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GOTZMANN LEXSEGMENT IDEALS

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In this paper we characterize the componentwise lexsegment ideals which are componentwise linear and the lexsegment ideals generated in one degree which are Gotzmann.

1. Introduction

Let k be a field and $S = k[x_1, \ldots, x_n]$ the ring of polynomials in n variables. We consider S to be standard graded, that is $\deg(x_i) = 1$ for all i and endowed with the graded lexicographical order with respect to $x_1 > \ldots > x_n$. Namely, if $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$ are two monomials in S, we have $u >_{lex} v$ if $\deg(u) > \deg(v)$ or $\deg(u) = \deg(v)$ and there exists $1 \leq s \leq n$ such that $a_i = b_i$ for all $i \leq s-1$ and $a_s > b_s$. We denote by $\mathfrak{m}$ the maximal graded ideal of S. Let $I \subset S$ be a graded ideal, $I = \bigoplus_{q \geq 0} I_q$. We denote by $H(I, -)$ its Hilbert function, that is $H(I, q) = \dim_k(I_q)$ for all $q \geq 0$, and by $I_{\langle q \rangle}$ the homogeneous ideal generated by the component of degree q of I.

In [7], J. Herzog and T. Hibi defined the componentwise linear ideals. Namely, a graded ideal I of S is called *componentwise linear* if, for each degree q, $I_{\langle q \rangle}$ has a linear resolution.

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Ideals with linear quotients are examples of componentwise linear ideals [3]. They were defined by J. Herzog and Y. Takayama in [9]. A graded ideal I of S *has linear quotients* if there exists a system of homogeneous generators $f_1,\dots,f_m$ of I such that the colon ideals $(f_1,\dots,f_{j-1}):(f_j)$ are generated by linear forms, for all $2 \le j \le m$. A. Soleyman Jahan and X. Zheng generalized the notion of ideals with linear quotients as follows: a graded ideal I *has componentwise linear quotients* if, for each degree q, $I_{\langle q\rangle}$ has linear quotients. They proved that any graded ideal with linear quotients has componentwise linear quotients [11, Theorem 2.7].

Since any ideal with linear quotients generated in one degree has a linear resolution [3], looking at the above definitions, one may note that any graded ideal with componentwise linear quotients is componentwise linear. In general, the converse does not hold.

Along the above definitions, one may consider the lexsegment ideals. We recall that a monomial ideal $I \subset S$ is called a lexsegment ideal if for each degree j, if $I_j \neq 0$, then I_j is generated by a lexsegment set of degree j, that is a set of monomials of degree j of the form

$$\mathscr{L}_j(u,v)=\{w\in\mathscr{M}_j \mid u\geq_{lex} w\geq_{lex} v\}.$$

for some monomials u,v of degree j, $u \geq_{lex} v$. We prove that, for this class of ideals, being componentwise linear is equivalent to having componentwise linear quotients. Next we consider a smaller class of lexsegment ideals that we call componentwise lexsegment ideals. For this subclass we require that, if d is the least degree of the minimal monomial generators, then, for any $j \geq d+1$, the j-degree component I_j is generated over k by the lexsegment set $\mathscr{L}_j(x_1^{j-d}u, x_n^{j-d}v)$ if $\mathscr{L}_d(u,v)$ generates I_d. In other words, each higher component I_j is generated over k by the lexsegment set of degree j determined by the first and the last monomials in the shadow of the lexsegment set which generates the previous component I_{j-1}.

For a componentwise lexsegment ideal I, we show that the property of being componentwise linear is equivalent to the condition that $I_{\langle d\rangle}$ has a linear resolution, where I_d is the first non-zero component of I.

Let d be a positive integer. Then any non-negative integer a has a unique representation of the form

$$a=\binom{a_d}{d}+\ldots+\binom{a_j}{j},$$

where $a_d > a_{d-1} > \ldots > a_j \geq j \geq 1$. This is called the *binomial* or *Macaulay expansion of a with respect to d*. For such an expansion of a with respect to d one defines

$$a^{\langle d\rangle}=\binom{a_d+1}{d+1}+\ldots+\binom{a_j+1}{j+1}.$$

It is customary to put $0^{\langle d\rangle} = 0$ for any $d > 0$.

