

DERIVED CATEGORIES OF TORIC FANO 3-FOLDS VIA THE FROBENIUS MORPHISM

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In [8, Conjecture 3.6], Costa and Miró-Roig state the following conjecture:

Every smooth complete toric Fano variety has a full strongly exceptional collection of line bundles. The goal of this article is to prove it for toric Fano 3-folds.

1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field K of characteristic 0, denote by $D^b(X) = D^b(\mathcal{O}_X\text{-mod})$ the derived category of bounded complexes of coherent sheaves of \mathcal{O}_X -modules. Recall that a coherent sheaf \mathcal{F} on a smooth projective variety X is *exceptional* if $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = K$ and $\text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}, \mathcal{F}) = 0$ for $i > 0$. An ordered collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$ of coherent sheaves on X is called an *exceptional* collection if each sheaf \mathcal{F}_i is exceptional and $\text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}_k, \mathcal{F}_j) = 0$ for $j < k$ and $i \geq 0$; moreover it is called a *strongly exceptional* collection if each \mathcal{F}_i is exceptional, $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}_k, \mathcal{F}_j) = 0$

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for $j < k$ and $\text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}_j, \mathcal{F}_k) = 0$ for $i \geq 1$ and for all j, k . A strongly exceptional collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$ of coherent sheaves on X is called *full* if $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n$ generate the bounded derived category $D^b(X)$.

We are interested to study the following general problem:

Problem 1.1. To characterize smooth projective varieties X which have a full strongly exceptional collection of coherent sheaves and, even more, if there is one made up of line bundles.

In this context there is an important conjecture [11] due to King :

Conjecture 1.2. Every smooth complete toric variety has a full strongly exceptional collection of line bundles.

Kawamata [10] proved that the derived category of a smooth complete toric variety has a full exceptional collection of objects, but the objects in these collections are sheaves rather than line bundles and the collection is only exceptional and not strongly exceptional. In the toric context, there are many contributions to the above conjecture. It turns out to be true for projective spaces [2], multiprojective spaces [6, Proposition 4.16], smooth complete toric varieties with Picard number ≤ 2 [6, Corollary 4.13] and smooth complete toric varieties with a splitting fan [6, Theorem 4.12].

Recently, in [12], Hille and Perling had constructed a counterexample at the King's Conjecture, precisely an example of smooth non Fano toric surface which does not have a full strongly exceptional collection made up of line bundles. In the Fano context, there are some numerical evidences (see [7]) which give support to the following conjecture due to Costa and Miró-Roig [8, Conjecture 3.6]:

Conjecture 1.3. Every smooth complete Fano toric variety has a full strongly exceptional collection of line bundles.

But the Fano hypothesis is not necessary, in fact, in Theorem 4.12 [6], Costa and Miró-Roig constructed full strongly exceptional collections of line bundles on families of smooth complete toric varieties none of which is entirely a Fano variety.

In this article we will prove the above conjecture for Fano toric 3-folds, more specifically:

Theorem 1.4 (Main Theorem 5.1). *All smooth toric Fano 3-folds have a full strongly exceptional collection made up of line bundles.*

In order to prove the above theorem, we will refer to the classification of Fano toric 3-folds due to Batyrev ([1]). In this classification there are some

known cases, i.e. some classes of Fano toric 3-folds are just known to have a full strongly exceptional collection made up of line bundles. For each unknown case we will use a method [3] due to Bondal and an algorithm [14] by Thomsen to produce a candidate to become a full strongly exceptional collection. Then using vanishing theorems we will check that these candidates are indeed a full strongly exceptional collection.

2. The Classification of Fano Toric 3-Folds

In this section we briefly recall the usual notation and terminology for toric varieties and present the classification of toric Fano 3-folds, due to Batyrev ([1]).

Definition 2.1. A complete *toric variety* of dimension n over an algebraically closed field K of characteristic 0 is a smooth variety X that contains a torus $T = (\mathbb{C}^*)^n$ as a dense open subset, together with an action of T on X , that extends the natural action of T on itself. Let N be a lattice in \mathbb{R}^n , by a *fan* Σ of strongly convex polyhedral cones in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ is meant a set of rational strongly convex polyhedral cones σ in $N_{\mathbb{R}}$ such that:

1. each face of a cone in Σ is also a cone in Σ ;
2. the intersection of two cone in Σ is a face of each.

We will assume that a fan is finite.

It is known that a complete toric variety X is characterized by a fan $\Sigma := \Sigma(X)$. Let $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice of N , if e_0, \dots, e_{n-1} is the basis of N , we indicate by $\hat{e}_0, \dots, \hat{e}_{n-1}$ the dual base. If σ is a cone in N , the dual cone σ^{\vee} is the subset of M defined as:

$$\sigma^{\vee} := \{\eta \in M \mid \eta(v) \geq 0 \text{ for all } v \in \sigma\}.$$

So we have a commutative semigroup $S_{\sigma} := \sigma^{\vee} \cap M$ and an open affine toric subvariety $U_{\sigma} := \text{Spec}(K[S_{\sigma}])$.

From a fan Σ , the toric variety $X(\Sigma)$ is constructed by taking the disjoint union of the affine toric subvarieties U_{σ} , one for each $\sigma \in \Sigma$ and gluing them. Conversely, any toric variety X can be realized as $X(\Sigma)$ for a unique fan Σ in N . The toric variety $X(\Sigma)$ is smooth if and only if any cone $\sigma \in \Sigma$ is generated by a part of a basis of N .

Definition 2.2. We put $\Sigma(i) := \{\sigma \in \Sigma \mid \dim \sigma = i\}$, for any $0 \leq i \leq n$. There is a unique generator $v \in N$ for any 1-dimensional cone $\sigma \in \Sigma(1)$ such that $\sigma \cap N = \mathbb{Z}_{\geq 0} \cdot v$ and it is called a *ray generator*. There is a one-to-one correspondence between the ray generators v_1, \dots, v_k and the principal toric divisors

D_j . Moreover, $D_1 \cap D_2 \cap \dots \cap D_k = 0$ if and only if the corresponding vectors v_1, v_2, \dots, v_k span a cone in Σ .

If X is a smooth toric variety of dimension n and m is the number of toric divisor of X (hence m is also the number of 1-dimensional rays generator in $\Sigma(X)$) then we have an exact sequence of \mathbb{Z} -modules:

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^m \longrightarrow \text{Pic}(X) \longrightarrow 0 .$$

Therefore we have that the Picard number of X is $\rho = m - n$ and the anticanonical divisor is given by $-K_X = D_1 + \dots + D_m$. The relations among the toric divisors are given by $\sum_{i=1}^m \langle u, v_i \rangle D_i = 0$ for u in a basis of $M = \text{Hom}(N; \mathbb{Z})$. A smooth toric Fano variety X is a smooth toric variety with anticanonical divisor $-K_X$ ample.

Definition 2.3. A set of toric divisors $\{D_1, \dots, D_k\}$ on $X(\Sigma)$ is called a *primitive set* if $D_1 \cap \dots \cap D_k = \emptyset$ but $D_1 \cap \dots \cap \hat{D}_j \cap \dots \cap D_k \neq \emptyset$ for all j , with $1 \leq j \leq k$. Equivalently, this means $\langle v_1, \dots, v_k \rangle \notin \Sigma$ but $\langle v_1, \dots, \hat{v}_j, \dots, v_k \rangle \in \Sigma$ for all j , with $1 \leq j \leq k$, and $P = \{v_1, \dots, v_k\}$ is called a *primitive collection*. If $S := \{D_1, \dots, D_k\}$ is a primitive set, the element $v := v_1 + \dots + v_k$ lies in the relative interior of a unique cone of Σ , if the cone is generated by v'_1, \dots, v'_s , then $v_1 + \dots + v_k = a_1 v'_1 + \dots + a_s v'_s$ with $a_i > 0$ is the corresponding *primitive relation*.

In terms of primitive collections and relations we have a nice criterion for checking if a smooth toric variety is Fano or not. A smooth toric variety $X(\Sigma)$ is Fano if and only if for every primitive relation

$$v_{i_1} + \dots + v_{i_k} - c_1 v_{j_1} - \dots - c_r v_{j_r} = 0$$

one has $k - \sum_{i=1}^r c_i > 0$. We recall that if we indicate by $K_0(X)$ the Grothendieck group, we know that its rank is equal to the number of maximal cones of Σ , in particular for a smooth toric Fano 3-folds $X(\Sigma)$ we can calculate it only knowing the number v of vertices of Σ i.e. $\text{rank}(K_0(X(\Sigma))) = 2v - 4$ and $\rho = v - 3$, where v is the number of ray generators of the toric variety X .

In the table below we give the list of 3-dimensional toric Fano varieties $V = V(P)$. We denote by S_i the Del Pezzo surface obtained blowing up of i points on \mathbb{P}^2 . The numbers ν, ρ and k_0 denote respectively the number of vertices, the Picard number and the rank of the Grothendieck group.

	Class of Toric Fano 3-folds	ν	ρ	k_0
Type I	\mathbb{P}^3	4	1	4
Type II	$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$	5	2	6
	$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$	5	2	6
	$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$	5	2	6
	$\mathbb{P}^2 \times \mathbb{P}^1$	5	2	6
	$\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, 1))$	6	3	8
	$\mathbb{P}_{S_1}(\mathcal{O} \oplus \mathcal{O}(l)), l^2 = 1$ on S_1	6	3	8
Type III	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	6	3	8
	$S_1 \times \mathbb{P}^1$	6	3	8
	$\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \otimes \mathcal{O}(1, -1))$	6	3	8
	$S_2 \times \mathbb{P}^1$	7	4	10
	$S_3 \times \mathbb{P}^1$	8	5	12
Type IV	$Bl_{\mathbb{P}^1}(\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1)))$	6	3	8
	$Bl_{\mathbb{P}^1}(\mathbb{P}^2 \times \mathbb{P}^1)$	6	3	8
	S_2 -bundle over \mathbb{P}^1	7	4	10
	S_2 -bundle over \mathbb{P}^1	7	4	10
	S_2 -bundle over \mathbb{P}^1	7	4	10
Type V	S_3 -bundle over \mathbb{P}^1	8	5	12

The following theorem is due to Costa and Miró -Roig (see Theorem 4.21 [6]):

Theorem 2.4. *Any smooth toric Fano 3-fold of Type I,II or III has a strongly exceptional collection made up of line bundles.*

In a more recent paper ([8, Proposition 2.5]) they also proved, using the same techniques we will use in this article, this theorem:

Theorem 2.5. *The smooth toric Fano 3-folds of Type V have a strongly exceptional collection made up of line bundles.*

3. Bondal’s Method and Thomsen’s Algorithm

We want to explain a method due to Bondal that we will use in the sequel. For that we recall briefly some definitions and facts.

Definition 3.1. Let X be a smooth projective variety defined over an algebraically closed field K of characteristic 0.

- A coherent sheaf \mathcal{F} on X is *exceptional* if $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = K$ and $\mathrm{Ext}_{\mathcal{O}_X}^i(\mathcal{F}, \mathcal{F}) = 0$ for $i > 0$;
- An ordered collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m)$ of coherent sheaves on X is an *exceptional collection* if each sheaf \mathcal{F}_i is exceptional and, for $j < k$ and $i \geq 0$, $\mathrm{Ext}_{\mathcal{O}_X}^i(\mathcal{F}_k, \mathcal{F}_j) = 0$;
- An exceptional collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m)$ is a *strongly exceptional collection* if in addition $\mathrm{Ext}_{\mathcal{O}_X}^i(\mathcal{F}_j, \mathcal{F}_k) = 0$ for $i \geq 1$ and $j \leq k$;
- An ordered collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m)$ of coherent sheaves on X is a *full (strongly) exceptional collection* if it is a (strongly) exceptional collection and $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m$ generate the bounded derived category $D^b(X)$.

Remark 3.2. The existence of a full strongly exceptional collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m)$ of coherent sheaves on a smooth projective variety X implies that the Grothendieck group $K_0(X) = K_0(\mathcal{O}_X\text{-mod})$ is isomorphic to \mathbb{Z}^{m+1} .

Given any smooth complete toric variety X , in [3] Bondal described a method to produce a collection of line bundles on X , which is expected to be full strongly exceptional for certain classes of toric Fano varieties. In particular he stated that for all the smooth toric Fano 3-folds but two these sequences were, indeed, strongly exceptional, and he does not say anything about the remaining cases. Now we show this method. Let X be a smooth complete toric variety of dimension n and T an n -dimensional torus acting on it. So for any integer $l \in \mathbb{Z}$, there is a well-defined toric morphism

$$\pi_l : X \rightarrow X$$

which restricts, on the torus T , to the Frobenius map

$$\pi_l : T \rightarrow T, \quad t \mapsto t^l.$$

This map is the factorization map with respect to the action of the group of l torsion of T . Let us fix a prime integer $p \gg 0$, $(\pi_p)_*(\mathcal{O}_X)^\vee$ is a vector bundle of rank p^n which splits into a sum of line bundles:

$$(\pi_p)_*(\mathcal{O}_X)^\vee = \bigoplus_{\chi} \mathcal{O}_X(D_\chi),$$

where the sum is taken over the group of characters of the p -torsion subgroup of T (see Theorem 1 and Proposition 2 [14]). Moreover,

$$c_1((\pi_p)_*(\mathcal{O}_X)^\vee) = \mathcal{O}_X\left(-\frac{p^{n-1}(p-1)}{2}K_X\right)$$

where K_X is the canonical divisor of X . Bondal [3] proved that the direct summands of $(\pi_p)_*(\mathcal{O}_X)^\vee$ generate the derived category $D^b(X)$ and Thomsen [14] described an algorithm for computing explicitly its decomposition. Let us summarize this for the case of a smooth complete toric variety X of dimension n , Picard number ρ rank of group of Grothendieck s .

