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SPLITTING TYPES OF SEMISTABLE BUNDLES ON $\mathbb{P}^2$

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We show that for $n \leq 5$ all generic splitting types of semistable vector bundles of rank n on $\mathbb{P}^2$ which are in principle possible by the theorem of Grauert-Mülich actually occur. We prove this by constructing examples for all possible splitting types.

1. Introduction

Fix an algebraically closed field K of characteristic zero. We consider semistable vector bundles on $\mathbb{P}^n$, $n \geq 2$. If one restricts such a bundle $\mathscr{E}$ to a line $L = \mathbb{P}^1 \subseteq \mathbb{P}^n$ one obtains a splitting

$$\mathscr{E}|_L = \bigoplus_{i=1}^{\operatorname{rk}\mathscr{E}} \mathscr{O}_L(a_i)$$

by a theorem of Grothendieck. One therefore obtains an ordered tuple

$$a_{\mathscr{E}}(L) = (a_1(L), \ldots, a_{\operatorname{rk}\mathscr{E}}(L)) \in \mathbb{Z}^{\operatorname{rk}\mathscr{E}}, \quad a_1(L) \geq \ldots \geq a_{\operatorname{rk}\mathscr{E}}(L)$$

which we call the *splitting type* of $\mathscr{E}$ on L.

The first natural question to ask is what happens if L varies. For this denote by $Gr(1,n)$ the Grassmanian of lines in $\mathbb{P}^n$. One has the following

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Theorem 1.1. *There is an open dense subset $U \subseteq Gr(1,n)$ such that $\mathscr{E}$ splits as*

$$a_{\mathscr{E}} = \inf_{L \in Gr(1,n)} a_{\mathscr{E}}(L),$$

where $\mathbb{Z}^{\operatorname{rk}\mathscr{E}}$ is endowed with the lexicographic ordering. This is called the generic splitting type *of $\mathscr{E}$.*

Proof. See [2, Ch. 1, Paragraph 3, Lemma 3.2.2]. □

Given a vector bundle $\mathscr{E}$ on $\mathbb{P}^n$, define the *slope* $\mu(\mathscr{E})$ as $\frac{c_1(\mathscr{E})}{\operatorname{rk}\mathscr{E}}$. We call the bundle $\mathscr{E}$ *semistable* if $\mu(\mathscr{F}) \leq \mu(\mathscr{E})$ for every nontrivial subsheaf $\mathscr{F}$ of $\mathscr{E}$. For more information about semistability see [2, Ch. 2, Paragraph 3].

One has a theorem of Grauert, Mülich and Spindler which imposes certain bounds on the generic splitting type for semistable bundles:

Theorem 1.2. *Let $\mathscr{E}$ be a semistable vector bundle of rank r on $\mathbb{P}^n$ with generic splitting type $a_{\mathscr{E}} = (a_1, \ldots, a_r)$. Then one has $0 \leq a_i - a_{i+1} \leq 1$ for $1 \leq i \leq r-1$.*

Proof. See [2, Ch. 2, Paragraph 2, Corollary 1]. □

It is now a natural question to ask what generic splitting types one can actually realise for semistable bundles of a given rank r. In particular it is interesting to ask whether all possible splitting types can actually occur.

From now on we will restrict our attention to $\mathbb{P}^2$. Let $M_{\mathbb{P}^2}(r, c_1, c_2)$ denote the moduli space of rank r semistable vector bundles with fixed Chern classes c_1 and c_2. One partial result in this direction is the following

Theorem 1.3. *There is an open dense subset $U \subseteq M_{\mathbb{P}^2}(r, c_1, c_2)$ such that any $\mathscr{E} \in U$ has generic splitting type*

$$(\underbrace{n+1, \ldots, n+1}_{s}, \underbrace{n, \ldots n}_{r-s})$$

where $s(n+1) + (r-s)n = c_1(\mathscr{E})$.

Proof. See [1, Corollary 2.5]. □

For $\operatorname{rk}\mathscr{E} < 4$ it is already known that all possible splitting types can actually occur. This is rather easy in the cases $\operatorname{rk}\mathscr{E} = 1, 2$. For the case of $\operatorname{rk}\mathscr{E} = 3$ see e.g. [1].