We recall the Gotzmann's persistence theorem [6].

Theorem 1.1. *Let $I \subset S$ be a homogeneous ideal generated by elements of degree at most d. If $H(I,d+1) = H(I,d)^{\langle d\rangle}$, then $H(I,q+1) = H(I,q)^{\langle q\rangle}$ for all $q \geq d$.*

Given a graded ideal $I \subset S$, there exists a unique lexicographic ideal I^{lex} such that I and I^{lex} have the same Hilbert function. The lexicographic ideal I^{lex} is constructed as follows. For each graded component I_j of I, one consider I_j^{lex} to be the k-vector space generated by the unique initial lexsegment $\mathscr{L}_j$ such that $|\mathscr{L}_j| = \dim_k(I_j)$. Let $I^{lex} = \bigoplus_j I_j^{lex}$. It is known that I^{lex} constructed as before is indeed an ideal.

A graded ideal $I \subset S$ generated in degree d is called a *Gotzmann ideal* if the number of generators of $\mathfrak{m}I$ is the smallest possible, namely it is equal to the number of generators of $\mathfrak{m}I^{lex}$. Therefore, by Gotzmann's persistence theorem, a graded ideal $I \subset S$ generated in degree d is Gotzmann if and only if I and $(I^{lex})_{\langle d\rangle}$ have the same Hilbert function.

J. Herzog and T. Hibi generalized this notion as follows: a graded ideal I of S is *a Gotzmann ideal* if all ideals $I_{\langle j\rangle}$ are Gotzmann ideals [7].

For graded Gotzmann ideals we have the following characterization in terms of (graded) Betti numbers [7].

Theorem 1.2. *Let $I \subset S$ be a graded ideal. The following conditions are equivalent:*

(a) $\beta_{ij}(S/I) = \beta_{ij}(S/I^{lex})$ for all i, j;

(b) $\beta_{1j}(S/I) = \beta_{1j}(S/I^{lex})$ for all j;

(c) $\beta_1(S/I) = \beta_1(S/I^{lex})$;

(d) I is a Gotzmann ideal.

Let I be a Gotzmann monomial ideal generated in degree d. From the above results it follows that I^{lex} is also generated in degree d and I has a linear resolution.

We aim at characterizing the lexsegment ideals generated in one degree which are Gotzmann.

For an integer $d \geq 2$, let $\mathscr{M}_d$ be the set of all monomials of degree d in S ordered lexicographically with $x_1 > x_2 > \ldots > x_n$.

A monomial ideal generated by an initial lexsegment of degree d, $\mathscr{L}^i(v) = \{w \in \mathscr{M}_d \mid w \geq v\}$, $v \in \mathscr{M}_d$, is called an *initial lexsegment ideal*.

Initial lexsegment ideals generated in one degree are obviously Gotzmann.

Arbitrary lexsegment ideals generated in one degree which have linear resolutions have been characterized in [1]. Their characterization distinguishes between completely and non-completely lexsegment ideals. In order to characterize the Gotzmann property of a lexsegment ideal generated in one degree, we also need to distinguish between these two classes of ideals. In the last two sections of this paper, we analyze these two classes.

Our paper gives a complete solution to a problem posed by Professor J. Herzog at the School of Research PRAGMATIC 2008 in Catania, July 2008.

2. Componentwise lexsegment ideals

In [10], H. Hulett and H.M. Martin defined a more general class of lexsegment ideals as follows: a monomial ideal I of S is called a *lexsegment ideal* if whenever $u, v \in I$ are monomials of the same degree and $u \geq_{lex} m \geq_{lex} v$, then $m \in I$.

We prove that, for this class of monomial ideals, the two notions, componentwise linear and componentwise linear quotients, are equivalent.

Theorem 2.1. *Let I be a lexsegment ideal. The ideal I is componentwise linear if and only if I has componentwise linear quotients.*

Proof. Firstly, let us assume that I has componentwise linear quotients. Since any ideal with linear quotients generated in one degree has a linear resolution, the statement follows by comparing the definitions.