We consider $\{\sigma_1, \dots, \sigma_s\}$ the set of maximal cones of the fan Σ associated to X and we denote by v_{i_1}, \dots, v_{i_n} the generators of σ_i . For each index $1 \leq i \leq s$, we denote by $A_i \in \text{GL}_n(\mathbb{Z})$ the matrix having as the j -th row the coordinates of v_{i_j} expressed in the basis e_1, \dots, e_n of N and by $B_i = A_i^{-1} \in \text{GL}_n(\mathbb{Z})$ its inverse. We indicate with w_{ij} the j -th column vector in B_i . Introducing the symbols X^{e^1}, \dots, X^{e^n} , we form the coordinate ring of the torus $T \subset X$:

$$R = K[(X^{e^1})^{\pm 1}, \dots, (X^{e^n})^{\pm 1}].$$

Moreover the coordinate ring of the open affine subvariety U_{σ_i} of X corresponding to the cone σ_i is the subring

$$R_i = K[X^{w_{i1}}, \dots, X^{w_{in}}] \subset R,$$

where $X^w := (X^{e^1})^{w_1} \dots (X^{e^n})^{w_n}$, if $w = (w_1, \dots, w_n)$, for simplicity we will usually write $X_{ij} := X^{w_{ij}}$. For each i and j , we denote by R_{ij} the coordinate ring of $\sigma_i \cap \sigma_j$ and we define $I_{ij} := \{v \in M_{n \times 1}(\mathbb{Z}) : X_i^v \text{ is a unit in } R_{ij}\}$ and $C_{ij} := B_j^{-1}B_i \in \text{GL}_n(\mathbb{Z})$, where we use the notation $X_i^v := (X_{i1})^{v_1} \dots (X_{in})^{v_n}$ with v a column vector with entries v_1, \dots, v_n .

For every $p \in \mathbb{N}$ and $w \in I_{ij}$, we define

$$P_p := \{v \in M_{n \times 1}(\mathbb{Z}) : 0 \leq v_i < p\}$$

and the maps

$$h_{ijp}^w : P_p \rightarrow R_{ij}, r_{ijp}^w : P_p \rightarrow P_p,$$

by means of the following equality:

$$C_{ij}v + w = p \cdot h_{ijp}^w(v) + r_{ijp}^w(v), \text{ for any } v \in P_p.$$

By [14, Lemma 2 and Lemma 3], these maps exist and they are unique.

Any toric Cartier divisor D on X can be represented in the form $\{(U_{\sigma_i}, X_i^{u_i})\}_{\sigma_i \in \Sigma}$, $u_i \in M_{n \times 1}(\mathbb{Z})$ (see [9, Chapter 3.3]). So if we fix one of these representants, we can define $u_{ij} = u_j - C_{ij}u_i$. If $\mathcal{O}_X(D) = \mathcal{O}_X$ is the trivial line bundle, then for any pair i, j , we have $u_{ij} = 0$. For any $p \in \mathbb{Z}$ and any toric Cartier divisor D on X , we fix a set $\{(U_{\sigma_i}, X_i^{u_i})\}_{\sigma_i \in \Sigma}$ representing D and we choose an index l of a cone $\sigma_l \in \Sigma$. Let $D_v, v \in P_p$, denote the Cartier divisor represented by the set $\{(U_{\sigma_i}, X_i^{h_i})\}_{\sigma_i \in \Sigma}$ where, by definition $h_i = h_i^v := h_{lip}^{u_i}(v)$.

Then, we have

$$(\pi_p)_*(\mathcal{O}_X(D))^\vee = \bigoplus_{v \in P_p} \mathcal{O}_X(D_v).$$

If $h_i = (h_{i_1}, \dots, h_{i_n})$ and $\alpha_{i_1}^j, \dots, \alpha_{i_n}^j$ are the entries of the j -th column vector of B_i , then by definition:

$$X_i^{h_i} = (X^{\hat{e}_1 \alpha_{i_1}^1} \dots X^{\hat{e}_n \alpha_{i_n}^1})^{h_{i_1}} (X^{\hat{e}_1 \alpha_{i_1}^2} \dots X^{\hat{e}_n \alpha_{i_n}^2})^{h_{i_2}} \dots (X^{\hat{e}_1 \alpha_{i_1}^n} \dots X^{\hat{e}_n \alpha_{i_n}^n})^{h_{i_n}},$$

and we indicate with:

$$l_{\sigma_i} := B_i \cdot h_i = (\alpha_{i_1}^1 h_{i_1} + \alpha_{i_1}^2 h_{i_2} + \dots + \alpha_{i_1}^n h_{i_n}) \hat{e}_1 + (\alpha_{i_2}^1 h_{i_1} + \alpha_{i_2}^2 h_{i_2} + \dots + \alpha_{i_2}^n h_{i_n}) \hat{e}_2 + \dots + (\alpha_{i_n}^1 h_{i_1} + \alpha_{i_n}^2 h_{i_2} + \dots + \alpha_{i_n}^n h_{i_n}) \hat{e}_n \in M = N^\vee.$$

In this notation, if D_v is the Cartier divisor represented by the set $\{(U_{\sigma_i}, X_i^{h_i})\}$, then

$$D_v = \beta_v^1 Z_1 + \dots + \beta_v^{n+\rho} Z_{n+\rho}$$

where $\beta_v^j = -l_{\sigma_k}(v_j)$, for any maximal cone σ_k containing the ray generator v_j associated to the toric divisor Z_j . We can observe that for any pair of maximal cones σ_k and σ_m containing v_j , we have $l_{\sigma_k}(v_j) = l_{\sigma_m}(v_j)$.

We will now construct a full strongly exceptional collection for all toric Fano 3–folds not covered in [6] and [8]. We will analyze case by case.

Note 3.3. From now on, when it will be clear wich variety we will be working with, we shall omit the subscript when denoting a line bundle associated to a given divisor D : hence we will write $\mathcal{O}(D)$ instead of $\mathcal{O}_X(D)$.

3.1. $\mathcal{D}_1 = Bl_{\mathbb{P}^1}(\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1)))$

Proposition 3.4. *Let p be a sufficiently large prime integer and indicate with $\pi_p : \mathcal{D}_1 \rightarrow \mathbb{P}^1$ the Frobenius morphism relative to p . Then the different summands of $(\pi_p)_*(\mathcal{O}_{\mathcal{D}_1})^\vee$ are the following:*

$$\begin{aligned} &\mathcal{O}, \mathcal{O}(Z_4 + Z_5), \mathcal{O}(2Z_4 + 2Z_5), \mathcal{O}(Z_6), \mathcal{O}(Z_5 + Z_6), \mathcal{O}(Z_4 + Z_5 + Z_6), \\ &\mathcal{O}(Z_4 + 2Z_5 + Z_6), \mathcal{O}(2Z_4 + 2Z_5 + Z_6), \mathcal{O}(Z_6 - Z_4). \end{aligned}$$

Proof. According to [1, Proposition 2.5.6], the primitive collections of \mathcal{D}_1 are the following:

$$\{v_3, v_6\}, \{v_4, v_6\}, \{v_3, v_5\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}.$$

On the other side, the primitive relations are:

- $v_3 + v_6 = 0$;
- $v_4 + v_6 = v_5$;
- $v_3 + v_5 = v_4$;
- $v_1 + v_2 + v_4 = 2v_3$;
- $v_1 + v_2 + v_5 = v_3$.

We take the following three maximal cones:

$$\sigma_1 = \{v_1, v_2, v_3\}, \sigma_2 = \{v_1, v_2, v_6\}$$

$$\sigma_3 = \{v_1, v_4, v_5\}$$

and, as a basis of \mathbb{Z}^3 : $e_1 = v_1, e_2 = v_2, e_3 = v_3$. Let \widehat{e}_i , for $i = 1, 2, 3$, be the dual basis. First of all we find the coordinates of v_4, v_5 and v_6 in the system we have taken and we obtain:

$$v_4 = (-1, -1, 2), v_5 = (-1, -1, 1), v_6 = (0, 0, -1)$$

and thus we get the three matrices:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Inverting these matrices we obtain:

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, B_3 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix}.$$

Let us now take a vector $v = (a_1, a_2, a_3)^t$ in P_p and let $d_i = C_{1i}v = (B_i)^{-1}B_1v = A_i v$ for $i = 1, 2, 3$. Then we have

$$d_2 = (a_1, a_2, -a_3), \text{ and } d_3 = (a_1, -a_1 - a_2 + 2a_3, -a_1 - a_2 + a_3).$$

Recall that $(\pi_p)_*(\mathcal{O}_X)^\vee = \bigoplus_{v \in P_p} \mathcal{O}_X(D_v)$, with

$$D_v = -l_{\sigma_3}(v_4)Z_4 - l_{\sigma_3}(v_5)Z_5 - l_{\sigma_2}(v_6)Z_6. \tag{1}$$

Let us see how D_v varies when we took different v 's in P_p .

Case 1: $v = 0$. In this case $D_v = 0$.

Case 2: $a_1 \neq 0, a_2 = a_3 = 0$.

Computing d_2 and d_3 we find that $d_2 = (a_1, 0, 0)$, while $d_3 = (a_1, -a_1, -a_1)$. Thus

$$d_2 = p \cdot (0, 0, 0) + (a_1, 0, 0) \text{ and } d_3 = p \cdot (0, -1, -1) + (a_1, p - a_1, p - a_1)$$

and consequently, $h_2 = 0$ and $h_3 = (0, -1, -1)$. It follows that l_{σ_2} is the zero operator and $l_{\sigma_3} = \widehat{e}_2$. Using the formulas (1) we gain:

$$D_v = -l_{\sigma_3}(v_4)Z_4 - l_{\sigma_3}(v_5)Z_5 = Z_4 + Z_5.$$

Case 3: $a_2 \neq 0, a_1 = a_3 = 0$.

In this case we have $d_2 = (0, a_2, 0)$ and $d_3 = (0, -a_2, -a_2)$. As in the previous situation we get

$$D_v = Z_4 + Z_5.$$

Case 4: $a_3 \neq 0, a_2 = a_1 = 0$.

Under the aforementioned hypothesis, $d_2 = (0, 0, -a_3)$, thus $h_2 = (0, 0, -1)$. The triple d_3 will instead be $(0, 2a_3, a_3)$. So we have now two possibilities for h_3 : $h_3 = (0, s, 0)$ with $s \in \{0, 1\}$ assuming the following values:

$$s = \begin{cases} 0 & \text{if } 2a_3 < p, \\ 1 & \text{if } 2a_3 \geq p. \end{cases}$$

The next step is to compute the operators l_{σ_2} and l_{σ_3} . We have $l_{\sigma_2} = \widehat{e}_3$ and $l_{\sigma_3} = s\widehat{e}_2 + \widehat{e}_3$. Concluding we got

$$D_v = -l_{\sigma_3}(v_4)Z_4 - l_{\sigma_3}(v_5)Z_5 - l_{\sigma_2}(v_6)Z_6 = -sZ_4 + Z_6 = \begin{cases} Z_6, \\ Z_6 - Z_4. \end{cases}$$

Case 5: $a_1, a_2 \neq 0, a_3 = 0$.

In this case $d_2 = (a_1, a_2, 0)$, that means that $h_2 = 0$ which implies that l_{σ_2} is the zero operator. On the other side $d_3 = (a_1, -a_1 - a_2, -a_1 - a_2)$ and, as in the previous case, we have two different possibilities for h_3 :

$$h_3 = \begin{cases} (0, -1, -1) & \text{If } -a_1 - a_2 \geq -(p-1), \\ (0, -2, -2) & \text{If } -a_1 - a_2 < -(p-1). \end{cases}$$

Thus we can write $h_3 = (0, -s, -s)$ where $s = 1, 2$. With this notation we can easily compute the operator $l_{\sigma_3} = s\widehat{e}_2$ and get

$$D_v = \begin{cases} Z_4 + Z_5, \\ 2Z_4 + 2Z_5. \end{cases}$$

Case 6: $a_1, a_3 \neq 0, a_2 = 0$.

Under these assumptions, $d_2 = (0, a_2, -a_3)$, $h_2 = (0, 0, -1)$ and, as we already calculated in the solution to the fourth case, $l_{\sigma_2} = \widehat{e}_3$. Since $d_3 = (a_1, -a_1 + 2a_3, -a_1 + a_3)$ we have several possibilities for h_3 , depending on the sign of $-a_1 + 2a_3$ and $-a_1 + a_3$:

Case 6.1: $-a_1 + a_3 \geq 0$.

If this is the case, then $h_3 = 0$ and l_{σ_3} is the zero operator. Thus we get

$$D_v = Z_6.$$

Case 6.2: $-a_1 + a_3 < 0, -a_1 + 2a_3 \geq 0$.

Under these hypothesis $h_3 = (0, 0, -1)$ and $l_{\sigma_3} = 2\widehat{e}_2 + \widehat{e}_3$. Consequently we get

$$D_v = Z_5 + Z_6.$$

Case 6.3: $-a_1 + 2a_3 < 0$.

In this case $h_3 = (0, -1, -1)$, and $l_{\sigma_3} = \widehat{e}_2$. Making all the computations we find out that

$$D_v = Z_4 + Z_5 + Z_6.$$

Case 7: $a_3, a_2 \neq 0, a_1 = 0$.

After having switched a_1 with a_2 , this case is completely identical to the previous one, and we get the same divisors D_v 's.

Case 8: $a_1, a_2, a_3 \neq 0$.

As before, in this case we have $h_2 = (0, 0, -1)$ and $l_{\sigma_2}(v_6) = -1$. Let us compute h_3 and l_{σ_3} .

Case 8.1: $-a_1 - a_2 + a_3 \geq 0$.

In this case $h_3 = 0$ and $D_v = Z_6$.

