LE MATEMATICHE
Vol. LXVI (2011) – Fasc. I, pp. 13–26
doi: 10.4418/2011.66.1.2

KOBER FRACTIONAL q-DERIVATIVE OPERATORS

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In the present paper, we define right and left sided Kober fractional q-derivative operators and show that these derivative operators are left inverse operators of Kober fractional q-integral operators. We obtain the images of generalized basic hypergeometric function and basic analogue of Fox H-function under these operators. We also deduce several interesting results involving q-analogues of some classical functions as special cases of our main findings.

1. Introduction

In the theory of q-calculus [4], $0 < |q| < 1$ the q-shifted factorial (q-analogue of the Pochhammer Symbol) is defined by

$$(a;q)_k = \begin{cases} \prod_{j=0}^{k-1} \left(1 - aq^j\right) & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \prod_{j=0}^{\infty} \left(1 - aq^j\right) & \text{if } k \to \infty \end{cases},$$

or equivalently $(a;q)_k = \frac{(a;q)_\infty}{\left(aq^k;q\right)_\infty}$ $(k \in N \text{ U } \{\infty\})$

Entrato in redazione: 15 aprile 2010

AMS 2010 Subject Classification: 33D15, 26A33.
Keywords: Kober fractional q-integral operators, Kober fractional q-derivative operators, Generalized basic hypergeometric function, Basic analogue of Fox H-function, Basic analogue of Bessel function.

Its extension to the real α is

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, (\alpha \in R) \tag{1}$$

Also, the q-analogue of the power function $(a-b)^\alpha$is

$$(a-b)^{(\alpha)} = a^\alpha \left({}^b/_a;q \right)_\alpha = a^\alpha \prod_{j=0}^{\infty} \left[\frac{1-\left({}^b/_a \right) q^j}{1-\left({}^b/_a \right) q^{j+\alpha}} \right]$$

$$= a^\alpha \frac{\left({}^b/_a;q \right)_\infty}{\left(q^\alpha {}^b/_a;q \right)_\infty}, (a \neq 0) \tag{2}$$

The q-gamma function is defined by

$$\Gamma_q(\alpha) = \frac{G(q^\alpha)}{G(q)} (1-q)^{1-\alpha} = (1-q)^{(\alpha-1)} (1-q)^{1-\alpha}, x \in R/\{0,-1,-2,...\} \tag{3}$$

where $G(q^\alpha) = \frac{1}{(q^\alpha;q)_\infty}$
Obviously $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$
The q-derivative of a function $f(x)$ is defined in the book by Gasper and Rahman [4]:
$(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}, (x \neq 0)$ and $(D_q f)(0) = \lim_{x\to 0} (D_q f)(x)$
where $D_q \to {}^d/_{dx}$, as $q \to 1$.
We have $D_q x^\mu = \frac{1-q^\mu}{1-q} x^{\mu-1} = [\mu]_q x^{\mu-1}$
The q-integral of a function is defined by in the book by Gasper and Rahman [4]:

$$\int_0^x f(t)\, d_q t = x(1-q) \sum_{k=0}^{\infty} q^k f\left(xq^k\right) \tag{4}$$

$$\int_x^\infty f(t)\, d_q t = x(1-q) \sum_{k=1}^{\infty} q^{-k} f\left(xq^{-k}\right) \tag{5}$$

The generalized basic hypergeometric series is given as follows [4]

$${}_r\phi_s \left[\begin{matrix} a_1,...,a_r \\ b_1,...,b_s \end{matrix} ;q,x \right] = \sum_{n=0}^{\infty} \frac{(a_1,...,a_r;q)_n}{(b_1,...,b_s;q)_n} \left[(-1)^n q^{\left(\begin{matrix} n \\ 2 \end{matrix} \right)} \right]^{1+s-r} \frac{x^n}{(q;q)_n} \tag{6}$$

with $\left(\begin{matrix} n \\ 2 \end{matrix} \right) = n(n-1)/2$ and $(a_1,...,a_r;q)_n = (a_1;q)_n \cdots (a_r;q)_n$

If $0<|q|<1$, series converges absolutely for all x if $r \leq s$ and for $|x|<1$ if $r=s+1$. Also if $|q|>1$, then the series converges absolutely for $|x|<|b_1,...,b_s|/|a_1,...,a_r|$.