Conversely, let I be a lexsegment ideal which is componentwise linear and let $d \geq 1$ be the lowest degree of the minimal monomial generators of I. For each $j \geq d$, the ideal $I_{\langle j\rangle}$ is a lexsegment ideal generated in one degree with a linear resolution. Therefore, for each $j \geq d$, $I_{\langle j\rangle}$ has linear quotients, by [5, Theorem 1.1 and Theorem 2.1]. The statement follows. $\square$

We consider a smaller class of lexsegment ideals, namely componentwise lexsegment ideals, and we characterize all the componentwise lexsegment ideals which are componentwise linear.

Definition 2.2. Let I be a monomial ideal in S and d the least degree of the minimal monomial generators. The ideal I is called *componentwise lexsegment* if, for all $j \geq d$, its degree j component I_j is generated, as k-vector space, by the lexsegment set $\mathscr{L}(x_1^{j-d}u, vx_n^{j-d})$.

Obviously, completely lexsegment ideals generated in one degree are componentwise lexsegment ideals as well.

Example 2.3. The ideal $I = (x_1x_3^2, x_2^3, x_1x_2^2x_3)$ is a componentwise lexsegment ideal. Indeed, one may note that I_3 is the k-vector space spanned by $\mathscr{L}(x_1x_3^2, x_2^3)$ and I_4 is the k-vector space generated by $\mathscr{L}(x_1^2x_3^2, x_2^3x_3)$. Since $\mathscr{L}(x_1^2x_3^2, x_2^3x_3)$ is a completely lexsegment set [12, Theorem 2.3], I_j is generated by the lexsegment set $\mathscr{L}(x_1^{j-2}x_3^2, x_2^3x_3^{j-3})$ for all $j \geq 4$.

One may easily find examples of lexsegment ideals which are not componentwise lexsegment.

Example 2.4. Let $I = (x_1x_2^2, x_1x_2x_3, x_1x_3^2, x_2^3, x_2^2x_3, x_1^3x_2, x_1^3x_3)$ be a monomial ideal in $k[x_1, x_2, x_3]$, where k is a field. We have that I_3 is the k-vector space spanned by the lexsegment set $\mathscr{L}_3(x_1x_2^2, x_2^2x_3)$, I_4 is the k-vector space generated by $\mathscr{L}_4(x_1^3x_2, x_2^2x_3^2)$ and $I_{\langle 4\rangle}$ is a completely lexsegment ideal. Therefore, I is a lexsegment ideal. Since $\mathscr{L}(x_1^2x_2^2, x_2^2x_3^2) \subsetneq \mathscr{L}_4(x_1^3x_2, x_2^2x_3^2)$, one gets that I is not componentwise lexsegment.

We characterize all the componentwise lexsegment ideals which are componentwise linear. Since this class of ideals is contained in the class of lexsegment ideals, it is clear that the notions componentwise linear and componentwise linear quotients are equivalent, by Theorem 2.1.

One may note that we always may assume $x_1 \mid u$ since otherwise we can study the ideal in a polynomial ring in a smaller number of variables.

Theorem 2.5. *Let I be a componentwise lexsegment ideal and $d \geq 1$ the lowest degree of the minimal monomial generators of I. Let $u, v \in \mathscr{M}_d$, $x_1|u$ be such that $I_{\langle d\rangle} = (\mathscr{L}(u,v))$. Then I is a componentwise linear ideal if and only if $I_{\langle d\rangle}$ has a linear resolution.*

Proof. If I is componentwise linear, the statement is straightforward. Therefore, we have to prove only that if $I_{\langle d\rangle}$ has a linear resolution then I is componentwise linear.

We separately treat the case of completely and of non-completely lexsegment ideals. Firstly, let us assume that $I_{\langle d\rangle}$ is a completely lexsegment ideal generated in one degree with a linear resolution. Hence $I = I_{\langle d\rangle}$. Since I is a completely lexsegment ideal generated in degree d with a linear resolution, the ideals generated by the shadows of $\mathscr{L}(u,v)$ are completely lexsegment ideals generated in one degree with linear resolutions. Therefore I is componentwise linear.