Case 8.2: $-(p-1) \leq -a_1 - a_2 + a_3 < 0, -a_1 - a_2 + 2a_3 \geq 0$.

In this case $h_3 = (0, 0, -1)$ and

$$D_v = Z_5 + Z_6.$$

Case 8.3: $-(p-1) \leq -a_1 - a_2 + a_3 \leq -a_1 - a_2 + 2a_3 < 0$.

In this case $h_3 = (0, -1, -1)$ and $l_{\sigma_3} = \widehat{e}_2$. It follows that

$$D_v = Z_4 + Z_5 + Z_6.$$

Case 8.4: $-(p-1) > -a_1 - a_2 + a_3, 0 > -a_1 - a_2 + 2a_3 \geq -(p-1)$.

In this case $h_3 = (0, -1, -2)$. Consequently $l_{\sigma_3} = 3\widehat{e}_2 + \widehat{e}_3$. Thus we have

$$D_v = Z_4 + 2Z_5 + Z_6.$$

Case 8.5: $-a_1 - a_2 + 2a_3 < -(p - 1)$.

Under this assumption $h_3 = (0, -2, -2)$ and $l_{\sigma_3} = 2\widehat{e}_2$. As a consequence we can compute

$$D_v = 2Z_4 + 2Z_5 + Z_6.$$

Putting together what we calculated in each case we obtain the statement. □

You may observe that the output of Thomsen’s algorithm consists of nine different line bundles. Since $\text{rk}(K_0(\mathcal{D}_1)) = 8$, there is no hope for this sequence to be full strongly exceptional. Thus we have first to find an eight items long full sequence and, afterward, prove that it is strongly exceptional. We accomplish the first of these two goals by proving the following proposition.

Proposition 3.5. *The bounded derived category $D^b(\mathcal{D}_1)$ is generated by the following line bundles:*

$$\begin{aligned} &\mathcal{O}, \mathcal{O}(Z_4 + Z_5), \mathcal{O}(2Z_4 + 2Z_5), \mathcal{O}(Z_6), \mathcal{O}(Z_5 + Z_6), \mathcal{O}(Z_4 + Z_5 + Z_6), \\ &\mathcal{O}(Z_4 + 2Z_5 + Z_6), \mathcal{O}(2Z_4 + 2Z_5 + Z_6). \end{aligned}$$

Proof. It is enough to prove that $\mathcal{O}(Z_6 - Z_4)$ is in the triangulated category generated by all the other line bundles. We do that by constructing an exact sequence in which $\mathcal{O}(Z_6 - Z_4)$ appears once, and all other objects in the sequence are direct sums of aforementioned line bundles. Let us consider the primitive collection $\{v_1, v_2, v_4\}$ and construct the Koszul complex associated to $\mathcal{O}(-Z_1) \oplus \mathcal{O}(-Z_2) \oplus \mathcal{O}(-Z_4)$:

$$\begin{aligned} 0 \leftarrow \mathcal{O}_Y \leftarrow \mathcal{O} \leftarrow \mathcal{O}(-Z_1) \oplus \mathcal{O}(-Z_2) \oplus \mathcal{O}(-Z_4) \leftarrow \\ \mathcal{O}(-Z_1 - Z_2) \oplus \mathcal{O}(-Z_2 - Z_4) \oplus \mathcal{O}(-Z_1 - Z_4) \leftarrow \mathcal{O}(-Z_1 - Z_2 - Z_4) \leftarrow 0 \end{aligned} \quad (2)$$

where Y denotes the intersection $Z_1 \cap Z_2 \cap Z_4$. Since we started out with a primitive collection, then $Y = \emptyset$ and (2) is exact.

Now we write all the divisors in the basis of $\text{Pic}(X)$ given by Z_4, Z_5 and Z_6 . Afterward we dualize the Koszul complex and twist it by $\mathcal{O}(Z_6 - Z_4)$. We obtain the following exact sequence:

$$\begin{aligned} 0 \rightarrow \mathcal{O}(Z_6 - Z_4) \rightarrow \mathcal{O}(Z_6 + Z_5)^2 \oplus \mathcal{O}(Z_6) \rightarrow \dots \\ \dots \rightarrow \mathcal{O}(Z_4 + Z_5 + Z_6)^2 \oplus \mathcal{O}(Z_4 + 2Z_5 + Z_6) \rightarrow \mathcal{O}(2Z_4 + 2Z_5 + Z_6) \rightarrow 0. \end{aligned} \quad (3)$$

Observe that all the line bundles in (3) but the first one are among those we picked as generators, hence the statement is proved. □

3.2. $\mathcal{D}_2 = Bl_{\mathbb{P}^1}(\mathbb{P}^2 \times \mathbb{P}^1)$

In this paragraph we are concerned with finding a set of generators for $D^b(\mathcal{D}_2)$ of cardinality 8, that is of cardinality the rank of the Grothendieck group $K_0(\mathcal{D}_2)$.

Proposition 3.6. *Let p be a sufficiently large prime integer and indicate with $\pi_p : \mathcal{D}_2 \rightarrow \mathcal{D}_2$ the Frobenius morphism relative to p . Then the different summands of $(\pi_p)_*(\mathcal{O}_{\mathcal{D}_2})^\vee$ are the following:*

$$\begin{aligned} \mathcal{O}, \mathcal{O}(Z_4 + Z_5), \mathcal{O}(2Z_4 + 2Z_5), \mathcal{O}(Z_6), \mathcal{O}(Z_5 + Z_6), \mathcal{O}(Z_4 + Z_5 + Z_6), \\ \mathcal{O}(Z_4 + 2Z_5 + Z_6), \mathcal{O}(2Z_4 + 2Z_5 + Z_6). \end{aligned}$$

Proof. From the classification of toric Fano 3-folds ([1, Proposition 2.5.6]) we know that the primitive collections of \mathcal{D}_2 are the same of \mathcal{D}_1 , while primitive relations are:

- $v_3 + v_6 = 0$;
- $v_4 + v_6 = v_5$;
- $v_3 + v_5 = v_4$;
- $v_1 + v_2 + v_4 = v_3$;
- $v_1 + v_2 + v_5 = 0$.

As we already did, choose (v_1, v_2, v_3) as a basis of \mathbb{Z}^3 , and take the following three maximal cones that covers all the vertices of the polytope defining the toric variety:

$$\begin{aligned} \sigma_1 = \{v_1, v_2, v_3\}, \quad \sigma_2 = \{v_1, v_2, v_6\}, \\ \sigma_3 = \{v_1, v_5, v_4\}. \end{aligned}$$

One can easily see that the matrices A_i but the last one unchanged respect to the ones we processed in the case of \mathcal{D}_1 , while

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

We need to compute its inverse:

$$B_3 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Now we take $v = (a_1, a_2, a_3)^t \in P_p$ and process it obtaining $d_3 = A_3 v = (a_1, a_1 - a_2, a_2 - a_3)^t$. Let us see how D_v changes if we take different v 's. It is useful to observe that, since A_1 and A_2 are unchanged, we do not need to be concerned with the computation of l_{σ_1} and l_{σ_2} because we can obtain them from the previous example. Almost the same computations we did for the \mathcal{S}_1 case show that

1. If all the a_i 's are null then $D_v = 0$.
2. If $a_1 \neq 0$, while $a_2 = a_3 = 0$, then $D_v = Z_4 + Z_5$.
3. If $a_2 \neq 0$, while $a_1 = a_3 = 0$, then $D_v = Z_4 + Z_5$.
4. If $a_3 \neq 0$, while $a_2 = a_1 = 0$, then $D_v = Z_6$.
5. If $a_1, a_2 \neq 0$, while $a_3 = 0$, the coordinate of l_{σ_3} in the dual basis are $(0, s, 0)$ with $s = 0, 1$. Thus we have two possibilities for D_v :

$$D_v = \begin{cases} Z_4 + Z_5, \\ 2Z_4 + 2Z_5. \end{cases}$$

6. If $a_1, a_3 \neq 0$, while $a_2 = 0$, then again we have two possibilities for D_v , since $l_{\sigma_3} = (0, 1, 1 - s)$, with $s = 0, 1$ in the dual basis. We obtain

$$D_v = \begin{cases} Z_5 + Z_6, \\ Z_4 + Z_5 + Z_6. \end{cases}$$

7. If $a_3, a_2 \neq 0$, while $a_1 = 0$, then l_{σ_3} is again $(0, 1, 1 - s)$ with $s = 0, 1$. Thus we get the same divisors we obtained in the previous case.
8. Finally, if all the a_i 's are not zero, we have to split our computation in four sub-cases, depending on the values $-a_1 - a_2$ and $-a_1 - a_2 + a_3$ will assume. We end up with four possibilities for D_v :

$$D_v = \begin{cases} 2Z_4 + 2Z_5 + Z_6, \\ Z_4 + 2Z_5 + Z_6, \\ Z_4 + Z_5 + Z_6, \\ Z_5 + Z_6. \end{cases}$$

The union of all these intermediate results implies the statement. □

3.3. \mathcal{E}_1, S_2 -bundle over \mathbb{P}^1 and $\mathcal{E}_2 = S_2$ -bundle over \mathbb{P}^1

Proposition 3.7. *Let p be a sufficiently large prime and indicate the Frobenius morphism relative to p by $\pi_p : \mathcal{E}_1 \rightarrow \mathcal{E}_1$. Then the distinct summands of $(\pi_p)_*(\mathcal{O}_{\mathcal{E}_1})^\vee$ are the following:*

$$\begin{aligned} & \mathcal{O}, \mathcal{O}(Z_7), \mathcal{O}(Z_4), \mathcal{O}(Z_4 + Z_7), \mathcal{O}(Z_4 + Z_5), \mathcal{O}(Z_4 + Z_5 + Z_7), \\ & \mathcal{O}(Z_1 + Z_5 + 2Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5 + 2Z_7). \end{aligned}$$

Proof. From [1, Proposition 2.5.9] we know that the primitive collections of \mathcal{E}_1 are:

$$\{v_2, v_4\}, \{v_3, v_5\}, \{v_1, v_3\}, \{v_2, v_5\}, \{v_1, v_4\}, \{v_6, v_7\},$$

and the ray generators of \mathcal{E}_1 satisfy the following relations:

- $v_2 + v_4 = 0$,
- $v_3 + v_5 = 0$,
- $v_1 + v_3 = v_2$,
- $v_2 + v_5 = v_1$,
- $v_1 + v_4 = v_5$,
- $v_6 + v_7 = v_1$.

As in the previous examples we choose three maximal cones such that they cover all the ray generators of the toric Fano 3-fold:

$$\begin{aligned} \sigma_1 &= \{v_2, v_3, v_6\}, & \sigma_2 &= \{v_4, v_5, v_6\}, \\ \sigma_3 &= \{v_2, v_1, v_7\}. \end{aligned}$$

Let us take $\mathfrak{B} = (v_2 = e_1, v_3 = e_2, v_6 = e_3)$ a basis of \mathbb{Z}^3 and denote with \widehat{e}_i the elements of the dual basis. For $i = 1, 2, 3$ we construct the matrices A_i and we get:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}.$$

Inverting these matrices we obtain:

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Now choose $v = (a_1, a_2, a_3)^t \in P_p$ and let

$$d_1 = A_1 v = (a_1, a_2, a_3), \quad d_2 = A_2 v = (-a_1, -a_2, a_3),$$

$$d_3 = A_3 v = (a_1, a_1 - a_2, a_1 - a_2 - a_3).$$

Defining as we did before h_i and l_{σ_i} , then we know that

$$D_v = -l_{\sigma_3}(v_1)Z_1 - l_{\sigma_1}(v_2)Z_2 - l_{\sigma_1}(v_3)Z_3 - l_{\sigma_2}(v_4)Z_4 + \\ -l_{\sigma_2}(v_5)Z_5 - l_{\sigma_1}(v_6)Z_6 - l_{\sigma_3}(v_7)Z_7.$$

Note that $A_1 v \in P_p$ for every v . So $h_1 = (0, 0, 0)$ for every v and l_{σ_1} is the zero operator. Thus we can write:

$$D_v = -l_{\sigma_3}(v_1)Z_1 - l_{\sigma_2}(v_4)Z_4 - l_{\sigma_2}(v_5)Z_5 - l_{\sigma_3}(v_7)Z_7.$$

To find all the D_v 's we have to see how the h_i 's change when v varies in P_p .

Case 1: $a_1 = a_2 = a_3 = 0$. As before in this case we have $D_v = 0$.

Case 2: $a_1 \neq 0, a_2 = a_3 = 0$.

In this case $d_2 = (-a_1, 0, 0)$ and $d_3 = (a_1, a_1, a_1)$. So we obtain that l_{σ_3} is the zero operator, while $l_{\sigma_2} = \hat{e}_1$. Putting this together and computing the coefficients $l_{\sigma_i}(v_j)$ we get that

$$D_v = Z_4.$$

Case 3: $a_2 \neq 0, a_1 = a_3 = 0$.

In this case we compute h_2 to be the vector $(0, -1, 0)$, and $h_3 = (0, -1, -1)$. As a consequence $l_{\sigma_2} = \hat{e}_2 = l_{\sigma_3}$. Then

$$D_v = Z_1 + Z_5 + Z_7.$$

Case 4: $a_3 \neq 0, a_1 = a_2 = 0$.

In this case l_{σ_2} is the zero operator. Computing l_{σ_3} we get instead that $h_3 = (0, 0, -1)$, $l_{\sigma_3}(v_1) = 0$ and $l_{\sigma_3}(v_7) = -1$. It follows that

$$D_v = Z_7.$$

Case 5: $a_1, a_2, \neq 0, a_3 = 0$.

