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ON THE NON-EXISTENCE OF ORTHOGONAL INSTANTON BUNDLES ON $\mathbb{P}^{2n+1}$

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In this paper we prove that there do not exist orthogonal instanton bundles on $\mathbb{P}^{2n+1}$. In order to demonstrate this fact, we propose a new way of representing the invariant, introduced by L. Costa and G. Ottaviani, related to a rank $2n$ instanton bundle on $\mathbb{P}^{2n+1}$.

1. Introduction

Instanton bundles were introduced in [4] on $\mathbb{P}^3$ and in [7] on $\mathbb{P}^{2n+1}$. Recently many authors have been dealing with them. The geometry of special instanton bundles is discussed in [8]. Authors of [3] study the stability of instanton bundles. They prove that all special symplectic instanton bundles on $\mathbb{P}^{2n+1}$ are stable and all instanton bundles on $\mathbb{P}^5$ are stable. Some computations involving instanton bundles can be done using computer programs, e.g. Macaulay, as it is presented in [2]. The $SL(2)$-action on the moduli space of special instanton bundles on $\mathbb{P}^3$ is described in [5]. In [6] it is shown that the moduli space of symplectic instanton bundles on $\mathbb{P}^{2n+1}$ is affine, by introducing the invariant that we use in this paper.

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As we see, instanton bundles were discussed in many papers. On $\mathbb{P}^3$ they have rank 2 and they are symplectic. On $\mathbb{P}^{2n+1}$, for $n \geq 2$, theoretically they can be divided in three groups: symplectic, orthogonal and neither symplectic nor orthogonal bundles. Lots of examples of symplectic bundles are known as well as of bundles from the third group. However, it was not known whether there exist orthogonal instanton bundles. We prove, see Theorem 3.3, that it's impossible for such bundles to exist. The question was posed by Trautmann in [9].

The paper is organised as follows. In the Preliminaries we give definitions of instanton bundle, orthogonal instanton bundle and symplectic instanton bundle. We state equivalent conditions for a bundle to be respectively orthogonal or symplectic. In the main part we prove the fact that there do not exist orthogonal instanton bundles on $\mathbb{P}^{2n+1}$ for an arbitrary n and arbitrary $k = c_2(E)$.

2. Preliminaries

Definition 2.1. Let X be a smooth projective variety. A *monad* on X is a complex of vector bundles:

$$0 \longrightarrow F \xrightarrow{\beta} G \xrightarrow{\alpha} H \longrightarrow 0$$

which is exact at F and at H.

Let V be a vector space of dimension $2n+2$ and $\mathbb{P}^{2n+1} = \mathbb{P}(V)$. Let U and I be vector spaces of dimension k and let W be a vector space of dimension $2n+2k$. We consider the following monad on $\mathbb{P}^{2n+1}$:

$$U \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(-1) \xrightarrow{\beta} W \otimes \mathcal{O}_{\mathbb{P}^{2n+1}} \xrightarrow{\alpha} I \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(1). \tag{1}$$

Note that α and β can be represented by two matrices A and B^t respectively, where A and B are $k \times (2n+2k)$-matrices with linear entries of maximal rank for every point in $\mathbb{P}^{2n+1}$ and $A \cdot B^t = 0$. The condition for A and B to be of maximal rank for every point in $\mathbb{P}^{2n+1}$ is equivalent to require that the map α is surjective and the map β is injective.

From now on, with a little abuse of notation, we will identify a linear map α with the matrix A associated to it, and we will simply write A.

Definition 2.2. We say that E is an *instanton bundle* (with quantum number k) if E is the cohomology of a monad as in (1), i.e.

$$E = \ker A / \operatorname{im} B^t.$$

Properties 2.3.

- *For an instanton bundle E we have that* $\operatorname{rk} E = 2n$.
- *The Chern polynomial of E is* $c_t = (1-t^2)^{-k} = 1 + kt^2 + \binom{k+1}{2} t^4 + \ldots$
  In particular, $c_1(E) = 0$ *and* $c_2(E) = k$.
- $E(q)$ *has natural cohomology in the range* $-2n-1 \leq q \leq 0$*, i.e.* $h^i(E(q)) \neq 0$ *for at most one* $i = i(q)$.