If $I_{\langle d\rangle} = (\mathscr{L}(u,v))$ is a non-completely lexsegment ideal generated in one degree with a linear resolution, then, by [1, Theorem 2.4], u and v must have the form

$$u = x_1x_{l+1}^{a_{l+1}} \cdots x_n^{a_n} \text{ and } v = x_lx_n^{d-1}$$

for some l, $2 \leq l < n$. Therefore, $\nu_1(u) = 1$ and $\nu_1(v) = 0$. Here, for a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$, we denoted by $\nu_i(m)$ the exponent of the variable x_i, that is $\nu_i(m) = a_i$.

If we look at the ends of the lexsegment $\mathscr{L}(x_1u, vx_n)$, we have $\nu_1(x_1u) = 2$, $\nu_1(vx_n) = 0$ and one may easily see that $(\mathscr{L}(x_1u, vx_n))$ is a completely lexsegment ideal. By [1, Theorem 1.3], $(\mathscr{L}(x_1u, vx_n))$ has a linear resolution. Since $(\mathscr{L}(x_1u, vx_n))$ is a completely lexsegment ideal with a linear resolution, the ideals generated by the shadows of $\mathscr{L}(x_1u, vx_n)$ are completely lexsegment ideals with linear resolutions. Therefore, I is componentwise linear. $\square$

3. Gotzmann completely lexsegment ideals

In this section we are going to characterize the completely lexsegment ideals generated in degree d which are Gotzmann.

Firstly we recall another operator connected with the binomial expansion of an integer.

Let $a = \binom{a_d}{d} + \ldots + \binom{a_j}{j}$, $a_d > a_{d-1} > \ldots > a_j \geq j \geq 1$, be the binomial expansion of a with respect to d. Then

$$a^{(d)} = \binom{a_d}{d+1} + \ldots + \binom{a_j}{j+1}.$$

We obviously have the following equality:

$$a^{\langle d\rangle} = a + a^{(d)}.$$

Lemma 3.1. *Let $c > b > 0$ be two integers. Let $b = \binom{b_d}{d} + \ldots + \binom{b_j}{j}$, $b_d > b_{d-1} > \ldots > b_j \geq j \geq 1$, and $c = \binom{c_d}{d} + \ldots + \binom{c_i}{i}$, $c_d > c_{d-1} > \ldots > c_i \geq i \geq 1$, be the d-binomial expansions of b and c. The following statements are equivalent:*

(i) $b^{(d)} = c^{(d)}$;

(ii) $j \geq 2$ *and* $c - b \leq j - 1$.

Proof. Let $b^{(d)} = c^{(d)}$. Since $c > b$, by [2, Lemma 4.2.7], there exists $s \leq d$ such that $c_d = b_d, \ldots, c_{s+1} = b_{s+1}$, and $c_s > b_s$. We obviously have $s + 1 \geq j$. Let us suppose that $s \geq j$. Since $c_s \geq b_s + 1$, we get:

$$\binom{c_s}{s+1} \geq \binom{b_s+1}{s+1} \geq \binom{b_s}{s+1} + \binom{b_{s-1}}{s} + \ldots + \binom{b_j}{j+1} + \binom{b_j}{j} >$$

$$> \binom{b_s}{s+1} + \binom{b_{s-1}}{s} + \ldots + \binom{b_j}{j+1}$$

This leads to the inequality $c^{(d)} > b^{(d)}$, which contradicts our hypothesis. Indeed, we have

$$c^{(d)} \geq \binom{c_d}{d+1} + \ldots + \binom{c_{s+1}}{s+2} + \binom{c_s}{s+1} >$$

$$> \binom{b_d}{d+1} + \ldots + \binom{b_{s+1}}{s+2} + \binom{b_s}{s+1} + \ldots + \binom{b_j}{j+1} = b^{(d)}.$$

Therefore we must have $s = j-1$. Hence $j \geq 2$ and c has the binomial expansion

$$c = \binom{c_d}{d} + \ldots + \binom{c_j}{j} + \binom{c_{j-1}}{j-1} + \ldots + \binom{c_i}{i}.$$

Using the equality $c^{(d)} = b^{(d)}$ we get

$$\binom{c_{j-1}}{j} + \ldots + \binom{c_i}{i+1} = 0,$$

which implies that $c_{j-1} = j-1, \ldots, c_i = i$. Therefore $c = b + j - i \leq b + j - 1$, which proves (ii).