In this case h_2 is always equal to $(-1, -1, 0)$ while we have two hypothesis for h_3 . Indeed, being $d_3 = (a_1, a_1 - a_2, a_1 - a_2)$, we have that $h_3 = (0, -s, -s)$ with

$$s = \begin{cases} 0 & \text{if } a_1 - a_2 \geq 0, \\ 1 & \text{otherwise.} \end{cases}$$

It follows that $l_{\sigma_2} = \widehat{e}_1 + \widehat{e}_2$ and $l_{\sigma_3} = s\widehat{e}_2$. Consequently we get

$$D_v = \begin{cases} Z_4 + Z_5, \\ Z_1 + Z_4 + Z_5 + Z_7. \end{cases}$$

Case 6: $a_1, a_3 \neq 0, a_2 = 0$.

In this case we have $h_2 = (-1, 0, 0)$ and, again, we get two possibilities for h_3 . More precisely

$$h_3 = \begin{cases} 0 & \text{if } a_1 - a_3 \geq 0, \\ (0, 0, -1) & \text{otherwise.} \end{cases}$$

As a consequence we obtain two different summands of $(\pi_p)_*(\mathcal{O}_{\mathcal{E}_1})^\vee$:

$$D_v = \begin{cases} Z_4, \\ Z_4 + Z_7. \end{cases}$$

Case 7: $a_3, a_2 \neq 0, a_1 = 0$.

In this case $h_2 = (0, -1, 0)$. Thus $l_{\sigma_2}(v_4) = 0$ and $l_{\sigma_2}(v_5) = -1$. It remains to compute l_{σ_3} . Being $d_3 = (0, -a_2, -a_2 - a_3)$ we have that $h_3 = (0, -1, -s)$ with $s = 1$ or $s = 2$. It follows that $l_{\sigma_3} = \widehat{e}_2 + (s-1)\widehat{e}_3$ and we end up with the following two divisors:

$$D_v = \begin{cases} Z_1 + Z_5 + Z_7, \\ Z_1 + Z_5 + 2Z_7. \end{cases}$$

Case 8: $a_1, a_2, a_3 \neq 0$.

In this case $h_2 = (-1, -1, 0)$ and $l_{\sigma_2}(v_4) = l_{\sigma_2}(v_5) = -1$. To compute h_3 we need to consider some different situations.

Case 8.1: $a_1 - a_2 - a_3 < -(p-1)$.

Under this assumption $a_1 - a_2 < 0$ and hence $h_3 = (0, -1, -2)$. It follows that $l_{\sigma_3}(v_1) = -1$, while $l_{\sigma_3}(v_7) = -2$. Putting this together with what was found earlier we get

$$D_v = Z_1 + Z_4 + Z_5 + 2Z_7.$$

Case 8.2: $0 > a_1 - a_2 - a_3 \geq -(p-1), a_1 - a_2 < 0$.

In this case $h_3 = (0, -1, -1)$ and $l_{\sigma_3}(v_1) = l_{\sigma_3}(v_7) = -1$. Then

$$D_v = Z_1 + Z_4 + Z_5 + Z_7.$$

Case 8.3: $0 > a_1 - a_2 - a_3 \geq -(p-1), a_1 - a_2 \geq 0$.

We have that $h_3 = (0, 0, -1)$, $l_{\sigma_3}(v_1) = 0$ and $l_{\sigma_3}(v_7) = -1$. So

$$D_v = Z_4 + Z_5 + Z_7.$$

□

Using the same methods and techniques, we were also able to prove the following proposition. Since the steps of the proof and all the calculations are pretty much the same as the ones in the previous case, we decided, for the sake of brevity, to omit them and to leave them to the reader.

Proposition 3.8. *Let p a sufficiently large prime and indicate the Frobenius morphism relative to p with $\pi_p : \mathcal{E}_2 \rightarrow \mathcal{E}_2$. Then the different summands of $(\pi_p)_*(\mathcal{O}_{\mathcal{E}_2})^\vee$ are the following:*

$$\begin{aligned} &\mathcal{O}, \mathcal{O}(Z_7), \mathcal{O}(Z_4), \mathcal{O}(Z_1 + Z_5), \mathcal{O}(Z_4 + Z_7), \mathcal{O}(Z_4 + Z_5), \mathcal{O}(Z_1 + Z_5 + Z_7), \\ &\mathcal{O}(Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5), \mathcal{O}(Z_1 + Z_4 + Z_5 + Z_7). \end{aligned}$$

3.4. $\mathcal{E}_4 = S_2$ -bundle over \mathbb{P}^1

Proposition 3.9. *Let p be a sufficiently large prime and indicate the Frobenius morphism relative to p with $\pi_p : \mathcal{E}_4 \rightarrow \mathcal{E}_4$. Then the different summands of $(\pi_p)_*(\mathcal{O}_{\mathcal{E}_4})^\vee$ are the following:*

$$\begin{aligned} &\mathcal{O}, \mathcal{O}(Z_7), \mathcal{O}(Z_4), \mathcal{O}(Z_1 + Z_5), \mathcal{O}(Z_4 + Z_7), \mathcal{O}(Z_4 + Z_5), \mathcal{O}(Z_1 + Z_5 + Z_7), \\ &\mathcal{O}(Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5), \mathcal{O}(Z_1 + Z_4 + Z_5 + Z_7). \end{aligned}$$

Proof. The primitive collections of \mathcal{E}_4 are the same of \mathcal{E}_1 while the primitive relations are the following:

$$\begin{aligned} v_2 + v_4 = 0, \quad v_3 + v_5 = 0, \quad v_1 + v_3 = v_2, \quad v_2 + v_5 = v_1, \\ v_1 + v_4 = v_5, \quad v_6 + v_7 = v_3. \end{aligned}$$

As before we choose (v_2, v_3, v_6) as a basis of \mathbb{Z}^3 , and we take the following three maximal cones that covers all the vertices of the polytope defining the toric variety, namely:

$$\sigma_1 = \{v_2, v_3, v_6\}, \quad \sigma_2 = \{v_4, v_5, v_6\}, \quad \sigma_3 = \{v_2, v_1, v_7\}.$$

Since only one primitive relation has changed, all the matrices A_i but the last one are the same as in the previous two cases, while

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Because of this our only concern is to calculate l_{σ_3} . In order to do so we just need to calculate the inverse of A_3 :

$$B_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

As usual we take $v = (a_1, a_2, a_3)^t \in P_p$ and we compute $d_3 = (a_1, a_1 - a_2, a_2 - a_3)$. Let us see how D_v changes if we take different v 's. Almost the same computations we did in the previous cases show that:

1. If all the a_i 's are null then $D_v = 0$.
2. If $a_1 \neq 0$, while $a_2 = a_3 = 0$, then $D_v = Z_4$.
3. If $a_2 \neq 0$, while $a_1 = a_3 = 0$, then $D_v = Z_1 + Z_5$.
4. If $a_3 \neq 0$, while $a_2 = a_1 = 0$, then $D_v = Z_7$.
5. If $a_1, a_2 \neq 0$, while $a_3 = 0$, the coordinates of l_{σ_3} in the dual basis are $(0, s, s)$ with $s = 0, 1$. Thus we have two possibilities for D_v :

$$D_v = \begin{cases} Z_4 + Z_5, \\ Z_1 + Z_4 + Z_5. \end{cases}$$

6. If $a_1, a_3 \neq 0$, while $a_2 = 0$, then $D_v = Z_4 + Z_7$.
7. If $a_3, a_2 \neq 0$, while $a_1 = 0$, then $l_{\sigma_3} = (0, 1, 1 + s)$ with $s = 0, 1$. It follows that

$$D_v = \begin{cases} Z_1 + Z_5, \\ Z_1 + Z_5 + Z_7. \end{cases}$$

8. Finally, if all the a_i 's are not zero, we get four possibilities for D_v :

$$D_v = \begin{cases} Z_4 + Z_5, \\ Z_1 + Z_4 + Z_5, \\ Z_4 + Z_5 + Z_7, \\ Z_1 + Z_4 + Z_5 + Z_7. \end{cases}$$

If we collect all cases we obtain exactly the divisors appearing in the statement above and hence the proposition is proved.

□

4. Vanishing Theorems

Given a full sequence of line bundles, $(\mathcal{L}_1, \dots, \mathcal{L}_n)$, in order to check if it is also strongly exceptional, we have to prove that:

- (i) $\text{Ext}_{\mathcal{O}_X}^i(\mathcal{L}_\alpha, \mathcal{L}_\beta) = 0$ for all $i > 0$ and for all α and β .
- (ii) $\text{Hom}(\mathcal{L}_\alpha, \mathcal{L}_\beta) = 0$ for all $\alpha > \beta$.

We briefly recall the definition of acyclic line bundle:

Definition 4.1. A line bundle \mathcal{L} on a smooth complete toric variety Y is said to be *acyclic* if

$$H^i(Y, \mathcal{L}) = 0 \quad \text{for every } i \geq 1.$$

We can substitute conditions (i) and (ii) by other two equivalent statements:

- (i)' $\mathcal{L}_\beta \otimes \mathcal{L}_\alpha^{-1}$ is acyclic for all α and β .
- (ii)' $\mathcal{L}_\beta \otimes \mathcal{L}_\alpha^{-1}$ has no global section for all $\alpha > \beta$.

Verify (ii)' is quite simple: in fact let Z_1, \dots, Z_m be the toric divisors on a toric variety X . For every $a = (a_1, \dots, a_m) \in \mathbb{Z}^m$ we can consider the subset $I_a = \{i_1, \dots, i_s\} \subseteq J = \{1, \dots, m\}$ such that $\rho \in I_a$ if and only if $a_\rho \geq 0$. Now to I_a , as well as to every subset of J , we associate the simplicial subcomplex, C_{I_a} , of the fan $\Sigma(X)$ which consists of the cones in $\Sigma(X)$ whose rays lie in I_a . It is a known fact that

Proposition 4.2. *With the above notation, given $D = \sum_\rho a_\rho Z_\rho$ a toric divisor on X then*

$$H^p(X, \mathcal{O}(D)) = \bigoplus_{a'} H_{\dim(X)-p}(C_{I_{a'}}, K)$$

where the sum is taken over all $a' = (a'_1, \dots, a'_m)$ such that D is linearly equivalent to $\sum_\rho a'_\rho Z_\rho$.

Proof. [4, Proposition 4.1] □

A consequence of this result was given in [5, Corollary 2.8]:

Corollary 4.3. *With the above notation, $H^0(X, \mathcal{O}(D))$ is determined only by $a' = (a'_1, \dots, a'_m)$ such that D is linearly equivalent to $\sum_\rho a'_\rho Z_\rho$, $a'_\rho \geq 0$. We call the toric divisors such as those toric effective divisors.*

Thus to verify that \mathcal{L} satisfies (ii)' write $\mathcal{L}_\beta \otimes \mathcal{L}_\alpha^{-1}$ as $\mathcal{O}(D)$ with D a T -Cartier divisor on the toric variety. Then it is enough to prove that it D is not linearly equivalent to a linear combination with non-negative coefficients of the principal toric divisors.

Condition (i)' is more laborious. In order to verify it we need some acyclicity criteria. The first one it is an easy consequence of a vanishing theorem by Mustața whose statement we briefly recall:

Theorem 4.4 (Mustața). *Let Y be a complete smooth toric variety, and be \mathcal{L} an ample line bundle on Y . If Z_{i_1}, \dots, Z_{i_k} are distinct toric divisors of Y , then the line bundle $\mathcal{L} \otimes \mathcal{O}_X(-Z_{i_1} - \dots - Z_{i_k})$ is acyclic.*

Proof. [13, Corollary 2.5] □

Remark 4.5. If X is a toric Fano variety, then all the line bundles of the form

$$\mathcal{O}_X \left(\sum_i \varepsilon_i Z_i \right)$$

with Z_i principal toric divisors and $\varepsilon_i \in \{0, 1\}$ are acyclic.

Observing the statement of condition (i)', it is evident that the hypothesis of Proposition 4.5 are too strong to apply to all the line bundles we need to be acyclic. Indeed, the set of divisors we have to check is "symmetric" in the sense that if D is one element of the set, than also $-D$ is an element. But (toric) effectiveness is not a symmetric propriety, thus if we can use the previous criterion to check the acyclicity of a line bundle \mathcal{L} , surely we will not be able to apply it to its dual. In conclusion we need a weaker criterion. In [4], Borisov and Hua gave sufficient and necessary conditions for a line bundle on a toric variety to be acyclic. In the next paragraphs we will explain their method.

Let us introduce the notion of forbidden set.

Definition 4.6. Let X a toric variety with m ray generators. For every proper subset $I \subsetneq J = \{1, \dots, m\}$ consider the associated simplicial complex C_I . We say that I is a *forbidden* set if C_I has a non-trivial homology.

Example 4.7. It is easy to see that $\{2, 4, 5\}$ is a forbidden set for \mathcal{E}_1 . In fact the simplicial complex C_I consists in 0 maximal cones, one face and 3 edges. Its (reduced) chain complex is

$$0 \rightarrow K \rightarrow K^3 \rightarrow K \rightarrow 0$$

that is obviously not exact. Then C_I has a non-trivial homology.

Thanks to the following result, forbidden sets play a key role in proving the acyclicity of line bundles on toric Fano varieties.

Proposition 4.8 (Borisov-Hua). *Let X a toric Fano variety and consider all forbidden sets $I \subset J$. For each of them consider the line bundles of the form*

$$\mathcal{O}_X \left(-\sum_{i \notin I} Z_i + \sum_{i \in I} a_i Z_i - \sum_{i \notin I} a_i Z_i \right) \quad (4)$$

with $a_i \in \mathbb{Z}_{\geq 0}$ for every $i \in J$. Then $\mathcal{L} \in \text{Pic}(X)$ is acyclic if and only if \mathcal{L} is not of the form (4).

Proof. [4, Proposition 4.3] □

We call the line bundles like (4) *forbidden line bundles* (or *forbidden forms*) relative to the set I . By abuse of notation we shall say that a divisor D is of a forbidden form relative to I or that it can be put in a forbidden form relative to I if it is linearly equivalent to a divisor F such that $\mathcal{O}(F)$ is a forbidden form.