The abnormal type of generalized basic hypergeometric series ${}_r\phi_s(.)$ is defined as

$$ {}_r\phi_s\left[\begin{array}{c} a_1,...,a_r \\ b_1,...,b_s \end{array};\begin{array}{c} q \\ q^{\alpha} \end{array},x\right]=\sum_{n=0}^{\infty}\frac{(a_1,...,a_r;q)_n}{(b_1,...,b_s;q)_n}q^{\alpha n(n+1)/2}\frac{x^n}{(q;q)_n} \tag{7}$$

where $\alpha>0$ and $0<|q|<1$.

The q-binomial series is given by

$$ {}_1\phi_0\left[\begin{array}{c} \alpha \\ - \end{array};q,x\right]=\frac{(\alpha x;q)_\infty}{(x;q)_\infty} \tag{8}$$

From (1) and (8), follows that

$$ (a-b)^{(\alpha)}=a^{\alpha}\,{}_1\phi_0\left[\begin{array}{c} q^{-\alpha} \\ - \end{array};q,\frac{b}{a}q^{\alpha}\right] \tag{9}$$

The q-analogues of the exponential function are given by ([3], [4]):

$$ E_q^x={}_0\phi_0(-;-;q,x)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^n}{(q;q)_n}=(x;q)_\infty \tag{10}$$

$$ e_q^x={}_1\phi_0(0;-;q,x)=\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}=\frac{1}{(x;q)_\infty} \tag{11}$$

In 1905, Jackson introduced the following q-analogue of the Bessel function ([6]; see also [4]):

$$ J_\nu^{(1)}(x;q)=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^{\nu}{}_2\phi_1\left(0,0;q^{\nu+1};q,-\frac{x^2}{4}\right),|x|<2 \tag{12}$$

The basic analogue of Fox H-function is defined by Saxena, Modi and Kalla [9], as follows

$$ H_{A,B}^{m_1,n_1}\left[x;q\left|\begin{array}{c} (a,\alpha) \\ (b,\beta) \end{array}\right.\right]$$

$$ =\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m_1}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n_1}G\left(q^{1-a_j+\alpha_j s}\right)\pi x^s}{\prod_{j=m_1+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n_1+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}ds$$

$$ =\frac{1}{2\pi i}\int_C\chi_q(s)\,x^s ds \tag{13}$$

where $0 \le m_1 \le B, 0 \le n_1 \le A; \alpha_j^{'}s$and $\beta_j^{'}s$ are all positive, the contour C is a line parallel to $Re\,(ws) = 0$, with indentations, if necessary, in such a manner that all poles of $G\left(q^{b_j-\beta_j s}\right), 1 \le j \le m_1$,are to the right, and those of $G\left(q^{1-a_j-\alpha_j s}\right), 1 \le j \le n_1$ are to the left of C. The integral converges, if $Re\,[s \log(x) - \log \sin \pi s] < 0$ for large values of |s|on the contour, that is, if $\left|\left\{\arg(x) - w_2 w_1^{-1} \log|x|\right\}\right| < \pi$, where$0 < |q| < 1, \log q = -w = -(w_1 + i w_2), w, w_1, w_2$ are definite quantities, $w_1$andw_2 being real.
We shall now express the generalized basic hypergeometric series ${}_{r+1}\phi_r$ in terms of basic analogue of Fox H-function. We start with the contour integral representation of generalized basic hypergeometric series ${}_{r+1}\phi_r$ as given in the book by Gasper and Rahman [4].

$$
\begin{aligned}
&{}_{r+1}\phi_r\left[\begin{array}{c} q^{a_1},\ldots,q^{a_{r+1}} \\ q^{b_1},\ldots,q^{b_r}\end{array};q,x\right] \\
&= \frac{(q^{a_1},\ldots,q^{a_{r+1}};q)_\infty}{\left(q,q^{b_1},\ldots q^{b_r};q\right)_\infty}\left(-\frac{1}{2\pi i}\right)\int_C \frac{\left(q^{1+s},q^{b_1+s},\ldots,q^{b_r+s};q\right)_\infty}{\left(q^{a_1+s},\ldots,q^{a_{r+1}+s};q\right)_\infty}\frac{\pi(-z)^s}{\sin \pi s}ds
\end{aligned}
\tag{14}
$$