Now, let $j \geq 2$ and $c \leq b + j - 1$. As in the first part of the proof, let $s \leq d$ be an integer such that $c_d = b_d, \ldots, c_{s+1} = b_{s+1}$, and $c_s > b_s$. If $s \geq j$, we get the following inequalities:

$$c = \binom{c_d}{d} + \ldots + \binom{c_{s+1}}{s+1} + \binom{c_s}{s} + \ldots + \binom{c_i}{i} \geq$$

$$\geq \binom{b_d}{d} + \ldots + \binom{b_{s+1}}{s+1} + \binom{b_s+1}{s} + \binom{c_{s-1}}{s-1} + \ldots + \binom{c_i}{i} \geq$$

$$\geq \binom{b_d}{d} + \ldots + \binom{b_{s+1}}{s+1} + \binom{b_s}{s} + \ldots + \binom{b_j}{j} + \binom{b_j}{j-1} + \binom{c_{s-1}}{s-1} + \ldots$$

$$\ldots + \binom{c_i}{i} = b + \binom{b_j}{j-1} + \binom{c_{s-1}}{s-1} + \ldots + \binom{c_i}{i} \geq b + j - i + s.$$

Since, by hypothesis, $c - b \leq j - 1$, we have $j - 1 \geq j - i + s$, thus $s \leq i - 1$, a contradiction. Hence, $s = j - 1$. Then we have:

$$c^{(d)} = \binom{c_d}{d+1} + \ldots + \binom{c_{s+1}}{s+2} + \binom{c_s}{s+1} + \ldots + \binom{c_i}{i+1} =$$

$$= \binom{b_d}{d+1} + \ldots + \binom{b_j}{j+1} + \binom{c_{j-1}}{s} + \ldots + \binom{c_i}{i+1} =$$

$$= b^{(d)} + \binom{c_{j-1}}{j} + \ldots + \binom{c_i}{i+1}$$

If we assume that $c_{j-1} \geq j$, then it follows that $\binom{c_{j-1}}{j-1} \geq j$. Looking at the d-binomial expansions of b and c, we get $c - b \geq j$, contradiction. Hence $c_{j-1} = j-1$. This equality implies also the equalities $c_k = k$, for all $i \leq k \leq j-2$. We obtain the following binomial expansion of c:

$$c = \binom{c_d}{d} + \ldots + \binom{c_j}{j} + \binom{j-1}{j-1} + \ldots + \binom{i}{i}.$$

Then

$$c^{(d)} = \binom{c_d}{d+1} + \ldots + \binom{c_j}{j+1} = \binom{b_d}{d+1} + \ldots + \binom{b_j}{j+1} = b^{(d)}.$$

$\square$

Lemma 3.2. *Let $c > 0$ be an integer with the binomial expansion*

$$c = \binom{c_d}{d} + \ldots + \binom{c_i}{i}, c_d > \ldots > c_i \geq i \geq 1.$$

The following statements are equivalent:

(a) $c^{(d)} = 0$*;*

(b) $c \leq d$.

Proof. Let $c \leq d$. Then c has the following binomial expansion with respect to d:

$$c = \binom{d}{d} + \ldots + \binom{i}{i}, \text{ for some } i \geq 1.$$

Hence $c^{(d)} = 0$.

Now let $c^{(d)} = 0$. We get

$$\binom{c_d}{d+1} + \ldots + \binom{c_i}{i+1} = 0, \text{ which implies}$$

$$c_d = d, \ldots, c_i = i.$$

It follows $c = d - (i-1) \leq d$. $\square$

Theorem 3.3. *Let $u, v \in \mathscr{M}_d$, $x_1 \mid u$ such that $I = (\mathscr{L}(u,v))$ is a completely lexsegment ideal of S which is not an initial lexsegment ideal. Let j be the exponent of the variable x_n in v and $a = |\mathscr{M}_d \setminus \mathscr{L}^i(u)|$. The following statements are equivalent:*

(a) I is a Gotzmann ideal;

(b) $a \geq \binom{n+d-1}{d} - (j+1)$.