In what follows we will prove that the full sequences we got in the previous sections are indeed strongly exceptional. Although Boris-Hua method's gives sufficient and necessary conditions for a line bundle to be acyclic, it needs a lot of tedious calculations and so we will use also the first criterion to lessen the number of divisors we need to check.

4.1. \mathcal{D}_1 and \mathcal{D}_2

Since we will need to apply Borisov-Hua's method, the first thing to do is find out which subsets of the set of vertices of \mathcal{D}_1 and \mathcal{D}_2 are forbidden. It turns out that there are eleven of them.

Proposition 4.9. *The forbidden sets for \mathcal{D}_i , $i = 1, 2$ are*

$$\begin{aligned} & \emptyset, \{3, 6\}, \{4, 6\}, \{3, 5\}, \{1, 2, 5\}, \{1, 2, 4\}, \{1, 2, 4, 5\}, \\ & \{1, 2, 3, 5\}, \{1, 2, 4, 6\}, \{3, 5, 6\}, \{3, 4, 6\}. \end{aligned}$$

Proof. We know from [7, Proposition 5.7] that the aforementioned sets are indeed the forbidden sets for \mathcal{D}_2 . Now, being a forbidden sets depends just upon the primitive collections of a variety and not by their relations. Thus the same result is true also for \mathcal{D}_1 , since \mathcal{D}_1 and \mathcal{D}_2 have the same primitive collections. □

Theorem 4.10. *The following is a full strongly exceptional collection for \mathcal{D}_1 :*

$$\begin{aligned} & \mathcal{O}, \mathcal{O}(Z_4 + Z_5), \mathcal{O}(2Z_4 + 2Z_5), \mathcal{O}(Z_6), \mathcal{O}(Z_5 + Z_6), \\ & \mathcal{O}(Z_4 + Z_5 + Z_6), \mathcal{O}(Z_4 + 2Z_5 + Z_6), \mathcal{O}(2Z_4 + 2Z_5 + Z_6). \end{aligned} \quad (5)$$

Proof. Proposition 3.5 tells us that the collection (5) is full. So, in order to prove it is also strongly exceptional, we have just to show that its line bundles satisfy the required vanishing. As first step we will prove that the following line bundles are acyclic:

$$\begin{aligned}
& \mathcal{O}(\pm(Z_4 + Z_5)), \mathcal{O}(\pm(2Z_4 + 2Z_5)), \mathcal{O}(\pm(Z_5 + Z_6)), \\
& \mathcal{O}(\pm(Z_4 + Z_5 + Z_6)), \mathcal{O}(\pm Z_6), \mathcal{O}(\pm(Z_4 + 2Z_5 + Z_6)), \\
& \mathcal{O}(\pm(2Z_4 + 2Z_5 + Z_6)), \mathcal{O}(\pm(-Z_4 - Z_5 + Z_6)), \mathcal{O}(\pm(-Z_4 + Z_6)), \quad (6) \\
& \mathcal{O}(\pm(-2Z_4 - 2Z_5 + Z_6)), \mathcal{O}(\pm(-2Z_4 - Z_5 + Z_6)), \mathcal{O}(\pm Z_5), \\
& \mathcal{O}(\pm(Z_4 + 2Z_5)), \mathcal{O}(\pm Z_4), \mathcal{O}(\pm(2Z_4 + Z_5)).
\end{aligned}$$

In order to shorten our argument, we shall split these invertible sheaves into four groups.

- a) $\mathcal{O}(Z_4 + Z_5), \mathcal{O}(2Z_4 + 2Z_5), \mathcal{O}(Z_6), \mathcal{O}(Z_5 + Z_6), \mathcal{O}(Z_4 + Z_5 + Z_6),$
 $\mathcal{O}(Z_4 + 2Z_5 + Z_6), \mathcal{O}(2Z_4 + 2Z_5 + Z_6), \mathcal{O}(-Z_4 + Z_6), \mathcal{O}(Z_5 + Z_6),$
 $\mathcal{O}(-2Z_4 - Z_5 + Z_6), \mathcal{O}(Z_5), \mathcal{O}(Z_4 + 2Z_5), \mathcal{O}(Z_4), \mathcal{O}(2Z_4 + Z_5);$
- b) $\mathcal{O}(-Z_4 - Z_5), \mathcal{O}(-2Z_4 - 2Z_5), \mathcal{O}(-Z_6), \mathcal{O}(-Z_4 - Z_5 - Z_6),$
 $\mathcal{O}(-2Z_4 - 2Z_5 - Z_6), \mathcal{O}(-Z_4 - Z_5 + Z_6), \mathcal{O}(2Z_4 + 2Z_5 - Z_6),$
 $\mathcal{O}(-2Z_4 - 2Z_5 + Z_6), \mathcal{O}(Z_4 + Z_5 - Z_6),$
- c) $\mathcal{O}(-Z_5 - Z_6), \mathcal{O}(-Z_4 - 2Z_5 - Z_6), \mathcal{O}(Z_4 - Z_6), \mathcal{O}(2Z_4 + Z_5 - Z_6),$
 $\mathcal{O}(-Z_5), \mathcal{O}(-Z_4 - 2Z_5), \mathcal{O}(-Z_4 - 2Z_5 + Z_6);$
- d) $\mathcal{O}(-Z_4), \mathcal{O}(-2Z_4 - Z_5).$

We claim that the line bundles in group a) are line bundles associated to toric effective divisors, whose coefficients of the principal toric divisors are either zero or one and hence they are acyclic due Remark 4.5. Indeed this is straightforward for those line bundles of the form $\mathcal{O}(aZ_4 + bZ_5 + cZ_6)$ with a, b, c already in $\{0, 1\}$. Let us see, as an example, that $-2Z_4 - Z_5 + Z_6$ is a divisor linearly equivalent to Z_3 ; all the other are proved by similar argument. First of all we want to write the generic divisor $D = \sum_{\rho=1}^6 a_\rho Z_\rho$ of \mathcal{D}_1 in the basis of $\text{Pic}(\mathcal{D}_1)$ given by Z_4, Z_5 and Z_6 , using the following relations:

$$Z_1 = Z_2 = Z_4 + Z_5, \quad Z_3 = -2Z_4 - Z_5 + Z_6.$$

We get:

$$D = (a_1 + a_2 - 2a_3 + a_4)Z_4 + (a_1 + a_2 - a_3 + a_5)Z_5 + (a_3 + a_6)Z_6. \quad (7)$$

Using the previous formula, we can easily be proved that $-2Z_4 - Z_5 + Z_6$ is linearly equivalent to Z_3 , and then $\mathcal{O}(-2Z_4 - Z_5 + Z_6) \simeq \mathcal{O}(Z_3)$.

To prove the acyclicity of the remaining line bundles we need to use Borisov-Hua's result.

Now denote with z_i for $i = 4, 5, 6$ the coefficient of Z_i in the representation of D ; all the divisors we have to check have the coefficient $z_6 \geq -1$, hence these divisors cannot be written in the forbidden forms relative to a set I which does not have neither 3 nor 6 among its elements. In fact the coefficient z_6 of the divisors in the forbidden forms relative to those sets is always less or equal to -2 . The sets we have still to check are:

$$\{3, 6\}, \{4, 6\}, \{3, 5\}, \{1, 2, 3, 5\}, \{1, 2, 4, 6\}, \{3, 5, 6\}, \{3, 4, 6\}.$$

For similar reasons we know that the line bundles in (4.1) cannot be put in the forbidden forms relative to a set I when it:

1. contains 3 but neither 1 nor 2 nor 4 (otherwise we will have $z_4 \leq -3$);
2. contains 3 but neither 1 nor 2 nor 5 (otherwise we will have $z_5 \leq -3$).

After this second cancellation we remain with

$$\{4, 6\}, \{1, 2, 3, 5\}, \{1, 2, 4, 6\}.$$

From (7) we can deduce that $a_4 - a_3 - a_5 = z_4 - z_5$, thus we have

$$a_4 - a_3 - a_5 = \begin{cases} 0 & \text{If we have } \mathcal{O}(D) \text{ in group b),} \\ 1 & \text{If we have } \mathcal{O}(D) \text{ in group c),} \\ -1 & \text{If we have } \mathcal{O}(D) \text{ in group d).} \end{cases} \quad (8)$$

It follows immediately that neither one of the divisors in those groups can be linearly equivalent to a line bundle in a forbidden form relative to $\{4, 6\}$ or $\{1, 2, 4, 6\}$, otherwise it should have $z_4 - z_5 \geq 2$. It remains to check that the divisors cannot be put in the forbidden forms relative to $\{1, 2, 3, 5\}$. The forbidden forms relative to this set have the difference $z_4 - z_5 \leq -1$. It follows from this and (8) that if $\mathcal{O}(D)$ can be put in one of this forbidden forms, then it is in group d) and $a_3 = a_5 = 0$. But $6 \notin \{1, 2, 3, 5\}$, hence we should have $z_6 = a_3 + a_6 \leq -1$ while none of the divisor in group d) have a negative z_6 . Thus we can conclude that all the divisors in groups a), b), c) and d) are acyclic.

To finish the proof we have still to check that the line bundles associated to the following divisors have no sections.

$$\text{a) } -Z_6, -Z_5 - Z_6, -Z_4 - Z_5 - Z_6, -Z_4 - 2Z_5 - Z_6, -2Z_4 - 2Z_5 - Z_6, \\ Z_4 + Z_5 - Z_6, Z_4 - Z_6, 2Z_4 + 2Z_5 - Z_6, 2Z_4 + Z_5 - Z_6;$$

$$\text{b) } -Z_4 - Z_5, -2Z_4 - 2Z_5, -Z_5, -Z_4 - 2Z_5, -Z_4, -2Z_4 - Z_5, -Z_5.$$

Looking at (7) is straightforward to see that all the divisors in group a), which have a negative z_6 , are not linearly equivalent to a toric effective divisor, hence their associated line bundles have not zero-cohomology. The divisors in b) are the opposite of some positive divisors and hence they cannot have sections and the statement is proved. \square

Theorem 4.11. *The following is a full strongly exceptional collection for \mathcal{D}_2 :*

$$\begin{aligned} \mathcal{O}, \mathcal{O}(Z_4 + Z_5), \mathcal{O}(2Z_4 + 2Z_5), \mathcal{O}(Z_6), \mathcal{O}(Z_5 + Z_6), \\ \mathcal{O}(Z_4 + Z_5 + Z_6), \mathcal{O}(Z_4 + 2Z_5 + Z_6), \mathcal{O}(2Z_4 + 2Z_5 + Z_6). \end{aligned} \quad (9)$$

Proof. Proposition 3.6 tells us that the collection (9) is full. So, in order to prove it also is strongly exceptional, we have just to show that its line bundles satisfy the required vanishing. As first step we will demonstrate that the following invertible sheaves, obtained as a difference of two line bundles in the sequence, are acyclic.

$$\begin{aligned} \mathcal{O}(\pm(Z_4 + Z_5)), \mathcal{O}(\pm(2Z_4 + 2Z_5)), \mathcal{O}(\pm(Z_5 + Z_6)), \\ \mathcal{O}(\pm(Z_4 + Z_5 + Z_6)), \mathcal{O}(\pm Z_6), \mathcal{O}(\pm(Z_4 + 2Z_5 + Z_6)), \\ \mathcal{O}(\pm(2Z_4 + 2Z_5 + Z_6)), \mathcal{O}(\pm(-Z_4 - Z_5 + Z_6)), \mathcal{O}(\pm(-Z_4 + Z_6)), \\ \mathcal{O}(\pm(-2Z_4 - 2Z_5 + Z_6)), \mathcal{O}(\pm(-2Z_4 - Z_5 + Z_6)), \mathcal{O}(\pm Z_5), \\ \mathcal{O}(\pm(Z_4 + 2Z_5)), \mathcal{O}(\pm Z_4), \mathcal{O}(\pm(2Z_4 + Z_5)). \end{aligned} \quad (10)$$

As we already did before, it is useful to split these line bundles in four groups

- a) $\mathcal{O}(Z_4 + Z_5), \mathcal{O}(2Z_4 + 2Z_5), \mathcal{O}(Z_6), \mathcal{O}(Z_5 + Z_6), \mathcal{O}(Z_4 + Z_5 + Z_6),$
 $\mathcal{O}(Z_4 + 2Z_5 + Z_6), \mathcal{O}(2Z_4 + 2Z_5 + Z_6), \mathcal{O}(-Z_4 + Z_6), \mathcal{O}(Z_5 + Z_6), \mathcal{O}(Z_5),$
 $\mathcal{O}(Z_4 + 2Z_5), \mathcal{O}(Z_4), \mathcal{O}(2Z_4 + Z_5);$
- b) $\mathcal{O}(-Z_4 - Z_5), \mathcal{O}(-2Z_4 - 2Z_5), \mathcal{O}(-Z_6), \mathcal{O}(-Z_4 - Z_5 - Z_6),$
 $\mathcal{O}(-2Z_4 - 2Z_5 - Z_6), \mathcal{O}(-Z_4 - Z_5 + Z_6), \mathcal{O}(2Z_4 + 2Z_5 - Z_6),$
 $\mathcal{O}(-2Z_4 - 2Z_5 + Z_6), \mathcal{O}(Z_4 + Z_5 - Z_6);$
- c) $\mathcal{O}(-Z_5 - Z_6), \mathcal{O}(-Z_4 - 2Z_5 + Z_6), \mathcal{O}(-Z_4 - 2Z_5 - Z_6), \mathcal{O}(Z_4 - Z_6),$
 $\mathcal{O}(2Z_4 + Z_5 - Z_6), \mathcal{O}(-Z_5), \mathcal{O}(-Z_4 - 2Z_5);$
- d) $\mathcal{O}(-Z_4), \mathcal{O}(-2Z_4 + -Z_5 + Z_6), \mathcal{O}(-2Z_4 - Z_5).$

As a first step we write the generic divisor $D = \sum_{\rho=1}^6 a_\rho Z_\rho$ of \mathcal{D}_2 in the basis of $\text{Pic}(\mathcal{D}_2)$ given by Z_4, Z_5 and Z_6 , using the following relations:

$$Z_1 = Z_2 = Z_4 + Z_5, \quad Z_3 = -Z_4 + Z_6.$$

We get:

$$D = (a_1 + a_2 - a_3 + a_4)Z_4 + (a_1 + a_2 + a_5)Z_5 + (a_3 + a_6)Z_6. \quad (11)$$

It is quite straightforward to see that all the line bundles in group a) are of the form $\mathcal{O}(E)$ with E a linear combination with coefficients 0 or 1 of the principal toric divisors, thus they are acyclic for Remark 4.5. It is all the same easy to verify that if the remaining line bundles are forbidden with respect to a set I , then, necessarily, we have $I = \{1, 2, 3, 5\}$ or $I = \{1, 2, 4, 6\}$. Now observe that

$$a_4 - a_5 - a_3 = \begin{cases} -1 & \text{If we have } \mathcal{O}(D) \text{ is in group d),} \\ 0 & \text{If we have } \mathcal{O}(D) \text{ is in group b),} \\ 1 & \text{If we have } \mathcal{O}(D) \text{ is in group c).} \end{cases}$$

Thus we can also eliminate $\{1, 2, 4, 6\}$, since all the forbidden forms relative to this set have $a_4 - a_3 - a_5 \geq 2$.