Applying the result$-\left(q^{1+s};q\right)_\infty = \frac{q^{-s}G(q^{-s})}{G(q^s)G(q^{1-s})}$, which can easily be obtained by some simple manipulations and expressing all the q-factorials in terms of q-gammas using (1), the ${}_{r+1}\phi_r$ can be expressed as

$$
{}_{r+1}\phi_r\left[\begin{array}{c} q^{a_1},\ldots,q^{a_{r+1}} \\ q^{b_1},\ldots,q^{b_r}\end{array};q,x\right]
$$

$$
= \frac{\prod_{j=1}^{r} G\left(q^{b_j}\right) G(q)}{\prod_{j=1}^{r+1} G(q^{a_j})}\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{r+1} G(q^{a_j+s})\, G(q^{-s})}{\prod_{j=1}^{r} G\left(q^{b_j+s}\right) G(q^s)\, G(q^{1-s})}\frac{\pi}{\sin \pi s}\left(-\frac{x}{q}\right)^s ds
$$

$$
= \frac{\prod\limits_{j=1}^{r} G\left(q^{b_j}\right) G(q)}{\prod\limits_{j=1}^{r+1} G(q^{a_j})} H_{r+1,r+2}^{1,r+1}\left[-\frac{x}{q};q\left|\begin{array}{c} (1-a_j,1) \\ (0,1),(1-b_j,1)(1,1)\end{array}\right.\right] \tag{15}
$$

The above result can be used to write the q-exponential function (11) and the q-Bessel function (12), in terms of basic analogue of H-function as follows

$$
e_q^x = G(q)\, H_{0,2}^{1,0}\left[-\frac{x}{q};q\left|\begin{array}{c} - \\ (0,1),(1,1)\end{array}\right.\right] \tag{16}
$$

$$
J_\nu^{(1)}(x;q) = [G(q)]^2 \left(\frac{x}{2}\right)^\nu H_{0,3}^{1,0}\left[\frac{x^2}{4q};q\left|\begin{array}{c} - \\ (0,1),(-\nu,1)(1,1)\end{array}\right.\right] \tag{17}
$$

The remaining paper is organized as follows. In Section 2, we present definitions and some properties of Kober fractional q-integral operators as these will be used in Section 3. In Section 3, we define right and left hand sided Kober fractional q-derivative operators and show that these are left inverse of Kober fractional q-integral operators. In Section 4, we obtain images of the power function, the generalized basic hypergeometric series and the basic analogue of H-function under right and left hand sided Kober fractional q-derivative operators. In the last Section 5, we obtain the images of q-analogues of some classical functions as special cases of the results established in Section 4.

2. Kober fractional q-integral operators

W.A. Al-Salam [2] and R.P. Agarwal [1] introduced several types of fractional q-integral operators and fractional q-derivative operators. Further study of various fractional q-integral and derivative operators is done by Isogawa et al. [5], Rajkovi et al. [8] and Saxena et al.([10], [9], [11]). In the present paper, we shall consider the following definition of the right and the left hand sided Kober fractional q-integral operators

$$\left(I_q^{\gamma,\delta} f\right)(x) = \frac{x^{-\gamma-\delta}}{\Gamma_q(\delta)} \int_0^x (x-tq)_{\delta-1} t^{\gamma} f(t)\, d_q t \tag{18}$$

$$\left(J_q^{\gamma,\delta} f\right)(x) = \frac{x^{\gamma} q^{-\gamma}}{\Gamma_q(\delta)} \int_x^{\infty} (t-x)_{\delta-1} t^{-\gamma-\delta} f\left(tq^{1-\delta}\right) d_q t \tag{19}$$

where δ is an arbitrary order of integration with $R(\delta) > 0$ and γ being real or complex.
For $q \to 1$, these operators (18) and (19), reduce to Kober operators $\left(I^{\gamma,\delta} f\right)(x)$ and $\left(J^{\gamma,\delta} f\right)(x)$ respectively as defined in [7].
For $\gamma = 0$, the operators (18) and (19), reduce to Riemann-Liouville fractional q-integral operator with a power weight and Weyl fractional q-integral operator respectively as

$$\left(I_q^{0,\delta} f\right)(x) = x^{-\delta} \left(I_q^{\delta} f\right)(x) = \frac{x^{-\delta}}{\Gamma_q(\delta)} \int_0^x (x-tq)_{\delta-1} f(t)\, d_q t, Re(\delta) > 0 \tag{20}$$

$$\left(J_q^{0,\delta} f\right)(x) = \left(J_q^{\delta} f\right)(x) = \frac{1}{\Gamma_q(\delta)} \int_x^{\infty} (t-x)_{\delta-1} t^{-\delta} f\left(tq^{1-\delta}\right) d_q t \tag{21}$$