Proof. Let $b = |\mathscr{M}_d \setminus \mathscr{L}^i(v)|$ and $w \in \mathscr{M}_d$ such that $|\mathscr{L}(u,v)| = |\mathscr{L}^i(w)|$. We denote $c = |\mathscr{M}_d \setminus \mathscr{L}^i(w)|$. Then $|\mathscr{L}^i(w)| = |\mathscr{L}^i(v)| - |\mathscr{L}^i(u)| + 1 = a - b + 1$, which yields:

$$\binom{n+d-1}{d} - c = a - b + 1,$$

that is

$$c = \binom{n+d-1}{d} - (a+1) + b. \tag{1}$$

Since I is completely, I is Gotzmann if and only if

$$|\mathscr{L}^i(wx_n)| = |\mathscr{L}(ux_1, vx_n)| = |\mathscr{L}^i(vx_n)| - |\mathscr{L}^i(ux_1)| + 1. \tag{2}$$

Since $x_1 \mid u$, we have $|\mathscr{L}^i(ux_1)| = |\mathscr{L}^i(u)|$. Therefore, the equality (2) is equivalent to

$$|\mathscr{L}^i(wx_n)| = |\mathscr{L}^i(vx_n)| - |\mathscr{L}^i(u)| + 1$$

that is

$$|\mathscr{M}_{d+1}| - c^{\langle d\rangle} = |\mathscr{M}_{d+1}| - b^{\langle d\rangle} - (|\mathscr{M}_d| - a) + 1.$$

Here we used the well known formula

$$|\mathscr{M}_{d+1} \setminus \mathrm{Shad}\mathscr{L}| = r^{\langle d\rangle},$$

where $\mathscr{L} \subset \mathscr{M}_d$ is an initial lexsegment and $r = |\mathscr{M}_d \setminus \mathscr{L}|$ [2]. Hence I is Gotzmann if and only if

$$c^{\langle d\rangle} = b^{\langle d\rangle} + \binom{n+d-1}{d} - a - 1. \tag{3}$$

By using (1), we obtain

$$c^{\langle d\rangle} = b^{\langle d\rangle} + c - b,$$

that is

$$c^{\langle d\rangle} - c = b^{\langle d\rangle} - b,$$

which is equivalent to

$$c^{(d)} = b^{(d)}. \tag{4}$$

Let us firstly consider the case $b=0$, that is $v=x_n^d$ and I is the final lexsegment determined by u. The equation (4) becomes

$$c^{(d)}=0. \tag{5}$$

By Lemma 3.2, $c^{(d)}=0$ if and only if $c\leq d$.

For the case $b>0$, the monomial v has the form

$$v=x_{l_1}\cdots x_{l_{d-j}}x_n^j,$$

for some $j\geq 0$ and $1\leq l_1\leq\ldots\leq l_{d-j}\leq n-1$. The d-binomial expansion of b is

$$b=\binom{n-l_1+d-1}{d}+\ldots+\binom{n-l_{d-j}+j}{j+1}.$$

By Lemma 3.1, the equality (4) holds if and only if $j\geq 1$ and $c-b\leq j$. Then we have obtained $c-b\leq j$ for any b. By (1), this inequality holds if and only if $\binom{n+d-1}{d}-(a+1)\leq j$, that is

$$a\geq\binom{n+d-1}{d}-(j+1).$$

$\square$

4. Gotzmann non-completely lexsegment ideals

Firstly, we recall the Taylor resolution. Let I be a monomial ideal of S with the minimal monomial generating set $G(I)=\{u_1,\ldots,u_r\}$. The Taylor resolution $(T_\bullet(I),d_\bullet)$ of I is defined as follows. Let L be the free S-module with the basis $\{e_1,\ldots,e_r\}$. Then $T_q(I)=\bigwedge^{q+1}L$ for $0\leq q\leq r-1$ and $d_q:T_q(I)\to T_{q-1}(I)$ for $1\leq q\leq r-1$ is defined as follows

$$d_q(e_{i_0}\wedge\ldots\wedge e_{i_q})=\sum_{s=0}^{q}(-1)^s\frac{\operatorname{lcm}(u_{i_0},\ldots,u_{i_q})}{\operatorname{lcm}(u_{i_0},\ldots,\check{u}_{i_s},\ldots,u_{i_q})}e_{i_0}\wedge\ldots\wedge\check{e}_{i_s}\wedge\ldots\wedge e_{i_q}.$$

