LE MATEMATICHE
Vol. LXIV (2009) – Fasc. II, pp. 99–116

DERIVED CATEGORY OF TORIC VARIETIES WITH PICARD NUMBER THREE

ARIJIT DEY - MICHAŁ LASOŃ - MATEUSZ MICHAŁEK

We construct a full, strongly exceptional collection of line bundles on the variety X that is the blow up of the projectivization of the vector bundle $\mathscr{O}_{\mathbb{P}^{n-1}} \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(b_1)$ along a linear space of dimension $n-2$, where b_1 is a non-negative integer.

1. Introduction

Let X be a smooth projective variety over the field of complex numbers $\mathbb{C}$ and let $D^b(X)$ be the derived category of bounded complexes of coherent sheaves of $\mathscr{O}_X$-modules. This category is an important algebraic invariant of X. In order to understand the derived category $D^b(X)$ one is interested in knowing a *strongly exceptional* finite collection of objects in $D^b(X)$ that generates the derived category $D^b(X)$.

The notion of "strongly exceptional" collection was first introduced by Gorodentsev and Rudakov [8] in order to study vector bundles on $\mathbb{P}^n$. An *exceptional* collection $\{F_0, F_1, \cdots, F_m\}$ of sheaves gives a functor F_E from the category of coherent sheaves $Coh(X)$ to the derived category $D^b(\mathscr{A} - module)$ of $\mathscr{A}$-modules, where $E = \oplus_{i=0}^{m} F_i$ and $\mathscr{A} = Hom(E,E)$. The functor F_E is extendable to the derived functor $D^b(F_E)$ from $D^b(X)$ to $D^b(\mathscr{A} - module)$. In

Entrato in redazione: 15 gennaio 2010

AMS 2000 Subject Classification: 14F05
Keywords: derived category, strongly exceptional collection, toric varieties

[3] Bondal proved that if $\{F_0, F_1, \cdots, F_m\}$ is a *full strongly exceptional* collection then the functor $D^b(F_E)$ is an equivalence of categories. The existence of a *full strongly exceptional collection* $\{F_0, F_1, \cdots, F_m\}$ of coherent sheaves on a smooth projective variety puts a strong condition on X namely the Grothendieck group $K^0(X)$ is isomorphic to $\mathbb{Z}^{m+1}$. In general for an arbitrary variety X, $K^0(X)$ has torsion but K^0 of a toric variety is a finitely generated free abelian group. So it is interesting to search for *full strongly exceptional collections* of sheaves in case of toric variety. For a smooth complete toric variety Kawamata [11] proved that the derived category $D^b(X)$ has a full collection of exceptional objects. In his collection, the objects are sheaves rather than line bundles and the collection is only exceptional and not strongly exceptional. For a smooth complete toric variety X, there is a well known construction due to Bondal which gives a (finite) full collection of line bundles of $D^b(X)$. In general Bondal's collection of line bundles need not be a strongly exceptional collection but one "hopes" that for huge families of toric varieties we will be able to choose a subset and order it in such a way that it becomes a full strongly exceptional collection. In [12] King made the following conjecture:

Conjecture 1.1. *For any smooth, complete toric variety X there exists a full, strongly, exceptional collection of line bundles.* [1]

This conjecture was disproved by Hille and Perling, in [10] they gave an example of a smooth, complete toric surface that does not have a full strongly exceptional collection of line bundles. The conjecture was reformulated by Costa and Miró-Roig:

Conjecture 1.2. *For every smooth, complete Fano toric variety there exists a full strongly exceptional collection of line bundles.*

This conjecture is still open and is supported by many numerical evidences. It has an affirmative answer when the Picard number of X is less than or equal to two. When the Picard number is one it is easy to see that X is isomorphic to projective space $\mathbb{P}^r$ and the collection $(\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(r))$ is a full strongly exceptional collection (this is a consequence of Beilinson's theorem). When the Picard number of X is two, the above question has affirmative answer and this is due to Costa and Miró-Roig [6]. When the Picard number is 3 the question is not fully resolved.

Toric varieties with Picard number three are completely classified by Batyrev [1] in terms of its *primitive collections*, he showed that any toric variety with Picard number 3 has 3 or 5 primitive collections. Toric varieties with 3

[1] Originally this conjecture was made in terms of existence of titling bundles whose direct summands are line bundles but it is easy to see that they are equivalent, see [6]

primitive collections are isomorphic to a projectivization of a decomposable bundle over a smooth toric variety W of a smaller dimension with Picard number 2, hence by [6] we have an affirmative answer to the conjecture. When the number of primitive collections is 5 the conjecture is still open. There are some partial results known in this case, for example recently Costa and Miró-Roig [5] answered the above conjecture affirmatively when X is a blow up of $\mathbb{P}^{n-r} \times \mathbb{P}^r$ along a multilinear subvariety of codimension 2. Motivated by this result, R. M. Miró-Roig and L. Costa (in the meeting P.R.A.G.MAT.I.C' 09) suggested us to investigate this question for a large family of toric varieties parameterized by positive integers b_1, n. In this note we consider a toric variety X which is a blow up of the projectivization of the vector bundle $\mathscr{O}_{\mathbb{P}^{n-1}} \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(b_1)$ on $\mathbb{P}^{n-1}$ along a linear space of dimension $n-2$, where b_1 is a positive integer. We are able to answer the conjecture affirmatively for this family of toric varieties (see Theorem 4.15). Note that not all our varieties are Fano, in fact X is Fano if $b_1 < n-1$.

We outline the structure of this paper. In §2, we briefly review the notions of strongly exceptional collection of sheaves and few basic facts about toric varieties which will be needed later on. In §3, we recall Batyrev's classification of toric varieties and we describe the family of toric varieties which we are interested in, in terms of fans and its primitive relations. In §4 we determine explicitly a full strongly exceptional collection of line bundles for this family of toric varieties.

2. Preliminaries

The goal of this section is to fix the notation and basic facts that we will use through this paper. We start by recalling the notions of exceptional sheaves, exceptional collection of sheaves, strongly exceptional collection of sheaves and full strongly exceptional collection of sheaves. Let X be a smooth projective variety over $\mathbb{C}$.