The operators given in (18) and (19), in view of (4) and (5) can be written as:

$$\left(I_q^{\gamma,\delta} f\right)(x) = \frac{(1-q)}{\Gamma_q(\delta)} \sum_{k=0}^{\infty} q^{k(1+\gamma)} \left(1-q^{k+1}\right)_{\delta-1} f\left(xq^k\right) \tag{22}$$

$$\left(J_q^{\gamma,\delta} f\right)(x) = \frac{(1-q)}{\Gamma_q(\delta)} \sum_{k=0}^{\infty} q^{k\gamma} \left(1-q^{k+1}\right)_{\delta-1} f\left(xq^{-k-\delta}\right) \tag{23}$$

The following properties hold good for ooperator (18).

$$\left(I_q^{\gamma,\delta} x^\lambda f\right)(x) = x^\lambda \left(I_q^{\gamma+\lambda,\delta} f\right)(x) \tag{24}$$

$$\left(I_q^{\gamma,\delta} I_q^{\gamma+\delta,\alpha} f\right)(x) = \left(I_q^{\gamma,\delta+\alpha} f\right)(x) \tag{25}$$

$$\left(I_q^{\gamma,\delta} I_q^{\alpha,\eta} f\right)(x) = \left(I_q^{\alpha,\eta} I_q^{\gamma,\delta} f\right)(x) \tag{26}$$

The above properties are also true for operator (19).

3. Kober fractional q-derivative operators

We define the right hand sided Kober fractional q-derivative operators of order δ, $Re(\delta) > 0$ as follows

$$\left(D_q^{\gamma,\delta} f\right)(x) = \prod_{j=1}^{n} \left([\gamma+j]_q + xq^{\gamma+j} D_q\right) \left(I_q^{\gamma+\delta,n-\delta} f\right)(x) \tag{27}$$

where $n = [R(\delta)] + 1, n \in N$

The left hand sided Kober fractional q-derivative operator is defined by

$$\left(P_q^{\gamma,\delta} f\right)(x) = \prod_{j=0}^{n-1} \left([\gamma+j]_q - xD_q\right) \left(J_q^{\gamma+\delta,n-\delta} f\right)(x) \tag{28}$$

Here the operators $I_q^{\gamma,\delta}$ and $J_q^{\gamma,\delta}$ are the right and the left hand sided Kober fractional q-integral operators of order δ defined by (18) and (19).

For $q \to 1$, the operators (27) and (28) reduce to right and left hand sided ordinary Kober fractional derivative operators as given in [7] respectively.

In this paper, we deal mainly with the right hand sided Kober fractional q-derivative operator $D_q^{\gamma,\delta}$. The case of the left hand sided derivative operator can be considered by analogy.

In the following theorem, we shall prove that, the Kober fractional q-derivative operators (27) and (28) act as left inverses of the Kober fractional q-integral operators (18) and (19) respectively.

Theorem 3.1. *Let $\gamma, \delta \in C$ and $Re(\delta) > 0$, then following are valid*

$$\left(D_q^{\gamma,\delta} I_q^{\gamma,\delta} f\right)(x) = f(x) \tag{29}$$

$$\left(P_q^{\gamma,\delta} J_q^{\gamma,\delta} f\right)(x) = f(x) \tag{30}$$

Proof. By using the definition (27) and properties (25) and (26), we obtain the relation

$$\left(D_q^{\gamma,\delta} I_q^{\gamma,\delta} f\right)(x) = \prod_{j=1}^{n} \left([\gamma+j]_q + xq^{\gamma+j} D_q\right) \left(I_q^{\gamma+\delta,n-\delta} I_q^{\gamma,\delta} f\right)(x)$$

$$= \prod_{j=1}^{n} \left([\gamma+j]_q + xq^{\gamma+j} D_q\right) \left(I_q^{\gamma,n} f\right)(x)$$