The augmentation $\varepsilon:T_0\to I$ is defined by $\varepsilon(e_i)=u_i$ for all $1\leq i\leq q$. It is known that, in general, the Taylor resolution is not minimal. M. Okudaira and Y. Takayama characterized all the monomial ideals with linear resolutions whose Taylor resolutions are minimal.

Theorem 4.1 ([13]). *Let I be a monomial ideal with linear resolution. The following conditions are equivalent:*

(i) The Taylor resolution of I is minimal;

(ii) $I = m \cdot (x_{i_1}, \ldots, x_{i_l})$ *for some* $1 \leq i_1 < \ldots < i_l \leq n$ *and for a monomial* m.

In [8], the componentwise linear monomial ideals whose Taylor resolutions are minimal are described.

Theorem 4.2 ([8]). *Let I be a componentwise linear monomial ideal of S. The following conditions are equivalent:*

(i) *The Taylor resolution of I is minimal;*

(ii) $\max\{m(u) : u \in G(I)\} = |G(I)|$;

(iii) *I is a Gotzmann ideal with* $|G(I)| \leq n$.

Now we can complete the characterization of non-completely lexsegment ideals which are Gotzmann.

Theorem 4.3. *Let* $u = x_t^{a_t} \cdots x_n^{a_n}$, $v = x_t^{b_t} \cdots x_n^{b_n}$ *be two monomials of degree d,* $u >_{lex} v$, $a_t \neq 0$, $t \geq 1$ *and* $I = (\mathscr{L}(u,v))$ *a non-completely lexsegment ideal. Then I is a Gotzmann ideal in S if and only if* $I = m(x_l, x_{l+1}, \ldots, x_{l+p})$ *for some* $t \leq l \leq n$, *some* $1 \leq p \leq n-l$ *and a monomial m.*

Proof. If $I = m(x_l, x_{l+1}, \ldots, x_{l+p})$ for some $t \leq l \leq n$, some $1 \leq p \leq n-l$ and a monomial m, then the ideal I is isomorphic to the monomial prime ideal $(x_l, x_{l+1}, \ldots, x_{l+p})$ and the Koszul complex of the sequence $x_l, x_{l+1}, \ldots, x_{l+p}$ is isomorphic to the minimal graded free resolution of I. Therefore I has a linear resolution and, by Theorem 4.1, the Taylor resolution of I is minimal. Since any ideal with a linear resolution is componentwise linear, it follows by Theorem 4.2 that I is a Gotzmann ideal.

Now it remains to prove that, if I is a Gotzmann ideal in S, then I has the required form.

Firstly, we prove that $\operatorname{projdim}(S/I) < n$. For this, we study the following cases.

Case I: $t = 1$, $b_1 = 0, a_1 = 1$. Since I is a non-completely lexsegment ideal which is Gotzmann, I has a linear resolution. Therefore, by [1, Theorem 2.4], u and v have the form

$$u = x_1 x_{l+1}^{a_{l+1}} \ldots x_n^{a_n} \text{ and } v = x_l x_n^{d-1}$$

for some l, $2 \leq l \leq n-1$. Since $x_n u <_{lex} x_1 v$, using [5, Proposition 3.2] we get $\operatorname{depth}(S/I) \neq 0$. Hence $\operatorname{projdim}(S/I) < n$.