Definition 2.1. *1. A coherent sheaf F on X is* **exceptional** *if $Hom(F,F) = \mathbb{C}$ and*
$Ext^i_{\mathscr{O}_X}(F,F) = 0$ for $i > 0$.

2. *An ordered collection $(F_0, F_1, \cdots, F_m)$ of coherent sheaves on X is an* **exceptional collection** *if each sheaf F_i is exceptional and*
$Ext^i_{\mathscr{O}_X}(F_k, F_j) = 0$ for $j < k$ and $i \geq 0$.

3. *An exceptional collection $(F_0, F_1, \cdots, F_m)$ of coherent sheaves on X is a* **strongly exceptional collection** *if $Ext^i_{\mathscr{O}_X}(F_j, F_k) = 0$ for $j \leq k$ and $i \geq 1$.*

4. *A (strongly) exceptional collection* $(F_0, F_1, \cdots, F_m)$ *of coherent sheaves on* X *is a* **full (strongly) exceptional collection** *if it generates the bounded derived category* $D^b(X)$ *of* X *i.e. the smallest triangulated category containing* $\{F_0, F_1, \cdots, F_m\}$ *is equivalent to* $D^b(X)$.

3. Toric varieties with Picard number three

In this section we introduce notation and facts concerning toric varieties that we use in our paper. An n dimensional toric variety X is a smooth, projective variety containing an n dimensional torus T (n copies of $\mathbb{C}^*$) together with an action on X and characterized by a fan Σ of strongly convex polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$, where N is a lattice $\mathbb{Z}^n$. We denote the $\mathbb{Z}$-basis of N by $e_1, \cdots, e_n$ and by $e_1^*, \cdots, e_n^*$ its dual basis in $M := Hom_{\mathbb{Z}}(N, \mathbb{Z})$. For every one dimensional cone $\sigma \in \Sigma$ there is a unique generator $v \in N$ such that $\sigma \cap N = \mathbb{Z}_{\geq 0} \cdot v$, it is called the ray generator. The set of all ray generators is denoted by R. To each ray generator $r \in R$ one can associate a toric divisor D_r (see also [7]). If the number of toric divisors is m then the Picard number of X is $m - n$ where n is the dimension of X. The anticanonical divisor $-K_X$ is given by $-K_X = \sum_{r \in R} D_r$. We say that X is Fano if $-K_X$ is ample.

Smooth, complete toric varieties with Picard number three have been classified by Batyrev in [1] according to their primitive relations. Let Σ be a fan in $N = \mathbb{Z}^n$.

Definition 3.1. *We say that a subset* $P \subset R$ *is a primitive collection if it is a minimal (with respect to inclusion) subset of* R *which does not span a cone in* Σ.

In other words a primitive collection is a subset of ray generators, such that all together they do not span a cone in Σ but if we remove any generator, then the rest spans a cone that belongs to Σ. To each primitive collection $P = \{x_1, \ldots, x_k\}$ we associate a primitive relation. Let $w = \sum_{i=1}^k x_i$ and $\sigma \in \Sigma$ be the cone of the smallest dimension that contains w. Let $y_1, \ldots, y_s$ be the ray generators of this cone. The toric variety of Σ was assumed to be smooth, so there are unique positive integers $n_1, \ldots, n_s$ such that

$$w = \sum_{i=1}^{s} n_i y_i.$$

Definition 3.2. *For each primitive collection* $P = \{x_1, \ldots, x_k\}$ *the linear relation:*

$$x_1 + \cdots + x_k - n_1 y_1 - \cdots - n_s y_s = 0$$

is called the primitive relation (associated to P*).*

Using the results of [9] and [14] Batyrev proved in [1] that for any smooth, complete n dimensional fan with $n+3$ generators its set of ray generators can be partitioned into l non-empty sets $X_0,\dots,X_{l-1}$ in such a way that the primitive collections are exactly sums of $p+1$ consecutive sets X_i (we use a circular numeration, that is we assume that $i \in \mathbb{Z}/l\mathbb{Z}$), where $l = 2p+3$. Moreover l is equal to 3 or 5. The number l is of course the number of primitive collections. In the case $l = 3$ the fan Σ is a splitting fan (that is any two primitive collections are disjoint). These varieties are well characterized, and we know much about full strongly exceptional collections of line bundles on them. The case of five primitive collections is much more complicated and is our object of study. For $l = 5$ we have the following result of Batyrev [1, Theorem 6.6].

Theorem 3.3. *Let $Y_i = X_i \cup X_{i+1}$, where $i \in \mathbb{Z}/5\mathbb{Z}$,*

$$X_0 = \{v_1,\dots,v_{p_0}\}, \quad X_1 = \{y_1,\dots,y_{p_1}\}, \quad X_2 = \{z_1,\dots,z_{p_2}\},$$

$$X_3 = \{t_1,\dots,t_{p_3}\}, \quad X_4 = \{u_1,\dots,u_{p_4}\},$$

where $p_0+p_1+p_2+p_3+p_4 = n+3$. Then any n-dimensional fan Σ with the set of generators $\bigcup X_i$ and five primitive collections Y_i can be described up to a symmetry of the pentagon by the following primitive relations with nonnegative integral coefficients $c_2,\dots,c_{p_2},b_1,\dots,b_{p_3}$:

$$v_1+\dots+v_{p_0}+y_1+\dots+y_{p_1}-c_2z_2-\dots-c_{p_2}z_{p_2}$$

$$-(b_1+1)t_1-\dots-(b_{p_3}+1)t_{p_3}=0,$$

$$y_1+\dots+y_{p_1}+z_1+\dots+z_{p_2}-u_1-\dots-u_{p_4}=0,$$

$$z_1+\dots+z_{p_2}+t_1+\dots+t_{p_3}=0,$$

$$t_1+\dots+t_{p_3}+u_1+\dots+u_{p_4}-y_1-\dots-y_{p_1}=0,$$

$$u_1+\dots+u_{p_4}+v_1+\dots+v_{p_0}-c_2z_2-\dots-c_{p_2}z_{p_2}-b_1t_1-\dots-b_{p_3}t_{p_3}=0.$$