Since $n \in N$, the relation

$$\prod_{j=1}^{n} \left([\gamma+j]_q + xq^{\gamma+j} D_q\right) \left(I_q^{\gamma,n} f\right)(x) = f(x) \tag{31}$$

can be proved by mathematical induction. For $n = 1$, we have

$$\left([\gamma+1]_q + xq^{\gamma+1} D_q\right) \left(I_q^{\gamma,1} f\right)(x)$$

$$= \left([\gamma+1]_q + xq^{\gamma+1} D_q\right)(1-q) \sum_{k=0}^{\infty} q^{k(1+\gamma)} f\left(xq^k\right)$$

$$= (1-q) \sum_{k=0}^{\infty} q^{k(1+\gamma)} \frac{(1-q^{\gamma+1})}{(1-q)} f\left(xq^k\right) + \left(q^{\gamma+1}-1\right) \sum_{k=0}^{\infty} q^{k(1+\gamma)} f\left(xq^k\right) + f(x)$$

$$= f(x)$$

Now, we have to prove the relation (31), for $n = m+1$ under the assumption that it holds true for $n = m$. For which, we have to see that the following relation holds.

$$\left([\gamma+m+1]_q + xq^{\gamma+m+1} D_q\right) \left(I_q^{\gamma,m+1} f\right)(x) = \left(I_q^{\gamma,m} f\right)(x) \tag{32}$$

By using (22), we can write the left side of (32) as

$$\left([\gamma+m+1]_q + xq^{\gamma+m+1} D_q\right) \frac{(1-q)}{\Gamma_q(m+1)} \sum_{k=0}^{\infty} q^{k(1+\gamma)} \left(1-q^{k+1}\right)_m f\left(xq^k\right)$$

$$= \frac{(1-q)}{\Gamma_q(m+1)} \left[\begin{array}{l} [\gamma+m+1]_q \sum_{k=0}^{\infty} q^{k(1+\gamma)} f\left(xq^k\right) \\ +xq^{\gamma+m+1} \sum_{k=0}^{\infty} q^{k(1+\gamma)} \left(1-q^{k+1}\right)_m \left[\frac{f\left(xq^k\right) - f\left(xq^{k+1}\right)}{x(1-q)}\right] \end{array} \right]$$

$$= \frac{1}{\Gamma_q(m+1)} \sum_{k=0}^{\infty} q^{k(1+\gamma)} \left(1-q^{k+1}\right)_{m-1} f\left(xq^k\right)$$
$$\left[\left(1-q^{k+m}\right)\left(1-q^{\gamma+1+m}\right) + q^{1+\gamma+m}\left(1-q^{k+m}\right) - q^m\left(1-q^k\right)\right]$$

$$= \frac{(1-q)}{\Gamma_q(m)} \sum_{k=0}^{\infty} q^{k(1+\gamma)} \left(1-q^{k+1}\right)_{m-1} f\left(xq^k\right) = \left(I_q^{\gamma,m} f\right)(x)$$

The result (30) , can be established on similar lines. □

4. Applications

In this section, we obtain images of the power function, the generalized basic hypergeometric series (6) and the basic analogue of Fox H-function (13) under the fractional q-derivative operators (27) and (28).

Theorem 4.1. *If* $\gamma, \delta, \mu \in C$ *and* $Re(\delta) > 0$, $Re(\gamma+\delta-\mu) > 0$, $Re(\gamma+\delta+\mu+1) > 0$, *then we have*

$$D_q^{\gamma,\delta} x^{\mu} = \frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+\mu+1)} x^{\mu} \tag{33}$$

$$P_q^{\gamma,\delta} x^{\mu} = \frac{\Gamma_q(\gamma+\delta-\mu)}{\Gamma_q(\gamma-\mu)} q^{\delta\mu} x^{\mu} \tag{34}$$

Proof. We first find the fractional q-integral $I_q^{\gamma,\delta}$ of x^{μ}.

$$I_q^{\gamma,\delta} x^{\mu} = \frac{(1-q)}{\Gamma_q(\delta)} \sum_{k=0}^{\infty} q^{k(\gamma+1)} \left(1-q^{k+1}\right)_{\delta-1} \left(xq^k\right)^{\mu}$$

$$= (1-q)^{\delta} x^{\mu} \sum_{k=0}^{\infty} q^{k(\gamma+\mu+1)} \frac{(q^{\delta};q)_k}{(q;q)_k}$$

$$= (1-q)^{\delta} x^{\mu} {}_1\phi_0\left(q^{\delta}; -; q^{\gamma+\mu+1}\right) = (1-q)^{\delta} x^{\mu} \frac{(q^{\gamma+\mu+\delta+1};q)_{\infty}}{(q^{\gamma+\mu+1};q)_{\infty}}$$

$$I_q^{\gamma,\delta} x^{\mu} = \frac{\Gamma_q(\gamma+\mu+1)}{\Gamma_q(\gamma+\mu+\delta+1)} x^{\mu} \tag{35}$$