In investigating the splitting type of a semistable bundle $\mathscr{E}$ we may assume that it is normalised, that is $-\operatorname{rk}\mathscr{E} + 1 \leq c_1(\mathscr{E}) \leq 0$. There are then precisely $2^{\operatorname{rk}\mathscr{E}-1}$ possible splitting types. Moreover, if $\mathscr{E}$ is semistable with a given splitting type then its dual $\mathscr{E}^\vee$ is again semistable and yields a bundle with first Chern class $c_1(\mathscr{E}^\vee) = -c_1(\mathscr{E})$. Normalising (that is twisting by $\mathscr{O}_{\mathbb{P}^2}(-1)$) if $c_1(\mathscr{E}) \neq 0$ one obtains a bundle $\mathscr{E}'$ with first Chern class $c_1(\mathscr{E}) - r$. Since pullback and dualising commute we may use this method to reduce the number of bundles to construct. More precisely we reduce to the first $\frac{\mathrm{rk}\mathscr{E}-2}{2}+2$ Chern classes if the rank of $\mathscr{E}$ is even and to the first $\frac{\mathrm{rk}\mathscr{E}-1}{2}+1$ Chern classes if the rank is odd.

2. Useful facts

Here we recall some methods to construct semistable bundles from given semistable bundles of lower rank.

Lemma 2.1. *Let $\mathscr{E}$ and $\mathscr{F}$ be semistable vector bundles with generic splitting types $(a_1,\dots,a_p)$ and $(b_1,\dots,b_q)$ respectively and let L be a generic line on $\mathbb{P}^n$, then:*

1. *$S^k\mathscr{E}$ is semistable and $S^k\mathscr{E}|_L = \bigoplus_{1\le i_1\le\ldots\le i_k\le p}\mathscr{O}_L(a_{i_1}+\ldots+a_{i_k})$.*
2. *$\Lambda^k\mathscr{E}$ is semistable and $\Lambda^k\mathscr{E}|_L = \bigoplus_{1<i_1<\ldots<i_k<p}\mathscr{O}_L(a_{i_1}+\ldots+a_{i_k})$.*
3. *$\mathscr{E}\otimes\mathscr{F}$ is semistable and $\mathscr{E}\otimes\mathscr{F}|_L = \bigoplus_{i=1}^p\bigoplus_{j=1}^q\mathscr{O}_L(a_i+b_j)$.*
4. *$\mathscr{E}\oplus\mathscr{F}$ is semistable if and only if $\mathscr{E}$ and $\mathscr{F}$ have equal slopes, i.e. $c_1(\mathscr{E})\,\mathrm{rk}\mathscr{F} = c_1(\mathscr{F})\,\mathrm{rk}\mathscr{E}$. Furthermore,*

$$(\mathscr{E}\oplus\mathscr{F})|_L = \bigoplus_{i=1}^{p}\mathscr{O}_L(a_i)\oplus\bigoplus_{i=1}^{q}\mathscr{O}_L(b_i).$$

Proof. The claim about the semistability follows from [8, Corollary 3.2.10] in the cases (1) - (3). For the semistability in (4) see [2, Ch. 2, Paragraph 1, Lemma 1.2.4 (ii)]. Since symmetric resp. exterior powers and pullbacks commute we reduce to the case of $\mathbb{P}^1$. Now, by [6, Proposition 2.5.4 and Proposition 2.5.13] we have that the sheafification functor $\tilde{}$ commutes with direct sums and tensor products. Thus the splitting behavior of (1) and (2) follows by [4, Corollary A2.3] and is clear in case of (3) and (4). □

Lemma 2.2. *The k-th symmetric power of the sheaf of differentials $\Omega_{\mathbb{P}^2}$ is semistable and splits on a generic line as*

$$\bigoplus_{i=0}^{k}\mathscr{O}_L(-i).$$

Proof. Note that the dualised Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^2}(1) \longrightarrow \mathscr{O}^3_{\mathbb{P}^2} \xrightarrow{A} \mathscr{O}_{\mathbb{P}^2}(1) \longrightarrow 0$$

remains exact after restricting to a general line L given by an equation $z = ax + by$. However, on L we have the exact sequence

$$0 \longrightarrow \mathscr{O}_L \oplus \mathscr{O}_L(-1) \xrightarrow{B} \mathscr{O}^3_L \xrightarrow{C} \mathscr{O}_L(1) \longrightarrow 0,$$

where

$$A = \begin{bmatrix} x & y & z \end{bmatrix}, \quad B = \begin{bmatrix} -a & -y \\ -b & x \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} x & y & ax+by \end{bmatrix}.$$

Thus $\Omega_{\mathbb{P}^2}(1)$ splits on a generic line as $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$. Furthermore, $\Omega_{\mathbb{P}^2}$ has no global sections, thus all its subbundles have negative slope. Hence $\Omega_{\mathbb{P}^2}(1)$ is semistable with splitting type $(0,-1)$, the rest of the claim follows from Lemma 2.1(1). $\square$