In our case we will be interested in varieties X with Picard number three that have the following sets X_i:

$$X_0 = \{v_1,\dots,v_{n-1}\}, X_1 = \{y\}, X_2 = \{z\}, X_3 = \{t\}, X_4 = \{u\} \tag{3.1}$$

So, from now on let us denote by X a smooth toric variety with Picard number 3 and primitive collections $X_0 \cup X_1$, $X_1 \cup X_2,\dots,X_4 \cup X_0$. We see that the cone $< v_1,\dots,v_{n-1},z >$ is in our fan, so as the variety is smooth, these ray generators form a basis $(e_1,\dots,e_n)$ of a lattice. In this basis, using the above primitive relations we see that all the considered ray generators are of the following form:

$$
\begin{aligned}
v_1 &= e_1, \dots, v_{n-1} = e_{n-1}, \\
t &= -e_n, \\
z &= e_n, \\
u &= -e_1 - \dots - e_{n-1} - be_n, \\
y &= -e_1 - \dots - e_{n-1} - (b+1)e_n.
\end{aligned} \tag{3.2}
$$

One can see that for any fixed dimension n we obtain an infinite number of smooth toric varieties parameterized by $b = b_1 \geq 0$, but only a finite number of them is Fano, namely for $b < n-1$ (because the sum of coefficients in each primitive relation has to be positive).

We need following basic facts about divisors on toric varieties. To each ray generator $r \in R$ we can associate a divisor D_r [7]. The relations (in the Picard group) among the divisors are given by the following equations:

$$\sum_{r \in R} e_i^*(r) D_r = 0,$$

what is

$$D_{v_1} - D_u - D_y = 0, \dots, D_{v_{n-1}} - D_u - D_y = 0,$$

$$D_z - D_t - bD_u - (b+1)D_y = 0.$$

Lemma 3.4. *The above linear relations imply that* $Pic(X) \cong \mathbb{Z}^3 = < D_v, D_y, D_t >$, *i.e. each divisor can be uniquely written in a form:*

$$eD_v + fD_y + gD_t,$$

where D_v is any fixed D_{v_i} (they are all linearly equivalent).

Proof. From the relations above it is obvious that each divisor is linearly equivalent to a divisor of this form. Let us assume that it has two such presentations. It means that they have to be linearly dependent:

$$eD_v + fD_y + gD_t = e'D_v + f'D_y + g'D_t + i(D_v - D_u - D_y)$$

$$+ j(D_z - D_t - bD_u - (b+1)D_y).$$

Since D_z occurs only on the right hand side in the above equality we have $j = 0$. Once $j = 0$ the divisor D_u occurs only on the right hand side so $i = 0$ and we get uniqueness. $\square$

4. Main theorem

In this section we prove that for smooth toric projective varieties X with Picard number 3 with sets of generators

$$X_0 = \{v_1, \dots, v_{n-1}\}, \quad X_1 = \{y\}, \quad X_2 = \{z\}, \quad X_3 = \{t\}, \quad X_4 = \{u\},$$

in the situation described in the previous section there exists a full strongly exceptional collection of line bundles in the derived category.

We proceed in several steps. First by pushing forward a trivial line bundle by a Frobenius morphism we obtain a vector bundle that splits into the direct sum of line bundles which by Bondal's result [2] generate $D^b(X)$. We can calculate the set C of these line bundles explicitly using the algorithm described in [15]. Then, we choose an ordered subset $C' \subset C$ and we prove that C' is strongly exceptional. Finally using Koszul complexes we prove that C and C' generate the same category, hence C' is also full.

4.1. Full collection

We fix a prime integer $p >> 0$. Let $F : X \to X$ be the p-th Frobenius morphism of our toric variety X, that is an extension of a morphism:

$$\begin{aligned} F: \quad & T \to T, \\ & t \to t^p \end{aligned}$$

where T is the torus of X. Using the results of [15] we can calculate the split of the push forward $F_*(\mathscr{O}_X)$. We will use similar notation as used in [15]. Let us recall the algorithm. We fix a basis of N. Let

$$R = k[(X^{e_1^*})^{\pm 1}, \dots, (X^{\hat{e}_n^*})^{\pm 1}],$$

be the coordinate ring of the torus T. To each cone $\chi_i \subset N$ of maximal dimension we associate a matrix A_i whose rows are ray generators in the chosen basis. Let $B_i = A_i^{-1}$ and $C_{ij} = B_j^{-1} B_i$. Let $w_{ij} = (w_{ij}^1, \dots, w_{ij}^n)$ be the j-th column of the matrix B_i. To each maximal cone $\chi_i \subset N$ one can also associate an open affine subvariety U_{χ_i} with the coordinate ring

$$R_i = k[X_{i1}, \dots, X_{in}] \subset R,$$

where we use the notation $X_{ij} = X^{w_{ij}} = (X^{e_1^*})^{w_{ij}^1} \dots (X^{e_n^*})^{w_{ij}^n}$. If we consider two cones $\chi_i, \chi_j \subset N$ then $\chi_i \cap \chi_j$ is a face of χ_i. Using [7, Proposition 2, p. 13] we see that there is a monomial M_{ij} such that $(R_i)_{M_{ij}}$ is the coordinate ring of $\chi_i \cap \chi_j$. Let

$$I_{ij} = \{g = (g_1, \dots, g_n) : X_i^g \text{ is a unit in } (R_i)_{M_{ij}}\}.$$

Let us also define the set

$$P_p = \{(g_1, \ldots, g_n) : 0 \le g_i < p)\}.$$

For $w \in I_{ji}$ one can define maps:

$$h^w_{ijp} : P_p \to I_{ji},$$

$$r^w_{ijp} : P_p \to P_p$$

determined for $g \in P_p$ by the equality

$$C_{ij} g + w = p h^w_{ijp}(g) + r^w_{ijp}(g). \tag{4.1}$$

Let us fix a Cartier divisor $D = \{(U_{\chi_i}, X_i^{u_i})\}$ and a line bundle $L \cong \mathscr{O}(D)$. From [15, Lemma 4] one gets, for each $g \in P_p$ and each cone χ_l a T-Cartier divisor $D_g = \{(U_{\chi_i}, X_i^{g_i})\}$, where $g_i = h^{u_{li}}_{lip}(g)$, that is independent from the choice of l. Moreover by [15, Theorem 1] taking all $g \in P_p$ one gets line bundles that form a split of the push forward by the Frobenius morphism $F_*(L)$. In our case the algorithm simplifies.