Now to obtain $D_q^{\gamma,\delta} x^{\mu}$, we use definition (27) and the result (35), to get

$$D_q^{\gamma,\delta} x^{\mu} = \prod_{j=1}^{n} \left([\gamma+j]_q + xq^{\gamma+j} D_q\right) \frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+n+\mu+1)} x^{\mu}$$

Thus, to prove (33), it remains to show that

$$\prod_{j=1}^{n} \left([\gamma+j]_q + xq^{\gamma+j} D_q\right) \frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+n+\mu+1)} x^{\mu} = \frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+\mu+1)} x^{\mu} \tag{36}$$

Since $n \in N$, (36) can be proved by the mathematical induction. For $n = 1$, we have

$$\left([\gamma+1]_q + xq^{\gamma+1}D_q\right)\frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+\mu+2)}x^{\mu}$$

$$= \frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+\mu+2)}\left(\frac{(1-q^{\gamma+1})}{1-q} + \frac{q^{\gamma+1}(1-q^{\mu})}{1-q}\right)x^{\mu}$$

$$= \frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+\mu+1)}x^{\mu}$$

Now, we have to prove (36), for $n = m+1$ under the assumption that it holds for $n = m$, i.e. we have

$$\prod_{j=1}^{m}\left([\gamma+j]_q + xq^{\gamma+j}D_q\right)\frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+m+\mu+1)}x^{\mu} = \frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+\mu+1)}x^{\mu} \quad (37)$$

Now,

$$\prod_{j=1}^{m+1}\left([\gamma+j]_q + xq^{\gamma+j}D_q\right)\frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+m+1+\mu+1)}x^{\mu}$$

$$= \prod_{j=1}^{m}\left([\gamma+j]_q + xq^{\gamma+j}D_q\right)\left([\gamma+m+1]_q + xq^{k+m+1}D_q\right)I_q^{\gamma+\delta,m+1-\delta}x^{\mu} \quad (38)$$

Using (32) for $f(x) = x^{\mu}$, the right side of (38) equal to

$$= \prod_{j=1}^{m}\left([\gamma+j]_q + xq^{\gamma+j}D_q\right)I_q^{\gamma+\delta,m-\delta}x^{\mu}$$

$$= \prod_{j=1}^{m}\left([\gamma+j]_q + xq^{\gamma+j}D_q\right)\frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+m+\mu+1)}x^{\mu}$$

$$= \frac{\Gamma_q(\gamma+\delta+\mu+1)}{\Gamma_q(\gamma+\mu+1)}x^{\mu}, \qquad (\text{using } (4.5))$$

Thus, we get that the result (36) is true for $n = m+1$. Hence the result (33). Similar steps can be used to prove (34), for left hand sided derivative. □

Theorem 4.2. *For* $\gamma, \mu, \lambda \in C, R(\delta) > 0$ and $Re(\gamma+\lambda+\delta) > 0$, we have

$$D_q^{\gamma,\delta}\left[x^{\lambda}{}_r\phi_s\left[\begin{matrix} a_1, a_2, ..., a_r \\ b_1, b_2, ..., b_s \end{matrix}; q, ax\right]\right]$$
$$= \frac{\Gamma_q(\gamma+\lambda+\delta+1)}{\Gamma_q(\gamma+\lambda+1)}x^{\lambda}{}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1, a_2, ..., a_r, q^{\gamma+\delta+\lambda+1} \\ b_1, b_2, ..., b_s, q^{\gamma+\lambda+1} \end{matrix}; q, ax\right] \quad (39)$$

and

$$P_q^{\gamma,\delta}\left[x^{-\lambda}{}_r\phi_s\left[\begin{array}{c}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{array};q,{}^{a}\!/_{x}\right]\right]$$
$$=\frac{\Gamma_q(\gamma+\lambda+\delta)}{\Gamma_q(\gamma+\lambda)}x^{-\lambda}q^{-\delta\lambda}{}_{r+1}\phi_{s+1}\left[\begin{array}{c}a_1,a_2,\ldots,a_r,q^{\gamma+\delta+\lambda}\\ b_1,b_2,\ldots,b_s,q^{\gamma+\lambda}\end{array};q,{}^{a}\!/_{xq^{\delta}}\right] \quad (40)$$