Now observe that for the line bundles $\mathcal{O}(\sum_{\rho} a_{\rho} Z_{\rho})$ in the forbidden form relative to $\{1, 2, 3, 5\}$ we have that $a_4 - a_3 - a_5 \leq -1$. Thus there is just one possibility for them to apply to the line bundles arising from the full sequence 9: we should have $a_4 = -1$ while $a_3 = a_5 = 0$; then $z_4 - z_5 = -1$ and the line bundles must be in group d). But since $6 \notin I$ these line bundle should have a negative z_6 too, and this does not apply to any of the invertible sheaves in group d). To finish the proof we have still to check that the line bundles associated to the following divisors have no sections.

- a) $-Z_6, -Z_5 - Z_6, -Z_4 - Z_5 - Z_6, -Z_4 - 2Z_5 - Z_6, -2Z_4 - 2Z_5 - Z_6, Z_4 + Z_5 - Z_6, Z_4 - Z_6, 2Z_4 + 2Z_5 - Z_6, 2Z_4 + Z_5 - Z_6;$
- b) $-Z_4 - Z_5, -2Z_4 - 2Z_5, -Z_5, -Z_4 - 2Z_5, -2Z_4 - Z_5, -Z_4, -Z_5.$

Looking at (11) is straightforward to see that all the divisors in group a), which have a negative z_6 , are not linearly equivalent to a toric effective divisor, hence their associated line bundles have not zero-cohomology. The divisors in group b), on the other hand, are opposite of positive divisors and hence they have no sections. Thus the statement is proved. \square

4.2. $\mathcal{E}_1, \mathcal{E}_2$ and \mathcal{E}_4

Again we need to find the forbidden sets. It can be proved the following statement:

Proposition 4.12. *The forbidden sets for \mathcal{E}_i , $i = 1, 2, 4$ are*

$$\emptyset, \{2, 4\}, \{3, 5\}, \{1, 3\}, \{2, 5\}, \{1, 4\}, \{6, 7\},$$

$$\begin{aligned} & \{2,4,5\}, \{2,4,1\}, \{3,5,1\}, \{3,5,2\}, \{1,3,4\}, \\ & \{1,3,6,7\}, \{3,5,6,7\}, \{2,4,6,7\}, \{1,4,6,7\}, \{2,5,6,7\}, \\ & \{1,3,5,6,7\}, \{2,4,5,6,7\}, \{1,3,5,6,7\}, \{1,2,4,6,7\}, \\ & \{1,3,4,6,7\}, \{2,3,5,6,7\}, \{1,2,3,4,5\} \end{aligned}$$

Proof. Certainly the simplicial complex associated to a set with just one element has a trivial reduced homology. Since the faces have all trivial homology, the only two-elements forbidden sets are the primitive collection. We want to show that the forbidden sets of cardinality three are precisely the unions of two primitive collections and so they are

$$\{2,4,5\}, \{2,4,1\}, \{3,5,1\}, \{3,5,2\}, \{1,3,4\}.$$

Indeed in a three element forbidden set can either contain:

1. zero primitive collections;
2. one primitive collection;
3. two primitive collections.

If the first is the case, than the set is a maximal cone and hence it has a trivial homology. If we are in the hypothesis of the second case, then the chain complex associated to the simplicial complex will look like:

$$0 \longrightarrow K^2 \longrightarrow K^3 \longrightarrow K \longrightarrow 0$$

that is an exact sequence. Thus the simplicial complex has a trivial reduced homology. Finally, if the set I is the union of two primitive collections, then its associated chain complex will be

$$0 \longrightarrow K \longrightarrow K^3 \longrightarrow K \longrightarrow 0$$

and hence it is a forbidden set.

Claim 1: The four-elements forbidden sets are the complementary of the three elements forbidden sets. Indeed let us suppose that I is the complementary of a maximal cone. Since all maximal cone are generated by either v_6 o v_7 , a complementary of a cone is necessarily of the following form

$$\{a_1, a_2, b, c\}$$

with $\{a_1, a_2\}$ the only one primitive collection contained in I . Thus its associated chain complex is

$$0 \longrightarrow K^2 \longrightarrow K^5 \longrightarrow K^4 \longrightarrow K \longrightarrow 0$$

that is obviously exact.

Suppose now that I is the complementary of J , a set of the form $\{p_1, p_2, q\}$ that contains just one primitive collection, namely $\{p_1, p_2\}$. Since all such I 's count either 6 or 7 among their element, we can consider two different cases:

1. both 6 and 7 are in I ,
2. $\{6, 7\} \not\subseteq I$

If both 6 and 7 are in J , then I will contain three primitive collections and its associated chain complex will be

$$0 \longrightarrow K^3 \longrightarrow K^4 \longrightarrow K \longrightarrow 0$$

that is exact, and hence I is not forbidden. It can happen that just one among 6 and 7 is an element of J . Under this assumption I will contain three primitive collections and will exist one of its elements that will not belong to any of these. In this case the chain complex associate to I will be

$$0 \longrightarrow K \longrightarrow K^4 \longrightarrow K^4 \longrightarrow K \longrightarrow 0$$

that is exact, as required. To prove the claim it remain to show that the complementary of the forbidden sets of cardinality equal to three are still forbidden sets. But it can be easily seen that these sets are of the form $\{a_1, a_2, b_1, b_2\}$ with $\{a_1, a_2\}$ and $\{b_1, b_2\}$ the only primitive collections in I . Hence the associated chain complex will look like

$$0 \longrightarrow K^4 \longrightarrow K^4 \longrightarrow K \longrightarrow 0$$

and it is obviously not exact.

Claim 2 The forbidden sets of cardinality five are the complementaries of the primitive collections.

Let us suppose that the set I is the complementary of a face, then there are two cases: or $\{6, 7\} \subseteq I$, or just one among 6 and 7 is in I . If we are in the first situation, then it can be seen that I contains two disjoint primitive collections and its associated chain complex is

$$0 \longrightarrow K^4 \longrightarrow K^8 \longrightarrow K^5 \longrightarrow K \longrightarrow 0$$

that is exact. Now assume that not both 6 or 7 are in I . Then I will contain three primitive collection that will cover just four of its five elements. Its associated chain complex will be

$$0 \longrightarrow K^3 \longrightarrow K^7 \longrightarrow K^5 \longrightarrow K \longrightarrow 0.$$

Conversely, suppose that I is the complementary of a primitive collection J . If $J = \{6, 7\}$ then the chain complex associated to I is

$$0 \longrightarrow K^5 \longrightarrow K^5 \longrightarrow K \longrightarrow 0$$

that is not exact.

If, otherwise J is different from $\{6, 7\}$, then I contains exactly 3 primitive collections: two of them cover three elements of the set, while the last one is disjoint from the previous. Knowing this data is straightforward to see that the chain complex associated to I is

$$0 \longrightarrow K^2 \longrightarrow K^7 \longrightarrow K^5 \longrightarrow K \longrightarrow 0.$$

It remains to prove that do not exist forbidden sets of cardinality 6. Again we can split the proof in two cases, depending on whether I contains $\{6, 7\}$ or not. In the first case I will contain four primitive collections: $\{6, 7\}$ and other three that will cover the remaining four elements of I . Knowing this it is easy to check that the chain complex associated to I is

$$0 \longrightarrow K^6 \longrightarrow K^{11} \longrightarrow K^6 \longrightarrow K \longrightarrow 0.$$

If otherwise I does not contain $\{6, 7\}$, then it will contain five primitive relations that will cover five of its 6 elements. Its associated chain complex will be of the form

$$0 \longrightarrow K^5 \longrightarrow K^{10} \longrightarrow K^6 \longrightarrow K \longrightarrow 0$$

that is again exact, and hence the proof is complete. □

Using the previous result we are now able to prove the next three propositions.

Theorem 4.13. *The following is a full strongly exceptional collection of line bundles for \mathcal{E}_1 .*

$$\begin{aligned} \mathcal{O}, \mathcal{O}(Z_7), \mathcal{O}(Z_4), \mathcal{O}(Z_4 + Z_7), \mathcal{O}(Z_4 + Z_5), \mathcal{O}(Z_1 + Z_5 + 2Z_7), \\ \mathcal{O}(Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5 + 2Z_7). \end{aligned} \tag{12}$$

Proof. We already know, by Bondal’s method, that this collection of line bundles generates the bounded derived category of \mathcal{E}_1 .

Our first step will be to check if the following line bundles satisfy the required

acyclicity.

$$\begin{aligned}
& \mathcal{O}(\pm(Z_7)), \mathcal{O}(\pm(Z_4)), \mathcal{O}(\pm(Z_4 + Z_7)), \mathcal{O}(\pm(Z_4 + Z_5))\mathcal{O}(\pm(Z_4 + Z_5)), \\
& \mathcal{O}(\pm(Z_1 + Z_5 + 2Z_7)), \mathcal{O}(\pm(Z_4 + Z_5 + Z_7)), \mathcal{O}(\pm(Z_1 + Z_4 + Z_5 + Z_7)), \\
& \mathcal{O}(\pm(Z_1 + Z_4 + Z_5 + 2Z_7)), \mathcal{O}(\pm(Z_4 - Z_7)), \mathcal{O}(\pm(Z_4 + Z_5 - Z_7)), \\
& \mathcal{O}(\pm(Z_1 - Z_5 + Z_7)), \mathcal{O}(\pm(Z_1 + Z_4 + Z_5)), \mathcal{O}(\pm(Z_5)), \quad (13) \\
& \mathcal{O}(\pm(Z_1 - Z_4 + Z_5 + 2Z_7)), \mathcal{O}(\pm(Z_5 + Z_7)), \mathcal{O}(\pm(Z_5 - Z_7)), \\
& \mathcal{O}(\pm(Z_1 - Z_4 + Z_5 + Z_7)), \mathcal{O}(\pm(Z_5 - Z_7)), \mathcal{O}(\pm(Z_1 - Z_4 + 2Z - 7)), \\
& \mathcal{O}(\pm(Z_1 + Z_7)), \mathcal{O}(\pm(Z_1 + 2Z_7)), \mathcal{O}(\pm(-Z_1 + Z_4 - Z_7)), \mathcal{O}(\pm(Z_1)).
\end{aligned}$$

As before, in order to ease the computation, we divide these line bundles in groups:

- a) $\mathcal{O}(Z_7), \mathcal{O}(Z_4), \mathcal{O}(Z_4 + Z_7), \mathcal{O}(Z_4 + Z_5), \mathcal{O}(Z_4 + Z_5 + Z_7),$
 $\mathcal{O}(Z_1 + Z_5 + 2Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5 + 2Z_7),$
 $\mathcal{O}(Z_4 - Z_7), \mathcal{O}(Z_4 + Z_5 - Z_7), \mathcal{O}(Z_1 + Z_5), \mathcal{O}(Z_1 + Z_4 + Z_5), \mathcal{O}(Z_5),$
 $\mathcal{O}(Z_5 + Z_7), \mathcal{O}(Z_1 + Z_5 + 2Z_7), \mathcal{O}(-Z_1 + Z_4 - Z_7), \mathcal{O}(Z_1 + 2Z_7),$
 $\mathcal{O}(-Z_1 + Z_4), \mathcal{O}(-Z_1 + Z_4 - Z_7), \mathcal{O}(Z_1), \mathcal{O}(Z_1 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_7);$
- b) $\mathcal{O}(-Z_4 - Z_5), \mathcal{O}(-Z_4 - Z_5 - Z_7), \mathcal{O}(-Z_1 - Z_5 - Z_7), \mathcal{O}(-Z_1 - Z_5 - 2Z_7),$
 $\mathcal{O}(-Z_1 - Z_4 - Z_5 - Z_7), \mathcal{O}(-Z_1 - Z_4 - Z_5 - 2Z_7), \mathcal{O}(-Z_4 - Z_5 + Z_7),$
 $\mathcal{O}(-Z_1 - Z_5), \mathcal{O}(-Z_1 - Z_4 - Z_5), \mathcal{O}(-Z_5), \mathcal{O}(-Z_1 + Z_4 - Z_5 - 2Z_7),$
 $\mathcal{O}(-Z_5 - Z_7), \mathcal{O}(-Z_1 + Z_4 - Z_5 - Z_7), \mathcal{O}(-Z_5 + Z_7);$
- c) $\mathcal{O}(-Z_4 + Z_7), \mathcal{O}(-Z_4), \mathcal{O}(Z_1 - Z_4 + Z_7), \mathcal{O}(-Z_1 - Z_7),$
 $\mathcal{O}(Z_1 - Z_4), \mathcal{O}(-Z_4 - Z_7), \mathcal{O}(-Z_1), \mathcal{O}(Z_1 - Z_4 + 2Z_7), \mathcal{O}(-Z_7);$
- d) $\mathcal{O}(Z_1 - Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 - Z_4 + Z_5 + 2Z_7), \mathcal{O}(Z_1 - Z_4 + Z_5),$
 $\mathcal{O}(-Z_1 - 2Z_7);$
- e) $\mathcal{O}(-Z_1 + Z_4 - 2Z_7);$
- f) $\mathcal{O}(Z_5 - Z_7).$

Given a divisor $D = z_1Z_1 + z_4Z_4 + z_5Z_5 + z_7Z_7$ on \mathcal{E}_1 , written in the basis (Z_1, Z_4, Z_5, Z_7) of $\text{Pic}(\mathcal{E}_1)$ we want to find conditions for another divisor $D' = \sum_{\rho=1}^7 a_\rho Z_\rho$ to be linear equivalent to D . In order to do that we write D' in the given basis using the following relations among the principal toric divisors:

$$Z_2 = -Z_1 + Z_4 - Z_7, \quad Z_3 = Z_1 + Z_5 + Z_7, \quad Z_6 = Z_7.$$

We gain

$$D' \simeq_{\text{lin}} (a_1 - a_2 + a_3)Z_1 + (a_2 + a_4)Z_4 + (a_5 + a_3)Z_5 + (-a_2 + a_3 + a_6 + a_7)Z_7. \quad (14)$$

Now, imposing equality among the coefficients of D' and D we have the conditions we were seeking.