Let us consider three matrices:

$$A_0 = Id_n, A_1 = \begin{bmatrix} & & & 0 \\ & Id_{n-1} & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} & & & 0 & 0 \\ & Id_{n-2} & & \vdots & \vdots \\ & & & 0 & 0 \\ -1 & \ldots & -1 & -1 & -b \\ -1 & \ldots & -1 & -1 & -b-1 \end{bmatrix}.$$

The matrices above correspond to following cones:

$$\sigma_0 =< v_1, \ldots, v_{n-1}, z >, \quad \sigma_1 =< v_1, \ldots, v_{n-1}, t >, \quad \sigma_2 =< v_1, \ldots, v_{n-2}, u, y > .$$

From 4.1 we get:

$$C_{0j} = A_j A_0^{-1} = A_j.$$

Let $g \in P_p$, as we are pushing forward trivial line bundle we want to calculate h^0_{0jp} and r^0_{0jp} that satisfy:

$$A_j g = p h^0_{0jp}(g) + r^0_{0jp}(g),$$

where $r^0_{0jm} \in P_p$. As $A_0 g = p \cdot 0 + g$, we see that D_g as a Cartier divisor on X is given by 1 on U_{σ_0}. On U_{σ_1} the divisor is given by:

$$\begin{cases} 1 & \text{if } g_n = 0 \\ X^{-w_{in}} & \text{if } g_n \neq 0, \end{cases}$$

and on U_{σ_2} by:

$$\begin{cases} X^{-sw_{j(n-1)}} X^{-sw_{jn}} & \text{if } g_n = 0 \\ X^{-sw_{j(n-1)}} X^{-sw_{jn}} \text{ or } X^{-sw_{j(n-1)}} X^{-(s+1)w_{jn}} & \text{if } g_n \neq 0, \end{cases}$$

where $s = g_1 + \cdots + g_{n-1} + bg_n$. We see that for $p >> 0$, $F_*(\mathscr{O}_X)$ splits into the direct sum of line bundles, all of which belong to one of the following three subsets:

$$B_1 = \{\mathscr{O}(-qD_u - (q+1)D_y - D_t) : q = 0, \ldots, n-1+b\}$$

$$B_2 = \{\mathscr{O}(-qD_u - qD_y - D_t) : q = 1, \ldots, n-1+b\}$$

$$B_3 = \{\mathscr{O}(-qD_u - qD_y) : q = 0, \ldots, n-1\}.$$

Proposition 4.1. *With the above notation the line bundles from the set $B_1 \cup B_2 \cup B_3$ generate the derived category $D^b(X)$.*

Proof. This is a direct consequence of Bondal's result from [2], that the split of the push forward of a trivial bundle by the p-th Frobenius morphism generates the derived category $D^b(X)$ for p sufficiently large. □

4.2. Forbidden subsets

In this subsection we want to characterize acyclic line bundles on X i.e. line bundles whose higher cohomologies vanishes. We will use this characterization to check if $Ext^i(L,M) = H^i(L^\vee \otimes M) = 0$ for $i > 0$.

Let Σ be an arbitrary fan in $N = \mathbb{Z}^n$ with the set of ray generators $x_1, ..., x_m$ and $\mathbb{P}_\Sigma$ be the toric variety associated to the fan Σ. For $I \subset \{1, \ldots, m\}$ let C_I be the simplicial complex generated by sets $J \subset I$ such that $\{x_i : i \in J\}$ generates the cone in Σ and for $r = (r_i : i = 1, \ldots, m)$. Let us define $Supp(r) := C_{\{i:\, r_i \geq 0\}}$.

From the result of Borisov and Hua [4] we have the following:

Proposition 4.2. *The cohomology $H^j(\mathbb{P}_\Sigma, L)$ is isomorphic to the direct sum over all $r = (r_i : i = 1, \ldots, m)$ such that $\mathscr{O}(\sum_{i=1}^m r_i D_{x_i}) \cong L$ of the $(n-j)$-th reduced homology of the simplicial complex $Supp(r)$.*

Definition 4.3. *A line bundle L on $\mathbb{P}_\Sigma$ is called acyclic if $H^i(\mathbb{P}_\Sigma, L) = 0$ for $i \geq 1$.*

Definition 4.4. *A proper subset I of $\{1,\dots,m\}$ is called a forbidden set if the simplicial complex C_I has nontrivial reduced homology.*

From Proposition 4.2 we have the following characterization of acyclic line bundles

Proposition 4.5. *A line bundle L on $\mathbb{P}_\Sigma$ is acyclic if it is not isomorphic to none of the following line bundles*

$$\mathscr{O}(\sum_{i\in I} r_i D_{x_i} - \sum_{i\notin I}(1+r_i)D_{x_i})$$

where $r_i \geq 0$ and I is a proper forbidden subset of $\{1,\dots,m\}$.

Hence to determine which bundles on $\mathbb{P}_\Sigma$ are acyclic it is enough to know which sets I are forbidden.

In case of simplicial complex C_I on the set of vertices I we also define a primitive collection as a minimal subset of vertices that do not form a simplex. A complex C_I is determined by its primitive collections, namely it contains simplices (subsets of I) that contain none of primitive collections.