Definitions 2.4. We say that an instanton bundle E is *symplectic* if there exists an isomorphism $\alpha : E \longrightarrow E^*$ such that $\alpha = -\alpha^*$.

We say that an instanton bundle E is *orthogonal* if there exists an isomorphism $\alpha : E \longrightarrow E^*$ such that $\alpha = \alpha^*$.

Having a symplectic instanton bundle is equivalent to the existence of a non-degenerate, skew-symmetric matrix J such that $B = A \cdot J$. After linear change of coordinates we may assume that J is of the form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and $A \cdot J \cdot A^t = 0$.

Having an orthogonal instanton bundle is equivalent to the existence of a non-degenerate, symmetric matrix J such that $B = A \cdot J$. We may assume that $J = I$ and $A \cdot J \cdot A^t = A \cdot A^t = 0$.

3. On the non-existence

Recalling the definition we gave before, having an orthogonal instanton bundle means having an application

$$W \otimes \mathcal{O}_{\mathbb{P}^{2n+1}} \xrightarrow{A} I \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(1) \tag{2}$$

with the additional properties that $A \cdot A^t = 0$, and A has maximal rank for every point in $\mathbb{P}^{2n+1}$.

Let us choose a basis for each vector space we are considering. In particular we have $i_1, \ldots, i_k$ as a basis of I, $w_1, \ldots, w_{2n+2k}$ as a basis of W and $v_0, \ldots, v_{2n+1}$ as a basis of V.

From the map defined in (2) we can induce a second map

$$W \xrightarrow{M} I \otimes V \tag{3}$$

over the global sections where M is a $(k(2n+2)) \times (2n+2k)$-matrix.

We want to construct such matrix in a particular way, i.e.

$$M := \left(\begin{array}{c} M_1 \\ \hline \vdots \\ \hline M_k \end{array} \right)$$

where, fixed j, the block M_j is a $(2n+2) \times (2n+2k)$-matrix "associated" to the element i_j of the basis of I. In other words the block M_j represents the following correspondence:

$$M_j : w \mapsto i_j \otimes M_j(w) \text{ with } i_j \text{ fixed.}$$

Using the matrix M it is possible to reconstruct the quadratic conditions given by $A \cdot A^t = 0$. Indeed, let us observe that the matrix A can be described by the following matrix

$$A = \left(\begin{array}{c} [x_0, \dots, x_{2n+1}] \cdot M_1 \\ \hline \vdots \\ \hline [x_0, \dots, x_{2n+1}] \cdot M_k \end{array} \right),$$

where x_i are the coordinates in V with respect to the basis $\{v_0, \dots, v_{2n+1}\}$.

The most natural idea is to consider the product $M \cdot M^t$, that is of the form

$$M \cdot M^t = \begin{pmatrix} M_1 \cdot M_1^t & M_1 \cdot M_2^t & \cdots & M_1 \cdot M_k^t \\ M_2 \cdot M_1^t & M_2 \cdot M_2^t & \cdots & M_2 \cdot M_k^t \\ \vdots & & & \vdots \\ M_k \cdot M_1^t & M_k \cdot M_2^t & \cdots & M_k \cdot M_k^t \end{pmatrix},$$

where each block $M_\alpha \cdot M_\beta^t$, for α, $\beta = 1, \dots, k$, is a square matrix representing a quadratic form in the variables $x_0, \dots, x_{2n+1}$. Because of this considerations, we can state the following equivalence of quadratic conditions

$$A \cdot A^t = 0 \iff M_\alpha \cdot M_\beta^t + M_\beta \cdot M_\alpha^t = 0, \text{ for all } \alpha, \beta = 1, \dots, k. \quad (4)$$

Let us observe, as in [6], that the vector spaces $W \otimes \mathcal{S}^n I$ and $\mathcal{S}^{n+1} I \otimes V$ have the same dimension $(2n+2k)\binom{k+n-1}{n} = (2n+2)\binom{k+n}{n+1}$, where we denote by $\mathcal{S}^n I$ the n-th symmetric power of I.

