 0.$$

Moreover if x_1 and x_2 belong to different sets $x_1 \in D_j$ and $x_2 \in D_l$ with $l > j$, then $\varphi(x_1) \neq \varphi(x_2)$ because $\varphi(x_2) - \varphi(x_1) \geq \mu(D_j \cap (x_1,1]) > 0$. From this follows that the sets $\varphi(D_i)$ are mutually disjoint. We note also that

$$\varphi(D_j) \subset [t_{j-1}, t_j]. \tag{3}$$

and $\varphi(\cup_i(D_j)) = \cup_i(\varphi(D_i)) \subset [0,1]$. Therefore φ is one-to-one and we can define $\varphi^{-1} : \varphi(\cup_i D_i) \to \cup_i D_i$.

We prove that the function φ is measurable and preserves the measure. As for measurability it is enough to note that for any $0 < c < 1$ there exist j and y such that

$$\{x : \varphi(x) < c\} = (\cup_{i=1}^{j-1} D_i) \cup (D_j \cap [0,y))$$

where j is chosen in such a way that $\sum_{i=1}^{j-1} \mu(D_i) \leq c < \sum_{i=1}^{j} \mu(D_i)$.

Because of σ-additivity of the measure and because the sets D_j are disjoint together with their images, it is enough to prove that φ is measure preserving mapping on each $D_j, j = 1, 2, \ldots$. So let j be fixed.

We shall use the following estimate (see [6], ch. VII, theorem 6.5): if a measurable function F is differentiable on a measurable set A then

$$\mu^*(F(A)) \leq \int_A |F'(x)| d\mu. \tag{4}$$

We apply the above estimation for a function φ_j defined on $[0, 1]$ by

$$\varphi_j(x) = \sum_{i=1}^{j-1} \mu(D_i) + \int_0^x \chi_{D_j} d\mu.$$

The function φ_j is continuous being the indefinite Lebesgue integral. We obviously have $\varphi_j([0, 1]) = [t_{j-1}, t_j]$ and so

$$\varphi_j(1) - \varphi_j(0) = \mu(D_j). \tag{5}$$

We note also that for $x \in D_j$ we have $\varphi_j(x) = \varphi(x)$. Since each point $x \in D_j$ is a point of density of D_j then $\varphi_j'(x) = 1$ for such x. Now using (4) for any measurable set M, $M \subset D_j$, we obtain

$$\mu^*(\varphi(M)) = \mu^*(\varphi_j(M)) \leq \int_M \chi_{D_j} d\mu = \mu(M). \tag{6}$$

In particular we have

$$\mu^*(\varphi(D_j)) \leq \mu(D_j). \tag{7}$$

Let $S_j = \{x \in [0, 1] : \varphi_j'(x) = 0\}$ and

$$P_j = \{x \in [0, 1] : 0 < \varphi_j'(x) < 1 \text{ or } \varphi_j'(x) \text{ does not exists}\}.$$

The Lebesgue density theorem implies that

$$\mu(S_j) = \mu([0, 1] \setminus D_j) \text{ and } \mu(P_j) = 0.$$

Applying (4) to the function φ_j and the set S_j we get

$$\mu(\varphi_j(S_j)) = 0. \tag{8}$$

The function φ_j being the indefinite Lebesgue integral is absolutely continuous and so has Lusin (N)-property, hence

$$\mu(\varphi_j(P_j)) = 0. \tag{9}$$

Now combining the (7), (8) and (9) we obtain

$$\mu(\varphi_j([0,1])) \le \mu^*(\varphi_j(D_j)) + \mu(\varphi_j(P_j)) + \mu(\varphi_j(S_j)) = \mu^*(\varphi(D_j)) \le \mu(D_j). \quad (10)$$

As φ_j is monotonic and continuous on $[0,1]$, so $\mu(\varphi_j([0,1])) = \varphi_j(1) - \varphi_j(0)$. Combining this with (5) and (10) we get

$$\mu(D_j) \le \mu^*(\varphi(D_j)) \le \mu(D_j).$$

Therefore we finally obtain

$$\mu^*(\varphi(D_j)) = \mu(D_j) = t_{j-1} - t_j. \quad (11)$$

Moreover $\varphi(D_j)$ is measurable. Indeed

$$\varphi(D_j) = \varphi_j(D_j) \supset \varphi_j([0,1]) \setminus (\varphi_j(P_j) \cup \varphi_j(S_j)) = [t_{j-1}, t_j] \setminus (\varphi_j(P_j) \cup \varphi_j(S_j)).$$

This together with (3) shows that $\varphi(D_j)$ coincides with the interval $[t_{j-1}, t_j]$ up to the set of measure zero and hence it is measurable. So we can rewrite (11) as

$$\mu(\varphi(D_j)) = \mu(D_j). \quad (12)$$

To get the same equality for any measurable M, $M \subset D_j$ we rewrite (6) for $D_j \setminus M$ obtaining $\mu^*(\varphi(D_j \setminus M)) \le \mu(D_j \setminus M)$. This together with (12) and the subadditivity of outer measure gives

$$\mu^*(\varphi(M)) \ge \mu(\varphi(D_j)) - \mu^*(\varphi(D_j \setminus M)) \ge \mu(D_j) - \mu(D_j \setminus M) = \mu(M).$$

Comparing this with (6) we obtain that $\mu^*(\varphi(M)) = \mu(M)$ for any $M \subset D_j$.