In case of our variety (described at the beginning of this section) we have $C_I = \{J \subset I : \widehat{Y_i} := \{j : x_j \in Y_i\} \not\subseteq J$ for $i = 1,\dots,5\}$, since Y_i are primitive collections. So sets $\widehat{Y_i}$ are primitive collections in the simplicial complex. The only difference between sets $\widehat{Y_i}$ and Y_i is that the first one is the set of indices of rays in the second one, so in fact they could be even identified. For our convenience we also define sets $\widehat{X_i} := \widehat{Y_i} \cap \widehat{Y_{i-1}}$ which are similarly sets of indices of sets X_i.

Lemma 4.6. *A primitive collection is a forbidden subset.*

Proof. Let I be a primitive collection with k elements. The chain complex of C_I is as follows

$$0 \to \mathbb{C}^{\binom{k}{k-1}} \to \mathbb{C}^{\binom{k}{k-2}} \to \dots \to \mathbb{C}^{\binom{k}{2}} \to \mathbb{C}^{\binom{k}{1}} \to \mathbb{C} \to 0,$$

which is not exact because the Euler characteristic is nonzero. □

Lemma 4.7. *A sum of two consecutive primitive collections is a forbidden subset.*

Proof. Let $I = \widehat{Y_i} \cup \widehat{Y_{i+1}} = \widehat{X_i} \cup \widehat{X_{i+1}} \cup \widehat{X_{i+2}}$, $|\widehat{X_i}| = k_1$, $|\widehat{X_{i+1}}| = k_2$, $|\widehat{X_{i+2}}| = k_3$ and $|I| = k$. Then chain complex of C_I is as follows

$$0 \to \mathbb{C}^{\binom{k}{k-1}-\binom{k_1}{k-1-k_2-k_3}-\binom{k_3}{k-1-k_1-k_2}} \to \dots \to \mathbb{C}^{\binom{k}{t}-\binom{k_1}{t-k_2-k_3}-\binom{k_3}{t-k_1-k_2}} \to \dots$$
$$\to \mathbb{C} \to 0$$

which is not exact because the Euler characteristic is nonzero. □

Lemma 4.8. *If a nonempty subset I is not a sum of primitive collections, then it is not forbidden.*

Proof. The simplicial homology of a simplicial complex is equal to the singular homology of this complex considered as a topological space (each simplex D, which is a d element set, can be changed into the convex hull of d linearly independent vectors in $\mathbb{R}^n$ that correspond to elements of this set). To avoid confusion with scalars let us name elements of $\{1,\dots,m\} \supset I$ as $\{x_1,\dots,x_m\}$. The above names are not by accident the same as rays of a fan, because this complex as a topological space can be realized by sum of convex hulls of sets of rays that form a cone in Σ and whose indices are contained in I.

There exists $a \in I$ such that a does not belong to any primitive collection which is contained in I. We can define a homotopy

$$H : [0,1] \times C_I \to C_I$$

which for $x = \alpha_1 x_1 + \dots + \alpha_m x_m$ ($\alpha_i \geq 0$, $\sum \alpha_i = 1$) gives

$$H(t,x) := t\alpha_1 x_1 + \dots + (1 - t + t\alpha_a)x_a + \dots + t\alpha_m x_m.$$

Of course $x \in C_I$ means that $S_x := \{i : \alpha_i > 0\} \subset I$ and $Y_i \nsubseteq S_x$, but then $S_x \cup \{a\}$ also satisfies this conditions, so $H(t,x) \in C_I$ and H is well defined. It is easy to observe that H is continuous $H(0,\cdot) = x_a$ and $H(1,\cdot) = id_{C_I}$. The complex C_I is homotopic to a point, so it has trivial reduced homologies. □

Lemma 4.9. *A sum of three consecutive primitive collections is not a forbidden subset.*

Proof. At the beginning of this proof we should give the same remark as in the proof of lemma 4.8. We have $I = \widehat{Y_i} \cup \widehat{Y_{i+1}} \cup \widehat{Y_{i+2}} = \widehat{X_i} \cup \widehat{X_{i+1}} \cup \widehat{X_{i+2}} \cup \widehat{X_{i+3}}$, so in our situation at least one of the sets $\widehat{X_{i+1}}$, $\widehat{X_{i+2}}$ has only one element. Without loss of generality we can assume that $\widehat{X_{i+2}} = \{x_c\}$ and also that $\widehat{X_i} = \{x_{a_1},\dots,x_{a_A}\}$, $\widehat{X_{i+1}} = \{x_{b_1},\dots,x_{b_B}\}$, $\widehat{X_{i+3}} = \{x_{d_1},\dots,x_{d_D}\}$. Let us define the homotopy

$$H : [0,1] \times C_I \to C_I$$

which for $x = \alpha_{a_1} x_{a_1} + \dots + \alpha_{b_1} x_{b_1} + \dots + \alpha_c x_c + \alpha_{d_1} x_{d_1} + \dots + \alpha_{d_D} x_{d_D}$ gives

$$H(t,x) := x + t\alpha_c x_{a_1} - t\alpha_c x_c.$$

If $\alpha_c = 0$ then $H(t,x) = x$. If $\alpha_c \neq 0$ then $S_{H(x,t)} \subset S_x \cup \{a_1\}$, but this set is also in our symplicial complex C_I, if contrary $a_2,\dots,a_A,b_1,\dots,b_B$ are in S_x so $\{b_1,\dots,b_B,c\} = Y_{i+1} \subset S_x$ a contradiction. So the homotopy H is well defined.

It is easy to observe that H is continuous, $H(0,\cdot) = id_{C_I}$ and $H(1,C_I)$ is a symplicial complex on vertices $x_{a_1},...,x_{b_B},x_{d_1},...,x_{d_D}$ with only one primitive collection $\{x_{a_1},...,x_{b_B}\}$. Hence in the same way as in Lemma 4.8 $H(1,C_I)$ can be contracted to a point x_{d_1}. This shows that C_I is homotopic to a point, so it has trivial reduced homologies. □

The above Lemmas match together to the following

Theorem 4.10. *The only forbidden subsets are primitive collections, their complements (these are exactly sums of two consecutive primitive collections) and the empty set.*

This gives us that in our situation

Corollary 4.11. *With the above notation a line bundle L is acyclic if and only if it is not isomorphic to any of the following line bundles*

$$\mathscr{O}(\alpha_1 D_v + \alpha_2 D_y + \alpha_3 D_z + \alpha_4 D_t + \alpha_5 D_u)$$

where exactly $2,3$ *or* 5 *consecutive* α *are negative and if* $\alpha_1 < 0$ *then* $\alpha_1 \leq -(n-1)$.