We can induce from

$$W \xrightarrow{M} I \otimes V$$

the morphisms

$$M \otimes id_{\mathcal{S}^n I} : W \otimes \mathcal{S}^n I \longrightarrow V \otimes I \otimes \mathcal{S}^n I,$$

$$id_V \otimes \pi : V \otimes I \otimes \mathcal{S}^n I \longrightarrow V \otimes \mathcal{S}^{n+1} I,$$

with π denoting the natural projection, and we consider the composition

$$Q = (id_V \otimes \pi) \circ (M \otimes id_{\mathcal{S}^n I}). \tag{5}$$

We recall a result stated in [6], that, in our situation, is the following

Theorem 3.1. *If we have an instanton bundle given by A, then* $\det Q \neq 0$*, i.e. Q is non-degenerate.* $\det Q$ *is* $SL(W) \times SL(I) \times SL(V)$*-invariant.*

As we did for M, we construct the matrix associated to Q. First of all we fix the lexicographic order induced by $i_1 > i_2 > \ldots > i_k$ on the elements of the basis of $\mathcal{S}^n I$ and on the elements of the basis of $\mathcal{S}^{n+1} I$. We label with $\zeta_1, \ldots, \zeta_s$ the elements of $\mathcal{S}^n I$ and by $\eta_1, \ldots, \eta_r$ the elements of $\mathcal{S}^{n+1} I$, where $s = \binom{k+n-1}{n}$ and $r = \binom{k+n}{n+1}$. We decompose Q in $r \cdot s$ blocks of dimension $(2n+2) \times (2n+2k)$. The block Q_{ij}, is the matrix that represents the linear map

$$w \otimes \zeta_j \longmapsto Q_{ij}(w) \otimes \eta_i,$$

for $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$.

We will see later, with more details, that each block Q_{ij} is either one of the M_j blocks or the matrix of all zeros.

Fact 3.2. *Considering the matrix Q we have constructed, there always exists a syzygy* $V^* \otimes \mathcal{S}^{n-1} I$ *with*

$$V^* \otimes \mathcal{S}^{n-1} I \xrightarrow{S} W \otimes \mathcal{S}^n I \xrightarrow{Q} V \otimes \mathcal{S}^{n+1} I$$

such that $Q \cdot S = 0$ *and* $S \neq 0$*.*

In particular, we are going to demonstrate that the map S, represented by a $\left((2n+2k)\binom{k+n-1}{n}\right) \times (2n+2)$-matrix of the form

$$S = \begin{pmatrix} M_1^t \\ \hline \vdots \\ \hline M_k^t \\ \hline 0 \\ \hline \vdots \\ \hline 0 \end{pmatrix} \tag{6}$$

will accomplish the property we are looking for.

We are going to abuse some language now, because remembering that Q is divided by blocks, from now on we will refer to a column of blocks of Q as a

column of Q and, the same way, to a row of blocks of Q as a row of Q.
We observe that each column of Q is related to an element of the basis of $\mathcal{S}^nI$ and each row is related to an element of the basis $\mathcal{S}^{n+1}I$. We advice to take a look at the next example to have a better understanding of all the constructions we made.