3. Calculating splitting types and verifying semistability

Given an $m \times n$ matrix A with degree k homogenous polynomials as entries, we consider a vector bundle $\mathscr{E}$ given by the kernel of a surjective map $\mathscr{O}^n_{\mathbb{P}^2} \xrightarrow{A} \mathscr{O}^m_{\mathbb{P}^2}(k)$. We can calculate the general splitting type of $\mathscr{E}$ by hand, however the calculations tend to be complicated and strictly numeric in nature. Thus we will use the computer program SINGULAR [5] and the following source code to calculate it:

```
ring R = (0,a,b),(x,y,z),dp;
matrix A[m][n] = the given matrix;
matrix B[m][1] = 0;
module M = std(modulo(A,B));
module N = std(subst(M,z,ax+by));
module Nx =0;
while(N<>Nx)
Nx=N;
N=std(quotient(N,x));
```

Note that SINGULAR deals with homogenous polynomials in $R = \mathbb{Q}[x,y,z]$, whereas we want to calculate sections in $\mathscr{O}^n_{\mathbb{P}^2}$. However on an affine chart, e.g. $x \neq 0$, a section can be obtained from a polynomial f by dividing through $x^{\deg f}$.

The vector bundle $\mathscr{E}$ corresponds to the matrix M. Substituting in M the variable z by $ax + by$ we obtain the module N which corresponds to the restriction $\mathscr{E}|_L$ to a general line given by the equation $z = ax + by$.

We consider the affine chart $x \neq 0$ and use the procedure `quotient` several times to divide by the biggest possible power of x.

According to Grothendieck's Theorem, $\mathscr{E}|_L$ splits into a direct sum of line bundles. Thus the minimal number of generators of N is equal to the rank of $\mathscr{E}|_L$ and each of them spans one of the line bundles from the direct sum.

Consequently, $\mathscr{E}|_L = \bigoplus_{i=1}^{\mathrm{rk}\mathscr{E}} \mathscr{O}_{\mathbb{P}^2}(-a_i)$, where a_i are the degrees of the consecutive generators.

Let $A(k_1,\dots,k_n)$ denote the $n \times (n+k_n+2)$ matrix with entries

$$a_{ij} = \begin{cases} x & j = i, \\ y & j = i+k_i+1, \\ z & j = i+k_i+2, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore let $A_{b_1,\dots,b_q}$ denote the matrix $A(\underbrace{0,\dots,0}_{b_1 \text{ terms}},\dots,\underbrace{q-1,\dots,q-1}_{b_q \text{ terms}})$. Finally, let $\mathscr{E}_{b_1,\dots,b_q}$ denote the vector bundle being the kernel of the morphism $\mathscr{O}^m_{\mathbb{P}^2} \longrightarrow \mathscr{O}^n_{\mathbb{P}^2}(1)$ given by $A_{b_1,\dots,b_q}$.

We denote by $\mathrm{Syz}(f_1,\dots,f_n)$ the vector bundle given as the kernel of a surjective map $\bigoplus_{1=1}^{n} \mathscr{O}_{\mathbb{P}^2}(-d_i) \xrightarrow{f_1,\dots,f_n} \mathscr{O}_{\mathbb{P}^2}$ where $f_1,\dots,f_n$ are suitable homogeneous polynomials of degrees $d_1,\dots d_n$.

Note that $\mathscr{E}_1 = \mathrm{Syz}(x,y,z)$ is the bundle $\Omega_{\mathbb{P}^2}$ defined by the Euler sequence.

We have used the above procedure to calculate the splitting types of several vector bundles of the shape $\mathscr{E}_{b_1,\dots,b_q}$. Our results can be found in the tables in the next section where we give examples of semistable bundles with given splitting type.

To confirm that the vector bundles we construct are semistable we used an algorithm based on a criterion of Hoppe(see [7]). This was developed and implemented in CoCoA[3] by Almar Kaid(see [9, Chapter 2]).

4. Main result

Using the methods described in previous sections we have checked that up to rank 5 every splitting type allowed by theorem 1.2 is actually realised by a semistable vector bundle. The following tables contain the list of all possible splitting types of normalised vector bundles together with examples of vector bundles having a given splitting type. In many cases there are various ways of obtaining a bundle with a given splitting type. We always choose the bundle which seems to be the easiest to construct, i.e. as a power or direct sum of bundles of smaller rank.