Proof. Since all D_{v_i} are linearly equivalent we match them together and as a consequence α_1 is the sum of all the coefficients of D_{v_i}. □

4.3. Strongly exceptional collection

We are looking for a full strongly exceptional collection. From the general theory we know that if it exists then its length should be equal to the rank of the Grothendieck group $K_0(X)$. In case of a smooth toric varieties the rank of this group is equal to the number of maximal cons in the fan.

In our case the maximal cones are n dimensional subsets of the set of all ray generators, except those subsets that contain a primitive collection. We want to calculate how many such subsets there are. First let us notice that at most 2 elements of such subset can be contained in $X_1 \cup X_2 \cup X_3 \cup X_4$, because otherwise it would contain a primitive collection. This means that we have got only two possibilities:

1) Exactly two elements of our subset are in this set. There are $\binom{n-1}{n-2} \cdot (\binom{4}{2} - 3) = 3(n-1)$ such subsets.

2) There is only one element of our subset that is in $X_1 \cup X_2 \cup X_3 \cup X_4$. We have got only two such subsets: $X_0 \cup X_2$ and $X_0 \cup X_3$.

All together we see that in our case there are $3n-1$ maximal cones. Let us choose the following ordered sequence of $3n-1$ line bundles from $B_1 \cup B_2 \cup B_3$:

$$\begin{aligned}
&\mathscr{O}(-(n-1+b)D_v - D_y - D_t), \mathscr{O}(-(n-1+b)D_v - D_t),\\
&\mathscr{O}(-(n-2+b)D_v - D_y - D_t), \mathscr{O}(-(n-2+b)D_v - D_t), \ldots,\\
&\mathscr{O}(-(b+1)D_v - D_t), \mathscr{O}(-bD_v - D_y - D_t),\\
&\mathscr{O}(-(n-1)D_v), \mathscr{O}(-(n-2)D_v), \ldots, \mathscr{O}.
\end{aligned} \tag{4.2}$$

We want to prove that this is a strongly exceptional collection. We know that for any line bundles L and M on X:

$$Ext^i(L,M) = H^i(L^\vee \otimes M).$$

First we want to prove that for any L and M in (4.2) $L^\vee \otimes M$ is acyclic. Let us write down line bundles of the form $L^\vee \otimes M$ where L and M are taken from (4.2).

$$Diff = \begin{cases}
(1) \quad \mathscr{O}(sD_v) & s = -(n-1), \ldots, n-1\\
(2) \quad \mathscr{O}(sD_v + D_t) & s = b+2-n, \ldots, n-1+b\\
(3) \quad \mathscr{O}(sD_v - D_t) & s = -(n-1+b), \ldots, n-b-2\\
(4) \quad \mathscr{O}(sD_v + D_y) & s = -(n-1), \ldots, n-2\\
(5) \quad \mathscr{O}(sD_v - D_y) & s = -(n-2), \ldots, n-1\\
(6) \quad \mathscr{O}(sD_v + D_y + D_t) & s = b-(n-1), \ldots, b+n-1\\
(7) \quad \mathscr{O}(sD_v - D_y - D_t) & s = -(b+n-1), \ldots, n-1-b
\end{cases}$$

From Corollary 4.11 we know that they are acyclic if they are not of the form

$$\mathscr{O}(\alpha_1 D_v + \alpha_2 D_y + \alpha_3 D_z + \alpha_4 D_t + \alpha_5 D_u) \cong$$

$$\cong \mathscr{O}((\alpha_1 + \alpha_5 + \alpha_3 b)D_v + (\alpha_2 + \alpha_3 - \alpha_5)D_y + (\alpha_3 + \alpha_4)D_t)$$

where exactly $2,3$ or 5 consecutive α are negative and if $\alpha_1 < 0$ then $\alpha_1 \leq -(n-1)$.

We will show that all line bundles of $Diff$ are not of this form. First let us observe that they are not of this form for all α negative since then the coefficient of D_t is less than or equal to -2. Let us suppose that they are of this form with exactly 2 or 3 consecutive α negative.

(1) The coefficient of D_y is 0 therefore $\alpha_2 + \alpha_3 = \alpha_5$. But α_2, α_3 and α_5 cannot have the same sign (we treat 0 as positive) so α_2 and α_3 have different signs. This means that α_3 and α_4 have the same sign, and as $\alpha_3 + \alpha_4 = 0$, they

both have to be equal to zero. So α_2 and as a consequence α_1 are negative hence the coefficient of D_v is less than or equal to $-n$, which is a contradiction.

(2) The coefficient of D_y is 0 so as before α_2 and α_3 have different signs. This means that α_3 and α_4 are of the same sign. We know that $\alpha_3 + \alpha_4 = 1$, so they both have to be positive and at most one equal to one. So α_2 and as a consequence α_1 is negative hence the coefficient of D_v is less than or equal to $-(n-1)+b$, which is a contradiction.

(3) As before α_2 and α_3 have different signs. α_3 cannot be positive since then α_4 and as a consequence coefficient of D_t would also be positive. So α_3 and as a consequence α_4 is negative hence the coefficient of D_t is less than or equal to -2, which is a contradiction.

(4) The coefficient of D_y is 1 therefore $\alpha_2 + \alpha_3 = \alpha_5 + 1$. But α_2, α_3 and α_5 cannot have the same sign, so α_2 and α_3 have different signs or $\alpha_2 = \alpha_3 = 0$ and $\alpha_5 = -1$.

First case: α_2 and α_3 have different signs. This means that α_3 and α_4 have the same sign, and as $\alpha_3 + \alpha_4 = 0$ we see that $\alpha_3 = \alpha_4 = 0$. So α_2 and as a consequence α_1 is negative hence the coefficient of D_v is less than or equal to $-n-1$, which is a contradiction.