In order to prove that the map S, defined in (6), is such that $Q \cdot S = 0$, we have only to look at the first k columns of Q. This means that we just need to check all possible passages from the basis elements $\left\{i_1^n, i_1^{n-1}i_2, \ldots, i_1^{n-1}i_k\right\} = \left\{i_1^{n-1}i_\alpha \mid \alpha = 1, \ldots, k\right\} = \mathcal{I} \subset \mathcal{S}^nI$ to all the element of the basis of $\mathcal{S}^{n+1}I$ of the form $\left\{i_1^{n-1}i_\alpha i_\beta \mid \alpha, \beta = 1, \ldots, k\right\}$. Let us observe that we are only considering all the elements of the two basis that contain the monomial i_1^{n-1}. We now have two options:

- We can get every element of $\mathcal{S}^{n+1}I$ of the type $i_1^{n-1}i_\alpha^2$, for $\alpha = 1, \ldots, k$, from an element of $\mathcal{I}$, only multiplying $i_1^{n-1}i_\alpha$ by i_α.
  This means that the only non-vanishing block of the row of Q corresponding to the element $i_1^{n-1}i_\alpha^2$ will be a block M_α in the α-th column. Multiplying this row by S we will obtain $M_\alpha \cdot M_\alpha^t$ which is zero for each α, because of the quadratic relations.

- We can get the elements of $\mathcal{S}^{n+1}I$ of the type $i_1^{n-1}i_\alpha i_\beta$, $\alpha \neq \beta$ in only two different ways: multiplying $i_1^{n-1}i_\alpha$ by i_β or multiplying $i_1^{n-1}i_\beta$ by i_α.
  This means that the only non-vanishing blocks of the row of Q corresponding to the element $i_1^{n-1}i_\alpha i_\beta$ will be a block M_α in the β-th column and a block M_β in the α-th column. Multiplying this row by S we will obtain $M_\alpha \cdot M_\beta^t + M_\beta \cdot M_\alpha^t$ which is zero for each α, β, because of the quadratic relations.

Because of everything we said, the Fact 3.2 is proved.

From Fact 3.2 we have that the kernel of Q is not zero, so $\det Q = 0$. This gives us a contradiction with Theorem 3.1.

Due to all our previous work, we can state

Theorem 3.3. *There do not exist orthogonal instanton bundles E on $\mathbb{P}^{2n+1}$, for each k, $n \in \mathbb{N}$.*

Example. We want to write the following simple case in order to give to the reader a better understanding of the previous proof.

It is known that for $n = 1$, see [7], there do not exist instanton bundles, so taking a simple case, we consider an instanton bundle E with $n = 2$ and $k = 4$. Let $i_1, \ldots, i_4$ be a basis of I, $v_0, \ldots, v_5$ be a basis of V and $w_1, \ldots, w_{12}$ be a basis of W. In this particular case we have a map

$$W \xrightarrow{M} I \otimes V$$

with M a 24×12-matrix. Remembering what we said before, M will have the form

$$M = \left(\begin{array}{c} M_1 \\ \hline M_2 \\ \hline M_3 \\ \hline M_4 \end{array} \right).$$

In this case, we consider the map

$$W \otimes \mathcal{S}^2 I \xrightarrow{Q} \mathcal{S}^3 I \otimes V$$

induced by M. Here Q is represented by a square matrix of order 120. Basing on what we have said, Q will have the following form

$$Q = \left(\begin{array}{|c|c|c|c|c|c|c|c|c|c||c}
\hline
M_1 & & & & & & & & & & i_1^3 \\ \hline
M_2 & M_1 & & & & & & & & & i_1^2 i_2 \\ \hline
M_3 & & M_1 & & & & & & & & i_1^2 i_3 \\ \hline
M_4 & & & M_1 & & & & & & & i_1^2 i_4 \\ \hline
 & M_2 & & & M_1 & & & & & & i_1 i_2^2 \\ \hline
 & M_3 & M_2 & & & M_1 & & & & & i_1 i_2 i_3 \\ \hline
 & M_4 & & M_2 & & & M_1 & & & & i_1 i_2 i_4 \\ \hline
 & & M_3 & & & & & M_1 & & & i_1 i_3^2 \\ \hline
 & & M_4 & M_3 & & & & & M_1 & & i_1 i_3 i_4 \\ \hline
 & & & M_4 & & & & & & M_1 & i_1 i_4^2 \\ \hline
 & & & & M_2 & & & & & & i_2^3 \\ \hline
 & & & & M_3 & M_2 & & & & & i_2^2 i_3 \\ \hline
 & & & & M_4 & & M_2 & & & & i_2^2 i_4 \\ \hline
 & & & & & M_3 & & M_2 & & & i_2 i_3^2 \\ \hline
 & & & & & M_4 & M_3 & & M_2 & & i_2 i_3 i_4 \\ \hline
 & & & & & & M_4 & & & M_2 & i_2 i_4^2 \\ \hline
 & & & & & & & M_3 & & & i_3^3 \\ \hline
 & & & & & & & M_4 & M_3 & & i_3^2 i_4 \\ \hline
 & & & & & & & & M_4 & M_3 & i_3 i_4^2 \\ \hline
 & & & & & & & & & M_4 & i_4^3 \\ \hline \hline
i_1^2 & i_1 i_2 & i_1 i_3 & i_1 i_4 & i_2^2 & i_2 i_3 & i_2 i_4 & i_3^2 & i_3 i_4 & i_4^2 &
\end{array} \right)$$