Using (14), it can easily be seen that all the line bundles in group a) are line bundles associated to toric effective divisors with coefficient $a_i \in \{1, 0\}$, and hence are acyclic. In order to check the acyclicity of the other line bundles we are going to use Borisov-Hua's criterion. Observe that:

1. The forbidden forms relative to a set I which does not contain neither 2 nor 4 as elements have the coefficient $z_4 \leq -2$.
2. The forbidden forms relative to a set I which does not contain neither 3 nor 5 satisfy $z_5 \leq -2$.
3. The forbidden forms relative to a set I such that $2 \in I$, but neither 1, nor 3 are in I satisfy $z_1 \leq -2$.
4. The forbidden forms relative to a set I with $2 \in I$ and such that its complementary I' contains $\{3, 6, 7\}$ as a subset have $z_7 \leq -3$.

Thus we can eliminate all the forbidden sets satisfying condition 1)-4). The set we have still to check are:

$$\{2, 3, 5\}, \{1, 3, 4\}, \{1, 3, 4, 6, 7\}, \{2, 3, 5, 6, 7\}, \{1, 2, 3, 4, 5\}.$$

We can eliminate $\{1, 3, 4\}$ and $\{1, 3, 4, 6, 7\}$ in the following way: suppose the some of the line bundles we are working with can be put in one of the forbidden forms relative to these sets. Since both sets $\{1, 3, 4\}$ and $\{1, 3, 4, 6, 7\}$ contains 1 and 3 but not 2 it follows that $z_1 = a_1 - a_2 + a_3 \geq 1$. The only possibility for our divisors is $z_1 = 1$ and hence $a_1 = a_3 = 0$ and $a_2 = -1$. But $5 \notin \{1, 3, 4\} \cup \{1, 3, 4, 6, 7\}$, thus $a_3 = 0$ would imply that $z_5 < 0$. Hence these line bundles should be in group b). But all the divisors in group b) have a non positive z_1 .

Now we will show that none among the divisors in b)-f) can be put in a forbidden form relative to $I = \{2, 3, 5\}$. Surely, since both a_3 and a_5 are positive, none of the divisors in b) (which have a negative z_5) is in the forbidden form relative to I . If we have a line bundle associated to a divisor D whose coefficient z_5 is null, then it can be put in one of the forbidden form relative to I if $a_3 = a_5 = 0$. As a consequence $z_7 = -a_2 + a_3 + a_6 + a_7 \leq -2$. Thus the line bundles in c) (which have $z_5 = 0$ and $z_7 \geq -1$) cannot be put in the required forbidden form. Let us see that neither one of the line bundles in e) can be of the forbidden form relative to I : if this would be the case, then we will have that $a_2 = 0$ and, since $4 \notin I$, $z_4 \leq -1$ that is impossible. Now we want to show that the line bundles in d) and f) are not in the forbidden form relative to I . In this case a_3 can be both

0 or -1. In any case we will have that $z_7 \leq -1$, and this is sufficient to eliminate all the bundles in d). As before we eliminate the invertible sheaf in e) because its z_4 is not negative.

In order to eliminate the set $I = \{1, 2, 3, 4, 5\}$ we proceed in a similar way. Observe now that again a_3 can be 0 or 1. If a_3 is 0, then $z_7 \leq -2$ and $z_1 \geq 0$ that is impossible, because the only invertible sheaf with $z_7 = -2$ is the one in e) and have $z_1 = -1$. Then $a_3 = 1$. But in this case we have $z_5 \geq 1$, $z_7 \leq -1$ and $z_1 \geq 1$ that is again impossible. It remains to check that none of the divisors in the list can be put in the forbidden form relative to $I = \{2, 3, 5, 6, 7\}$. Again $a_3 = 0, 1$. If $a_3 = 0$, then $z_1 \leq -1$ it follows that $z_1 = -1$ and $a_2 = 0$. As a consequence $z_4 \leq -1$ but none of the divisor we have has both a negative z_1 and a negative z_4 . Now suppose that $a_3 = 1$. Then $z_5 = 1$ and $z_1 \leq 0$. All the line bundles with a positive z_5 (group d) and f)) have a non-negative z_1 . It follows that $z_1 = 0$ and, in particular $z_4 \leq -1$ but this is impossible.

Finally, to finish proving the proposition, we have to check that the following divisors are not linearly equivalent to any toric effective divisor.

$$\begin{aligned} \text{a) } & -Z_4, -Z_4 - Z_7, -Z_4 - Z_5, -Z_4 - Z_5 - Z_7, -Z_1 - Z_5 - 2Z_7, \\ & -Z_1 - Z_4 - Z_5 - Z_7, -Z_1 - Z_4 - Z_5 - 2Z_7, -Z_4 + Z_7, -Z_4 - Z_5 + Z_7, -Z_1 - \\ & Z_5 + Z_7, -Z_1 - Z_4 - Z_5, -Z_5, -Z_5 - Z_7, -Z_1 + Z_4 - Z_5 - 2Z_7, \\ & -Z_1 - Z_5 - Z_7, -Z_5 + Z_7, -Z_1 + Z_4 - Z_5 - Z_7, -Z_1 - Z_5, Z_1 - Z_4 + Z_7; \end{aligned}$$

$$\text{b) } -Z_7, -Z_1 + Z_4 - 2Z_7, -Z_1 - Z_7, -Z_1 - 2Z_7, -Z_1.$$

Observe that the divisors in a), which have a negative z_4 or a negative z_5 cannot possibly be linear equivalent to a toric effective divisor. For what it concerns group b), it can be deduced from (14) that it is a necessary condition in order for a divisor D to be linearly equivalent to a toric effective divisor that $z_1 \geq -a_2 \geq -z_4$ and $z_7 \geq -a_2 \geq -z_4$. It is easy to see that none of the divisors in group b) satisfies this condition. \square

Theorem 4.14. *The following is a full strongly exceptional collection for \mathcal{E}_2 :*

$$\mathcal{O}, \mathcal{O}(Z_7), \mathcal{O}(Z_4), \mathcal{O}(Z_1 + Z_5), \mathcal{O}(Z_1 + Z_5 + Z_7), \mathcal{O}(Z_4 + Z_7), \mathcal{O}(Z_4 + Z_5),$$

$$\mathcal{O}(Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5), \mathcal{O}(Z_1 + Z_4 + Z_5 + Z_7).$$

Proof. We already know that the collection is full. Thus we have just to show that it is strongly exceptional. First of all we will prove the vanishing of the

higher cohomology of the following line bundles:

$$\begin{aligned}
& \mathcal{O}(\pm(Z_7)), \mathcal{O}(\pm(Z_4)), \mathcal{O}(\pm(Z_1 + Z_5)), \mathcal{O}(\pm(Z_1 + Z_5 + Z_7)), \\
& \mathcal{O}(\pm(Z_4 + Z_7)), \mathcal{O}(\pm(Z_4 + Z_5)), \mathcal{O}(\pm(Z_4 + Z_5 + Z_7)), \mathcal{O}(\pm(Z_1 + Z_4 + Z_5)), \\
& \mathcal{O}(\pm(Z_1 + Z_4 + Z_5 + Z_7)), \mathcal{O}(\pm(Z_4 - Z_7)), \mathcal{O}(\pm(Z_1 + Z_5 - Z_7)), \\
& \mathcal{O}(\pm(Z_4 + Z_5 - Z_7)), \mathcal{O}(\pm(Z_1 + Z_4 + Z_5 - Z_7)), \mathcal{O}(\pm(Z_1 + Z_5 - Z_4)), \quad (15) \\
& \mathcal{O}(\pm(Z_5)), \mathcal{O}(\pm(Z_5 + Z_7)), \mathcal{O}(\pm(Z_1 - Z_4 + Z_5 - Z_7)), \mathcal{O}(\pm(Z_4 - Z_1)), \\
& \mathcal{O}(\pm(Z_1 - Z_4 - Z_7)), \mathcal{O}(\pm(Z_4 + Z_7)), \mathcal{O}(\pm(Z_1 - Z_4 + Z_7)), \mathcal{O}(\pm(Z_5 - Z_7)), \\
& \mathcal{O}(\pm(Z_1 + Z_7)), \mathcal{O}(\pm(Z_1 - Z_7)), \mathcal{O}(\pm(Z_1)), \mathcal{O}(\pm(-Z_1 + Z_4 - Z_5 + Z_7)).
\end{aligned}$$

As usual, it is better to split all these invertible sheaves into four groups.

- a) $\mathcal{O}, \mathcal{O}(Z_7), \mathcal{O}(Z_4), \mathcal{O}(Z_1 + Z_5), \mathcal{O}(Z_1 + Z_5 + Z_7), \mathcal{O}(Z_4 + Z_7),$
 $\mathcal{O}(Z_4 + Z_5), \mathcal{O}(Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5), \mathcal{O}(Z_1 + Z_4 + Z_5 + Z_7),$
 $\mathcal{O}(Z_5), \mathcal{O}(Z_5 + Z_7), \mathcal{O}(Z_4 + Z_7), \mathcal{O}(Z_1), \mathcal{O}(Z_1 + Z_7), \mathcal{O}(Z_4 - Z_7),$
 $\mathcal{O}(-Z_1 + Z_4), \mathcal{O}(-Z_1 + Z_4 - Z_7), \mathcal{O}(-Z_1 + Z_4 + Z_7);$
- b) $\mathcal{O}(-Z_1 + Z_4 - Z_5), \mathcal{O}(-Z_1 + Z_4 - Z_5 + Z_7), \mathcal{O}(-Z_1 + Z_4 - Z_5 - Z_7),$
 $\mathcal{O}(-Z_5), \mathcal{O}(-Z_1 - Z_5), \mathcal{O}(-Z_1 - Z_5 - Z_7), \mathcal{O}(-Z_1 - Z_5 + Z_7),$
 $\mathcal{O}(-Z_5 - Z_7), \mathcal{O}(-Z_5 + Z_7), \mathcal{O}(-Z_4 - Z_5), \mathcal{O}(-Z_4 - Z_5 - Z_7),$
 $\mathcal{O}(-Z_1 - Z_4 - Z_5), \mathcal{O}(-Z_1 - Z_4 - Z_5 - Z_7), \mathcal{O}(-Z_4 - Z_5 + Z_7),$
 $\mathcal{O}(-Z_1 - Z_4 - Z_5 + Z_7);$
- c) $\mathcal{O}(+Z_1 + Z_4 + Z_5 - Z_7), \mathcal{O}(Z_4 + Z_5 - Z_7), \mathcal{O}(Z_5 - Z_7), \mathcal{O}(Z_1 + Z_5 - Z_7),$
 $\mathcal{O}(Z_1 - Z_4 + Z_5), \mathcal{O}(Z_1 - Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 - Z_4 + Z_5 - Z_7);$
- d) $\mathcal{O}(-Z_7), \mathcal{O}(-Z_1 - Z_7), \mathcal{O}(-Z_1), \mathcal{O}(Z_1 - Z_7), \mathcal{O}(-Z_1 + Z_7), \mathcal{O}(-Z_4),$
 $\mathcal{O}(-Z_4 - Z_7), \mathcal{O}(-Z_4 + Z_7), \mathcal{O}(Z_1 - Z_4), \mathcal{O}(Z_1 - Z_4 - Z_7),$
 $\mathcal{O}(Z_1 - Z_4 + Z_7), \mathcal{O}(-Z_1 - Z_5 + Z_7), \mathcal{O}(-Z_5 - Z_7), \mathcal{O}(Z_1 - Z_4 - Z_7),$
 $\mathcal{O}(-Z_1 - Z_4 - Z_5 - Z_7),$
 $\mathcal{O}(Z_1 - Z_4 + Z_5 - Z_7).$

Let $D = \sum_{\rho=1}^7 a_\rho Z_\rho$ be any divisor on \mathcal{E}_2 . We want to write it in the basis given by Z_1, Z_4, Z_5 and Z_7 using the following relations:

$$Z_2 = -Z_1 + Z_4 - Z_7, \quad Z_3 = Z_1 + Z_5, \quad Z_6 = Z_7.$$

We get

$$D \simeq_{\text{lin}} (a_1 + a_3 - a_2)Z_1 + (a_3 + a_5)Z_5 + (a_2 + a_4)Z_4 + (a_6 + a_7 - a_2)Z_7. \quad (16)$$

We indicate with (z_1, z_4, z_5, z_7) the coordinate of D in the chosen basis.