Second case: $\alpha_2 = \alpha_3 = 0$ and $\alpha_5 = -1$. We have also $\alpha_4 = 0$ so α_1 and α_5 are negative hence the coefficient of D_v is less than or equal to $-n$, which is a contradiction.

(5) The coefficient of D_y is -1 therefore $\alpha_2 + \alpha_3 = \alpha_5 - 1$. But α_2, α_3 and α_5 cannot have the same sign so α_2 and α_3 have different signs. This means that α_3 and α_4 have the same sign, and as $\alpha_3 + \alpha_4 = 0$ we see that $\alpha_3 = \alpha_4 = 0$. So α_2 and as a consequence α_1 is negative hence the coefficient of D_v is less than or equal to $-(n-1)$, which is a contradiction.

(6) The coefficient of D_y is 1 therefore $\alpha_2 + \alpha_3 = \alpha_5 + 1$. But α_2, α_3 and α_5 cannot have the same sign, so α_2 and α_3 have different signs or $\alpha_2 = \alpha_3 = 0$ and $\alpha_5 = -1$.

First case: Assume that α_2 and α_3 have different signs. In this case α_3 and α_4 are of the same sign and as $\alpha_3 + \alpha_4 = 1$ they have to be positive and at most one. So α_2 and as a consequence α_1 is negative hence the coefficient of D_v is less than or equal to $-n+b$, which is a contradiction.

Second case: Assume $\alpha_2 = \alpha_3 = 0$. We have $\alpha_4 = 1$ so α_1 and α_5 are negative hence the coefficient of D_v is less than or equal to $-n$, which is a contradiction.

(7) The coefficient of D_y is -1 therefore $\alpha_2+\alpha_3=\alpha_5-1$. But α_2,α_3 and α_5 cannot have the same sign so α_2 and α_3 have different signs. α_3 cannot be positive since then α_4 and as a consequence the coefficient of D_t would also be positive. So α_3 and as a consequence α_4 is negative. Hence the coefficient of D_t is less than or equal to -2, which is a contradiction.

To have a strongly exceptional collection it remains to prove that any pair of two line bundles L_i and L_j for $i<j$ from our ordered sequence satisfies $0=Ext^0(L_j,L_i)=H^0(L_j^\vee\otimes L_i)$. This is equivalent to showing that $L_j^\vee\otimes L_i$ has no global sections so from [4] it is not of the form

$$(*)\quad \mathscr{O}(\alpha_1D_v+\alpha_2D_y+\alpha_3D_z+\alpha_4D_t+\alpha_5D_u)\cong$$

$$\cong\mathscr{O}((\alpha_1+\alpha_5+\alpha_3b)D_v+(\alpha_2+\alpha_3-\alpha_5)D_y+(\alpha_3+\alpha_4)D_t)$$

with all α_i nonnegative. Let us partition our ordered collection into two collections:

$$Col_1=(\mathscr{O}(-(n-1+b)D_v-D_y-D_t),\mathscr{O}(-(n-1+b)D_v-D_t),$$
$$\mathscr{O}(-(n-2+b)D_v-D_y-D_t),\mathscr{O}(-(n-2+b)D_v-D_t),$$
$$\ldots,\mathscr{O}(-(b+1)D_v-D_t),\mathscr{O}(-bD_v-D_y-D_t))$$

and

$$Col_2=(\mathscr{O}(-(n-1)D_v),\mathscr{O}(-(n-2)D_v),\ldots,\mathscr{O}).$$

If we take a difference of an element from Col_1 and Col_2, then the coefficient of D_t is negative so the difference is not of the form (*). If we take the difference of two elements from Col_2 then the coefficient of D_v is negative, hence it is not of the form (*). If we take the difference of two elements from Col_1, then either the coefficients of D_v is negative or the difference is equal to $-D_y$. The divisor $-D_y$ is not of the form (*), because α_5 would have to be strictly positive, hence the coefficient of D_v would not be zero.

We have proven:

Proposition 4.12. *With the above notation the following ordered sequence of line bundles in X*

$$\mathscr{O}(-(n-1+b)D_v-D_y-D_t),\mathscr{O}(-(n-1+b)D_v-D_t),\ldots$$

$$\ldots,\mathscr{O}(-(b+1)D_v-D_t),\mathscr{O}(-bD_v-D_y-D_t),\mathscr{O}(-(n-1)D_v),\ldots,\mathscr{O}$$

is a strongly exceptional collection.

□

4.4. Generating a derived category

Finally, we will prove that the strongly exceptional collection given in Proposition 4.12 is also full. As already mentioned it is enough to prove that it generates all line bundles of the set $B_1 \cup B_2 \cup B_3$. In order to show that we need following two lemmas.

Lemma 4.13. *Let k be any integer. Line bundles $\mathcal{O}(-kD_v - D_y - D_t), \dots, \mathcal{O}(-(n-1+k)D_v - D_y - D_t), \mathcal{O}(-(k+1)D_v - D_t), \dots, \mathcal{O}(-(n-1+k)D_v - D_t)$ generate $\mathcal{O}(-kD_v - D_t)$ in the derived category.*

Proof. We consider the Koszul complex for $\mathcal{O}(D_y), \mathcal{O}(D_{v_1}), \dots, \mathcal{O}(D_{v_{n-1}})$:

$$0 \to \mathcal{O}(-(n-1)D_v - D_y) \to \cdots \to \mathcal{O}(-D_v)^{n-1} \oplus \mathcal{O}(-D_y) \to \mathcal{O} \to 0.$$

By tensoring it with $\mathcal{O}(-kD_v - D_t)$ we obtain:

$$0 \to \mathcal{O}(-(n-1+k)D_v - D_y - D_t) \to \cdots \to$$

$$\to \mathcal{O}(-(1+k)D_v - D_t)^{n-1} \oplus \mathcal{O}(-kD_v - D_y - D_t) \to \mathcal{O}(-kD_v - D_t) \to 0.$$