where the empty entries represent 6×12-matrices of zeros. We observe that in the bottom and right part of the matrix we can see, respectively, the elements of the basis of $\mathcal{S}^2 I$ related to the columns of Q and the elements of the basis of $\mathcal{S}^3 I$ related to the rows of Q.

Taking in mind the quadratic conditions expressed in (4), it is immediate to check that

$$Q \cdot S = Q \cdot \begin{pmatrix} M_1^t \\ \hline M_2^t \\ \hline M_3^t \\ \hline M_4^t \\ \hline 0 \\ \hline \vdots \\ \vdots \\ \hline 0 \end{pmatrix} = 0,$$

with $S \neq 0$. We can conclude that for the case $k = 4$ and $n = 2$ there do not exist orthogonal instanton bundles.

Corollary 3.4. *Any self-dual instanton bundle E is symplectic.*

Proof. The bundle E is simple by [3]. Then

$$1 = h^0(E \otimes E^*) = h^0(S^2 E) + h^0(\wedge^2 E).$$

Theorem 3.3 implies that $h^0(S^2 E) = 0$ so we have $h^0(\wedge^2 E) = 1$ and E is symplectic. □

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REFERENCES

[1] V. Ancona, *I fibrati istantoni sugli spazi proiettivi. Teoria, metodi algoritmici e risultati sperimentali*, Volumi INdAM, Aracne, Roma, 1996.

[2] V. Ancona - G. Anzidei - P. Breglia - G. Ottaviani - A. Pizzotti, *Mathematical instanton bundles on projective spaces: an algorithmic approach*, Differential operators and mathematical physics. Papers of the colloquium, Coimbra, Portugal, June 19-22. Coimbra: Universidade de Coimbra, Departamento de Matematica. Textos Mat. Ser. B 24 (1-3) (2000), 143–159.

[3] V. Ancona - G. Ottaviani, *Stability of special instanton bundles on* $\mathbb{P}^{2n+1}$, Trans. Amer. Math. Soc. 341 (2) (1994), 677–693.

[4] M. Atiyah - V. Drinfeld - N. Hitchin - Yu. Manin, *Construction of instantons*, Phys. Lett. 65A (1978), 185–187.

[5] L. Costa - G. Ottaviani, *Group action on instanton bundles over* $\mathbb{P}^3$, Math. Nachr. 246/247 (2002), 31–46.

[6] L. Costa - G. Ottaviani, *Nondegenerate Multidimensional Matrices and Instanton Bundles*, Trans. Amer. Math. Soc. 355 (1) (2002), 49–55.

[7] Ch. Okonek - H. Spindler, *Mathematical instanton bundles on* $\mathbb{P}^{2n+1}$, J. Reine Angew. Math. 364 (1986).

[8] H. Spindler - G. Trautmann, *Special instanton bundles on* $\mathbb{P}^{2N+1}$*, their geometry and their moduli*, Math. Ann. 286 (1-3) (1990), 559–592.

[9] G. Trautmann, *Generalities about Instanton Bundles, unpublished note*, (1994).

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