From this, (12) and the fact that the mapping φ is one-to-one on D_j we get

$$\mu^*(\varphi(D_j) \setminus \varphi(M)) = \mu^*(\varphi(D_j \setminus M)) = \mu(D_j \setminus M) = \mu(D_j) - \mu(M) =$$
$$= \mu(\varphi(D_j) - \mu^*(\varphi(M)).$$

Considering $\varphi(M)$ as a subset of measurable set $\varphi(D_j)$ we can interpret the above equality as Lebe sgue criterium for measurability of $\varphi(M)$. So we have proved that φ is a measure preserving mapping on D_j and therefore on whole $\cup_i D_i$.

We also have

$$\mu(\varphi(\cup_i D_i)) = \mu(\cup_i(\varphi(D_i))) = \sum_i \mu(\varphi(D_i)) = \sum_i \mu(D_i) = 1.$$

So both functions φ and φ^{-1} are mapping $[0,1]$ onto $[0,1]$, up to a set of measure zero. We show now that $\psi_\alpha := \varphi^{-1}$ is the function we are looking for.

To prove that ψ_α is also measure preserving mapping it is enough to check that the pre-image of any measurable set under our mapping φ is measurable. So, let $\varphi(E)$ be a measurable set then $\varphi(E) = A \cup B$ where A is a Borel set and $\mu(B) = 0$. Then $E = \varphi^{-1}(A) \cup \varphi^{-1}(B)$, with $\varphi^{-1}(A)$ measurable as pre-image of a Borel set under measurable mapping. We can also find a Borel set G such that $B \subset G$ and $\mu(G) = 0$. Therefore $\varphi^{-1}(B) \subset \varphi^{-1}(G)$ with $\varphi^{-1}(G)$ measurable. Since we know that φ is measure preserving mapping on the class of measurable sets we get $\mu(G) = \mu^*(\varphi(\varphi^{-1}(G))) = \mu(\varphi^{-1}(G))$. This implies $\mu(\varphi^{-1}(B)) = 0$ and then $\varphi^{-1}(B)$ is measurable. This proves the measurability of E. As φ is one-to-one on $[0,1]$ up to the set of measure zero and is measure preserving mapping we obtain that $\psi_\alpha = \varphi^{-1}$ is also measure preserving mapping.

The function $f(\psi_\alpha(y))$ is defined almost everywhere on $[0,1]$. As the Lebesgue integral is invariant under measure preserving mapping we get

$$\int_{t_{j-1}}^{t_j} f(\psi_\alpha(y))d\mu_y = \int_{\varphi(D_j)} f(\psi_\alpha(y))d\mu_y = \int_{D_j} f(x)d\mu_x = c_j.$$

Therefore we get $\int_0^{t_n} f(\psi(y))d\mu_y = \sum_{k=1}^n c_k$.

So, having in mind that $\sum_{n=1}^{+\infty} c_n = \alpha$, we obtain

$$\lim_{n\longrightarrow\infty} \int_0^{t_n} f(\psi)d\mu_y = \lim_{n\longrightarrow\infty} \sum_{k=1}^n c_k = \alpha.$$

Considering now any t, $0 < t < 1$, there exists n such that $t_{n-1} < t < t_n$ and the interval (t_{n-1}, t_n) is the image of D_j, up to a set of measure zero. As the function $f(\psi)$ keeps the sign on $[t_{n-1}, t_n]$, then the value of $\int_0^t f(\varphi^{-1}(y))d\mu_y$ is between the values $\int_0^{t_{n-1}} f(\varphi^{-1}(y))d\mu_y$ and $\int_0^{t_n} f(\varphi^{-1}(y))d\mu_y$, and we conclude

$$\lim_{t\longrightarrow 1} \int_0^t f(\varphi^{-1}(y))d\mu_y = \alpha$$

proving that improper Lebesgue integral of function $f(\psi(y))$ on $[0,1]$ is equal to α. Now applying Theorem 2.1 we complete the proof of Theorem 2.2. $\square$

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GIUSEPPE RAO
Department of Mathematic and Applications
Palermo University
e-mail: rao@math.unipa.it

FRANCESCO TULONE
Department of Mathematic and Applications
Palermo University
e-mail: tulone@math.unipa.it