We can easily see that all the line bundles in a) can be associated to a toric effective divisor with the coefficients $a_i \in \{1, 0\}$ (and hence acyclic due to Remark 4.5). For example

$$-Z_1 + Z_4 + Z_7 \simeq_{\text{lin}} Z_2 + Z_6 + Z_7.$$

For all the other divisors we need Borisov-Hua criterion.

It is easy to see that the remaining line bundles cannot be forbidden with respect to a set I unless I is among the following three sets:

$$\{1, 3, 4\}, \{1, 3, 4, 6, 7\}, \{2, 3, 5, 6, 7\}.$$

First of all we eliminate $\{1, 3, 4\}, \{1, 3, 4, 6, 7\}$: in both these sets appear indices 1 and 3 and does not appear 2. This means that coefficient z_1 is greater or equal to 1. The only possibility is that $z_1 = 1$, hence $a_1 = a_3 = 0$ and $a_2 = -1$. Since neither 5 is in the previous sets, then $-1 \leq z_5 = a_3 + a_5 \leq -1$. The divisors with $z_5 = -1$ are the ones of group b). You can observe that neither one of these has a positive z_1 .

It remains to eliminate $I = \{2, 3, 5, 6, 7\}$.

All the divisors of the group b) are not of a forbidden form relative to this set. In fact all this divisors have $z_5 = -1$ and the forbidden divisors relative to I have $z_5 \geq 0$. Neither the divisor of group d), that have $z_5 = 0$ cannot be of the forbidden forms relative to I . Indeed, if we impose to the forbidden forms the condition to have $z_5 = 0$ we get that, necessarily, $a_3 = a_5 = 0$. Thus $a_1 = z_1 + a_2 \geq z_1$. Since $a_1 \leq -1$ we get that $a_1 = z_1 = -1$. But, on the other side we have that $z_4 = a_2 + a_4 \leq -1$. None of the divisor of group d) have both z_1 and z_4 negative. Finally we can see that also the divisors of group c) are not in of the forbidden forms relative to I : since both a_3 and a_5 are non negative, we have that $a_3 \leq 1$. It follows that $z_1 = a_1 + a_3 - a_2 \leq 0$. But all the divisors in c) have a non negative z_1 coefficient, thus there is just one possibility: $z_1 = 0$. This yields that $a_1 = -1, a_3 = 1$ and $a_2 = 0$. The last equality implies that $z_4 = a_4 \leq -1$, but this cannot be since all the divisors in c) who have $z_1 = 0$ have a non negative z_4 too.

Now, to prove the statement, we just need to show that the following divisors are not linearly equivalent to a toric effective divisor.

- a) $-Z_4, -Z_4 - Z_7, -Z_4 - Z_5, -Z_4 - Z_5 - Z_7, -Z_1 - Z_4 - Z_5,$
 $-Z_1 - Z_4 - Z_5 - Z_7, -Z_4 + Z_7, -Z_4 - Z_5 + Z_7, -Z_1 - Z_4 - Z_5 + Z_7,$
 $Z_1 - Z_4 + Z_5 - Z_7, Z_1 - Z_4, Z_1 - Z_4 - Z_7, -Z_1 - Z_4, Z_1 + Z_5 - Z_4,$
 $Z_1 + Z_7 - Z_4, -Z_4 + Z_7;$
- b) $-Z_1 - Z_5, -Z_1 - Z_5 - Z_7, -Z_1 - Z_5 + Z_7, -Z_1 + Z_4 - Z_5, -Z_5, -Z_5 - Z_7,$
 $-Z_1 - Z_5 - Z_7, -Z_5 + Z_7;$

c) $-Z_7, -Z_1, -Z_1 - Z_7, Z_7 - Z_1$.

Looking at (16) it is obvious that the divisors in a) and b), which have or $z_4 = -1$ or $z_5 = -1$ cannot be toric effective. The divisors in the last group have $z_4 = 0$ and or z_1 or z_7 equal to -1 . If they were equivalent to a toric effective divisor, then $a_2 > 0$ but this is impossible since we would have $z_4 = a_2 + a_4 > 0$. \square

Theorem 4.15. *The following is a full strongly exceptional collection for \mathcal{E}_4 :*

$$\begin{aligned} &\mathcal{O}, \mathcal{O}(Z_7), \mathcal{O}(Z_4), \mathcal{O}(Z_1 + Z_5), \mathcal{O}(Z_1 + Z_5 + Z_7), \mathcal{O}(Z_4 + Z_7), \mathcal{O}(Z_4 + Z_5), \\ &\mathcal{O}(Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5), \mathcal{O}(Z_1 + Z_4 + Z_5 + Z_7). \end{aligned}$$

Proof. We already know that the collection is full. Thus we have just to show that it is strongly exceptional. First of all we will prove the acyclicity of the following line bundles:

$$\begin{aligned} &\mathcal{O}(\pm(Z_7)), \mathcal{O}(\pm(Z_4)), \mathcal{O}(\pm(Z_1 + Z_5)), \mathcal{O}(\pm(Z_1 + Z_5 + Z_7)), \\ &\mathcal{O}(\pm(Z_4 + Z_7)), \mathcal{O}(\pm(Z_4 + Z_5)), \mathcal{O}(\pm(Z_4 + Z_5 + Z_7)), \mathcal{O}(\pm(Z_1 + Z_4 + Z_5)), \\ &\mathcal{O}(\pm(Z_1 + Z_4 + Z_5 + Z_7)), \mathcal{O}(\pm(Z_4 - Z_7)), \mathcal{O}(\pm(Z_1 + Z_5 - Z_7)), \\ &\mathcal{O}(\pm(Z_4 + Z_5 - Z_7)), \mathcal{O}(\pm(Z_1 + Z_4 + Z_5 - Z_7)), \mathcal{O}(\pm(Z_1 + Z_5 - Z_4)), \quad (17) \\ &\mathcal{O}(\pm(Z_5)), \mathcal{O}(\pm(Z_5 + Z_7)), \mathcal{O}(\pm(Z_1 - Z_4 + Z_5 - Z_7)), \mathcal{O}(\pm(Z_4 - Z_1)), \\ &\mathcal{O}(\pm(Z_1 - Z_4 - Z_7)), \mathcal{O}(\pm(Z_4 + Z_7)), \mathcal{O}(\pm(Z_1 - Z_4 + Z_7)), \mathcal{O}(\pm(Z_5 - Z_7)), \\ &\mathcal{O}(\pm(Z_1 + Z_7)), \mathcal{O}(\pm(Z_1 - Z_7)), \mathcal{O}(\pm(Z_1)), \mathcal{O}(\pm(-Z_1 + Z_4 - Z_5 + Z_7)). \end{aligned}$$

As in the previous case we shall split all the line bundles above into four groups.

- a) $\mathcal{O}, \mathcal{O}(Z_7), \mathcal{O}(Z_4), \mathcal{O}(Z_1 + Z_5), \mathcal{O}(Z_1 + Z_5 + Z_7), \mathcal{O}(Z_4 + Z_7),$
 $\mathcal{O}(Z_4 + Z_5), \mathcal{O}(Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 + Z_4 + Z_5), \mathcal{O}(Z_1 + Z_4 + Z_5 + Z_7),$
 $\mathcal{O}(Z_5), \mathcal{O}(Z_5 + Z_7), \mathcal{O}(Z_4 + Z_7), \mathcal{O}(Z_1), \mathcal{O}(Z_1 + Z_7), \mathcal{O}(-Z_1 + Z_4),$
 $\mathcal{O}(-Z_1 + Z_4 + Z_7), \mathcal{O}(+Z_1 + Z_4 + Z_5 - Z_7), \mathcal{O}(Z_4 + Z_5 - Z_7), \mathcal{O}(Z_5 - Z_7),$
 $\mathcal{O}(Z_1 + Z_5 - Z_7);$
- b) $\mathcal{O}(-Z_1 + Z_4 - Z_5), \mathcal{O}(-Z_1 + Z_4 - Z_5 + Z_7), \mathcal{O}(-Z_1 + Z_4 - Z_5 - Z_7),$
 $\mathcal{O}(-Z_5), \mathcal{O}(-Z_1 - Z_5), \mathcal{O}(-Z_1 - Z_5 - Z_7), \mathcal{O}(-Z_1 - Z_5 + Z_7),$
 $\mathcal{O}(-Z_5 - Z_7), \mathcal{O}(-Z_5 + Z_7), \mathcal{O}(-Z_4 - Z_5), \mathcal{O}(-Z_4 - Z_5 - Z_7),$
 $\mathcal{O}(-Z_1 - Z_4 - Z_5), \mathcal{O}(-Z_1 - Z_4 - Z_5 - Z_7), \mathcal{O}(-Z_4 - Z_5 + Z_7),$
 $\mathcal{O}(-Z_1 - Z_4 - Z_5 + Z_7);$
- c) $\mathcal{O}(Z_1 - Z_4 + Z_5), \mathcal{O}(Z_1 - Z_4 + Z_5 + Z_7), \mathcal{O}(Z_1 - Z_4 + Z_5 - Z_7);$
- d) $\mathcal{O}(-Z_7), \mathcal{O}(-Z_1 + Z_4 - Z_7), \mathcal{O}(Z_4 - Z_7), \mathcal{O}(-Z_1 - Z_7), \mathcal{O}(-Z_1),$
 $\mathcal{O}(Z_1 - Z_7), \mathcal{O}(-Z_1 + Z_7), \mathcal{O}(-Z_4), \mathcal{O}(-Z_4 - Z_7), \mathcal{O}(-Z_4 + Z_7),$
 $\mathcal{O}(Z_1 - Z_4), \mathcal{O}(Z_1 - Z_4 - Z_7), \mathcal{O}(Z_1 - Z_4 + Z_7).$

As in the previous cases, our first step will be to write the generic divisor $D = \sum_{\rho=1}^7 a_{\rho} Z_{\rho}$ in the basis $\text{Pic}(\mathcal{E}_4)$ given by Z_1, Z_4, Z_5 and Z_7 using the following relations among the generators:

$$Z_2 = -Z_1 + Z_4, \quad Z_3 = Z_1 + Z_5 - Z_7, \quad Z_6 = Z_7.$$

We get

$$D \simeq_{\text{lin}} (a_1 + a_3 - a_2)Z_1 + (a_3 + a_5)Z_5 + (a_2 + a_4)Z_4 + (a_6 + a_7 - a_3)Z_7. \quad (18)$$

As usual we denote with (z_1, z_4, z_5, z_7) the coordinates of D in the chosen basis. A tedious computation shows that all the line bundle in a) are line bundles associated to a divisor whose coefficients a_i are either 0 or 1 (and hence acyclic thanks to Remark 4.5). For all the other divisors we need Borisov-Hua criterion.

As in the case of \mathcal{E}_2 it can be observed that none of the line bundles above can be of any of the forbidden forms relative to a forbidden set I unless $I = \{1, 3, 4, 6, 7\}$.

Observe that 1 and 3 are among the elements of I and $2 \notin I$. This implies that $z_1 = 1$, hence $a_1 = a_3 = 0$ and $a_2 = -1$. Since $5 \notin I$ $-1 \leq z_5 = a_3 + a_5 \leq -1$. The divisors with $z_5 = -1$ are the ones of group b) and none of these has a positive z_1 .

Now, to prove the statement, we just need to show that the following divisors are not linearly equivalent to a toric effective divisor.

- a) $-Z_4, -Z_4 - Z_7, -Z_4 - Z_5, -Z_4 - Z_5 - Z_7, -Z_1 - Z_4 - Z_5,$
 $-Z_1 - Z_4 - Z_5 - Z_7, -Z_4 + Z_7, -Z_4 - Z_5 + Z_7, -Z_1 - Z_4 - Z_5 + Z_7,$
 $Z_1 - Z_4 + Z_5 - Z_7, Z_1 - Z_4, Z_1 - Z_4 - Z_7, -Z_1 - Z_4, Z_1 + Z_5 - Z_4,$
 $Z_1 + Z_7 - Z_4, -Z_4 + Z_7;$
- b) $-Z_1 - Z_5, -Z_1 - Z_5 - Z_7, -Z_1 - Z_5 + Z_7, -Z_1 + Z_4 - Z_5, -Z_5, -Z_5 - Z_7,$
 $-Z_1 - Z_5 - Z_7, -Z_5 + Z_7;$
- c) $-Z_7, -Z_1, -Z_1 - Z_7, Z_7 - Z_1.$

Looking at (16) it is obvious that the divisors in a) and b), which have or negative z_4 or a negative z_5 cannot be toric effective. The divisors in the last group have both z_4 and z_5 equal to zero, while at least one among z_1 and z_7 is equal to -1. Thus it is straightforward to see that they cannot be linearly equivalent to a toric effective divisor. \square

5. Conclusions

Collecting all the results we obtained so far we are able to enunciate:

Theorem 5.1 (Main Theorem). *All toric Fano 3-folds have a full strongly exceptional collection made up of line bundles.*

Proof. As mentioned in the introduction to this paper, there were just five varieties left out: \mathcal{D}_1 , \mathcal{D}_2 , \mathcal{E}_1 , \mathcal{E}_2 and \mathcal{E}_4 . In theorem 4.10 we proved that \mathcal{D}_1 admits a full strongly exceptional collection. In theorems 4.11, 4.13, 4.14 and 4.15 we showed that the same thing is true for the other four left. \square

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