Rank 3:

c_1	splitting type	example
0	$(0,0,0)$	$\mathscr{O}_{\mathbb{P}^2}^3$
0	$(1,0,-1)$	$S^2\Omega_{\mathbb{P}^2}(1)$
-1	$(0,0,-1)$	$\mathrm{Syz}(x^4,y^4,z^4,xyz^2)(-3)$
-2	$(0,-1,-1)$	$\mathrm{Syz}(x^4,y^4,z^4,xyz^2)^\vee(2)$

Rank 4:

c_1	splitting type	example
0	$(0,0,0,0)$	$\mathscr{O}_{\mathbb{P}^2}^4$
0	$(1,0,0,-1)$	$S^2\Omega_{\mathbb{P}^2}(1)\oplus\mathscr{O}_{\mathbb{P}^2}$
-1	$(0,0,0,-1)$	$\mathrm{Syz}(x^5,y^5,z^5,x^3yz,xy^3z)(-4)$
-1	$(1,0,-1,-1)$	$\mathrm{Syz}(x^5,y^5,z^5,x^2y^2z,xyz^3)(-4)$
-2	$(0,0,-1,-1)$	$\Omega_{\mathbb{P}^2}\oplus\Omega_{\mathbb{P}^2}$
-2	$(1,0,-1,-2)$	$S^3\Omega_{\mathbb{P}^2}(1)$
-3	$(0,-1,-1,-1)$	$\mathrm{Syz}(x^5,y^5,z^5,x^3yz,xy^3z)^\vee(3)$
-3	$(0,0,-1,-2)$	$\mathrm{Syz}(x^5,y^5,z^5,x^2y^2z,xyz^3)^\vee(3)$

Rank 5:

c_1	splitting type	example
0	$(0,0,0,0,0)$	$\mathscr{O}_{\mathbb{P}^2}^5$
0	$(1,0,0,0,-1)$	$S^2\Omega_{\mathbb{P}^2}(1)\oplus\mathscr{O}_{\mathbb{P}^2}^2$
0	$(1,1,0,-1,-1)$	$\mathscr{E}_{2,2,3,3}(2)$
0	$(2,1,0,-1,-2)$	$S^4\Omega_{\mathbb{P}^2}(2)$
-1	$(0,0,0,0,-1)$	$\mathrm{Syz}(x^6,y^6,z^6,x^3y^2z,xy^3z^2,xyz^4)(-5)$
-1	$(1,0,0,-1,-1)$	$\mathrm{Syz}(x^6,y^6,z^6,x^3y^3,x^2y^2z^2,xyz^4)(-5)$
-1	$(1,1,0,-1,-2)$	$\mathscr{E}_{1,2,3,5}(2)$
-2	$(0,0,0,-1,-1)$	$\mathrm{Syz}(x^7,y^7,z^7,x^4y^3,y^4z^3,x^3z^4)(-6)$
-2	$(1,0,-1,-1,-1)$	$\mathrm{Syz}(x^7,y^7,z^7,x^3y^3z,x^2y^2z^3,xyz^5)(-6)$
-2	$(1,0,0,-1,-2)$	$\mathscr{E}_{1,2,3,6}(2)$
-3	$(0,0,-1,-1,-1)$	$\mathrm{Syz}(x^7,y^7,z^7,x^4y^3,y^4z^3,x^3z^4)^\vee(5)$
-3	$(0,0,0,-1,-2)$	$\mathrm{Syz}(x^7,y^7,z^7,x^3y^3z,x^2y^2z^3,xyz^5)^\vee(5)$
-3	$(1,0,-1,-1,-2)$	$\mathscr{E}_{1,2,3,6}^\vee(-3)$
-4	$(0,-1,-1,-1,-1)$	$\mathrm{Syz}(x^6,y^6,z^6,x^3y^2z,xy^3z^2,xyz^4)^\vee(4)$
-4	$(0,0,-1,-1,-2)$	$\mathrm{Syz}(x^6,y^6,z^6,x^3y^3,x^2y^2z^2,xyz^4)^\vee(4)$
-4	$(1,0,-1,-2,-2)$	$\mathscr{E}_{1,2,3,5}^\vee(-3)$

We also do have promising results for rank six. Mostly by considering bundles of the type $\mathscr{E}_{b_1,\dots,b_q}$ but unfortunately the matrices are too large for CoCoA to handle.

We have not yet developed a systematic way to construct semistable bundles of type $\mathscr{E}_{b_1,\ldots,b_q}$ with a given splitting type. However, based on the computations we have made we state the following conjecture:

Conjecture. Every splitting type allowed by Theorem 1.2 is actually realised by a semistable vector bundle of type $\mathscr{E}_{b_1,\ldots,b_q}$.

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