Case II: $t = 1$, $0 < b_1 < a_1$. Since I is a non-completely lexsegment ideal, we must have $b_1 = a_1 - 1$. Now, if I does not have a linear resolution, I is not Gotzmann. The ideal I has a linear resolution if and only if $J = (\mathscr{L}(u', v'))$ has a linear resolution, where $u' = u/x_1^{b_1}$ and $v' = v/x_1^{b_1}$. One may easy check that J is a non-completely lexsegment ideal. Therefore J has a linear resolution if and only if u' and v' have the form $u' = x_1 x_{l+1}^{a_{l+1}} \dots x_n^{a_n}$ and $v' = x_l x_n^{d-1}$ for some l, $2 \le l \le n-1$ [1, Theorem 2.4] and this implies $u = x_1^{a_1} x_{l+1}^{a_{l+1}} \dots x_n^{a_n}$ and $v' = x_1^{a_1 -1} x_l x_n^{d-1}$. Since $x_n u <_{lex} x_1 v$, using [5, Proposition 3.2] we get depth$(S/I) \neq 0$. Hence projdim$(S/I) < n$.

Case III: $t = 1$, $a_1 = b_1 > 0$. Since $x_n u <_{lex} x_1 v$, we have that depth$(S/I) \neq 0$ by [5, Proposition 3.2]. Hence projdim$(S/I) < n$.

Case IV: $t > 1$. We obviously have $x_n u <_{lex} x_1 v$ and, by [5, Proposition 3.2], depth$(S/I) \neq 0$. Therefore projdim$(S/I) < n$.

We may conclude that projdim$(S/I) < n$ in all the cases.

Let $w \in \mathscr{M}_d$ be a monomial such that $|\mathscr{L}(u,v)| = |\mathscr{L}^i(w)|$. Since I is a Gotzmann ideal, I^{lex} is generated in degree d, that is $I^{lex} = (\mathscr{L}^i(w))$. By [7, Corollary 1.4], I and I^{lex} have the same Betti numbers. In particular, we have

$$\operatorname{projdim}(I) = \operatorname{projdim}(I^{lex}).$$

Since projdim$(S/I) < n$, we have projdim$(S/I^{lex}) < n$. The ideal I^{lex} is stable in the sense of Eliahou and Kervaire, thus there exists $j < n$ such that $w = x_1^{d-1} x_j$. Therefore, $|\mathscr{L}(u,v)| = j < n$. By the hypothesis, I is a Gotzmann ideal and I is componentwise linear since it has a linear resolution. By Theorem 4.2, the Taylor resolution of I is minimal. The conclusion follows by Theorem 4.1 and taking into account that I is a lexsegment ideal. □

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REFERENCES

[1] A. Aramova - E. De Negri - J. Herzog, *Lexsegment ideals with linear resolutions*, Illinois J. Math. **42(3)** (1998), 509–523.

[2] W. Bruns - J. Herzog, *Cohen–Macaulay Rings*, Cambridge University Press, Cambridge, 1993.

[3] A. Conca - J. Herzog, *Castelnuovo–Mumford regularity of products of ideals*, Collect. Math. **54** (2003), 137–152.

[4] S. Eliahou - M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra **129** (1990), 1–25.

[5] V. Ene - A. Olteanu - L. Sorrenti, *Properties of lexsegment ideals*, To appear in Osaka J. Math. (arXiv.org:0802.1279).

[6] G. Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. **158** (1978), 61–70.

[7] J. Herzog - T. Hibi, *Componentwise linear ideals*, Nagoya Math. J. **153** (1999), 141–153.

[8] J. Herzog - T. Hibi - S. Murai - Y. Takayama, *Componentwise linear ideals with minimal or maximal Betti numbers*, Ark. Math. **46** (2008), 69–75.

[9] J. Herzog - Y. Takayama, *Resolutions by mapping cones*, Homology, Homotopy and Applications **4(2)** (2002), 277–294.

[10] H. A. Hulett - H. M. Martin, *Betti numbers of lexsegment ideals*, J. Algebra **275(2)** (2004), 629–638.

[11] A. Soleyman Jahan - X. Zheng, *Pretty clean monomial ideals and linear quotients*, (arXiv.org:0707.2914).

[12] E. De Negri - J. Herzog, *Completely lexsegment ideals*, Proc. Amer. Math. Soc. **126(12)** (1998), 3467–3473.

[13] M. Okudaira - Y. Takayama, *Monomial ideals with linear quotients whose Taylor resolutions are minimal*, (arXiv.org:0610168).

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