All sheaves that appear in this exact sequence, apart from the last one, are exactly $\mathcal{O}(-kD_v - D_y - D_t), \dots, \mathcal{O}(-(n-1+k)D_v - D_y - D_t), \mathcal{O}(-(k+1)D_v - D_t), \dots, \mathcal{O}(-(n-1+k)D_v - D_t)$, so indeed we can generate $\mathcal{O}(-kD_v - D_t)$. $\square$

Lemma 4.14. *Let k be any integer. Line bundles $\mathcal{O}(-(k+1)D_v - D_y - D_t), \dots, \mathcal{O}(-(n-1+k)D_v - D_y - D_t), \mathcal{O}(-(k+1)D_v - D_t), \dots, \mathcal{O}(-(n+k)D_v - D_t)$ generate $\mathcal{O}(-kD_v - D_y - D_t)$ in the derived category.*

Proof. The proof is similar to the last one. We have to consider the Koszul complex for line bundles $\mathcal{O}(D_u), \mathcal{O}(D_{v_1}), \dots, \mathcal{O}(D_{v_{n-1}})$:

$$0 \to \mathcal{O}(-(n-1)D_v - D_u) \to \cdots \to \mathcal{O}(-D_v)^{n-1} \oplus \mathcal{O}(-D_u) \to \mathcal{O} \to 0$$

we dualize it and we tensor with $\mathcal{O}(-(n+k)D_v - D_t)$. $\square$

Summarizing, we have proved:

Theorem 4.15. *Let X be a smooth, complete, n dimensional toric variety with Picard number three, ray generators $X_0 \cup \cdots \cup X_4$, where*

$$X_0 = \{v_1, \dots, v_{n-1}\}, \quad X_1 = \{y\}, \quad X_2 = \{z\}, \quad X_3 = \{t\}, \quad X_4 = \{u\},$$

primitive collections $X_0 \cup X_1$, $X_1 \cup X_2, \dots, X_4 \cup X_0$ and primitive relations:

$$v_1 + \cdots + v_{n-1} + y - (b+1)t = 0,$$

$$y+z-u=0,$$

$$z+t=0,$$

$$t+u-y=0,$$

$$u+v_1+\cdots+v_{n-1}-bt=0,$$

where b is a positive integer.

Then the collection

$$\mathscr{O}(-(n-1+b)D_v-D_y-D_t),\mathscr{O}(-(n-1+b)D_v-D_t),\ldots$$

$$\ldots,\mathscr{O}(-(b+1)D_v-D_t),\mathscr{O}(-bD_v-D_y-D_t),\mathscr{O}(-(n-1)D_v),\ldots,\mathscr{O}$$

is a full strongly exceptional collection of line bundles.

Proof. We already know that this is a strongly exceptional collection from Proposition 4.12. Inductively using lemmas 4.13 and 4.14 we can prove that it generates sets B_1 and B_2. The set B_3 is already in our collection, hence our collection is also full. □

Acknowledgements

We would like to thank very much L. Costa and R.M. Miró-Roig for introducing us to the subject. We are also grateful to them for sharing their many useful ideas and meticulous reading of a preliminary draft of this paper.

REFERENCES

[1] V. V. Batyrev, *On the classification of smooth projective toric varieties*, Tôhoku Math. J. 43 (1991), 569–585.

[2] A. I. Bondal, *Derived categories of toric varieties*, Oberwolfach Report 5/2006, 284–286.

[3] A. I. Bondal, *Representation of associative algebras and coherent sheaves*, Math. USSR Izvestiya 34 (1) (1990), 23–42.

[4] L. Borisov - Z. Hua, *On the conjecture of King for smooth toric Deligne-Mumford stacks*, Adv. Math. 221 (2009), 277–301.

[5] L. Costa - R. M. Miró-Roig, *Derived category of toric varieties with small Picard number*, Preprint.

[6] L. Costa - R. M. Miró-Roig, *Tilting sheaves on toric varieties*, Math. Z. 248 (4) (2004), 849–865.

[7] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies, Princeton Univeristy Press 1993.

[8] A. L. Gorodentsev - A. N. Rudakov, *Exceptional vector bundles on projective spaces*, Duke Math. J. 54 (1) (1987), 115–130.

[9] B. Gronbaum, *Convex polytopes*, John Wiley and Sons, London-New York-Sidney, 1967.

[10] L. Hille - M. Perling, *A counterexample to King's conjecture*, Compos. Math. 142 (2006), 1507–1521.

[11] Y. Kawamata, *Derived categories of toric varieties*, Michigan Math. J. 54 (3) (2006), 517–535.

[12] A. King, *Titling bundles on some rational surfaces*, Preprint at http://www.maths.bath.ac.uk/~masadk/papers/.

[13] P. Kleinschmidt, *A classification of toric varieties with few generators*, Aequationes Math. 35 (1988), 254–266.

[14] T. Oda - H. S. Park, *Linear Gale transforms and Gelfand-Kapranov-Zelevinskij decompositions*, Tôhoku Math. J. 43 (1991), 375–399.

[15] J. F. Thomsen, *Frobenius Direct Images of Line Bundles on Toric Varieties*, J. Algebra 226 (2000), 865–874.

ARIJIT DEY
Max-Planck Institut fur Mathematik
Vivatsgasse 7, 53111 Bonn, Deutchland
e-mail: `arijit@mpim-bonn.mpg.de`

MICHAŁ LASOŃ
Mathematical Institute of the Polish Academy of Sciences
Św. Tomasza 30, 31-027 Kraków, Poland
Faculty of Mathematics and Computer Science
Jagiellonian University, 30-387 Kraków, Poland
e-mail: `mlason@op.pl`

MATEUSZ MICHAŁEK
Mathematical Institute of the Polish Academy of Sciences
Św. Tomasza 30, 31-027 Kraków, Poland
Institut Fourier, Universite Joseph Fourier
100 rue des Maths, BP 74, 38402 St Martin d'He'res, France
e-mail: `wajcha2@poczta.onet.pl`