Proof. In view of (6) and (33), the left side of (39) becomes

$$\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n}{(b_1;q)_n(b_2;q)_n\ldots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$$
$$\times\frac{\Gamma_q(\gamma+\lambda+\delta+1+n)}{\Gamma_q(\gamma+\lambda+1+n)}\frac{x^{\lambda+n}}{(q;q)_n}(a)^n$$
$$=(1-q)^{-\delta}\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n}{(b_1;q)_n(b_2;q)_n\ldots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$$
$$\times\frac{\left(q^{\gamma+\lambda+\delta+1};q\right)_n}{\left(q^{\gamma+\lambda+1};q\right)_n}\frac{\left(q^{\gamma+\lambda+1};q\right)_\infty}{\left(q^{\gamma+\lambda+\delta+1};q\right)_\infty}\frac{x^{\lambda+n}}{(q;q)_n}(a)^n$$
$$=\frac{\Gamma_q(\gamma+\lambda+\delta+1)}{\Gamma_q(\gamma+\lambda+1)}x^{\lambda}$$
$$\times\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n}{(b_1;q)_n(b_2;q)_n\ldots(b_s;q)_n}\frac{\left(q^{\gamma+\lambda+\delta+1};q\right)_n}{\left(q^{\gamma+\lambda+1};q\right)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}(ax)^n$$

which is equal to the right side of (39).
Result (40) can be proved on similar lines. □

Theorem 4.3. *For $\gamma,\mu,\delta\in C$,$\lambda>0$ and$R(\delta)>0$, we have*

$$D_q^{\gamma,\delta}\left[x^{\mu}H_{A,B}^{m_1,n_1}\left[\rho x^{\lambda};q\left|\begin{array}{c}(a,\alpha)\\(b,\beta)\end{array}\right.\right]\right]$$
$$=(1-q)^{-\delta}x^{\mu}H_{A+1,B+1}^{m_1,n_1+1}\left[\rho x^{\lambda};q\left|\begin{array}{c}(-\gamma-\delta-\mu,\lambda)(a,\alpha)\\(b,\beta)(-\gamma-\mu,\lambda)\end{array}\right.\right] \quad (41)$$
$$D_q^{\gamma,\delta}\left[x^{\mu}H_{A,B}^{m_1,n_1}\left[\rho x^{-\lambda};q\left|\begin{array}{c}(a,\alpha)\\(b,\beta)\end{array}\right.\right]\right]$$

$$= (1-q)^{-\delta} x^{\mu} H_{A+1,B+1}^{m_1+1,n_1}\left[\rho x^{-\lambda};q \left| \begin{array}{c} (a,\alpha)(1+\gamma+\mu,\lambda) \\ (1+\gamma+\delta+\mu,\lambda)(b,\beta) \end{array}\right.\right] \tag{42}$$

$$p_q^{\gamma,\delta}\left[x^{\mu} H_{A,B}^{m_1,n_1}\left[\rho x^{\lambda};q \left| \begin{array}{c} (a,\alpha) \\ (b,\beta) \end{array}\right.\right]\right]$$

$$= (1-q)^{-\delta} x^{\mu} q^{\delta\mu} H_{A+1,B+1}^{m_1+1,n_1}\left[\rho \left(xq^{\delta}\right)^{\lambda};q \left| \begin{array}{c} (a,\alpha)(\gamma-\mu,\lambda) \\ (\gamma+\delta-\mu,\lambda)(b,\beta) \end{array}\right.\right] \tag{43}$$

$$P_q^{\gamma,\delta}\left[x^{\mu} H_{A,B}^{m_1,n_1}\left[\rho x^{-\lambda};q \left| \begin{array}{c} (a,\alpha) \\ (b,\beta) \end{array}\right.\right]\right]$$

$$= (1-q)^{-\delta} x^{\mu} q^{\delta\mu} H_{A+1,B+1}^{m_1,n_1+1}\left[\rho \left(xq^{\delta}\right)^{-\lambda};q \left| \begin{array}{c} (1-\gamma-\delta+\mu,\lambda)(a,\alpha) \\ (b,\beta)(1-\gamma+\mu,\lambda) \end{array}\right.\right] \tag{44}$$

where $0 \le m_1 \le B, 0 \le n_1 \le A, Re\left[s\log(x) - \log\sin\pi s\right] < 0, 0 < |q| < 1.$

Proof. By using the definitions (13) and (33), the left side of (41) becomes

$$D_q^{\gamma,\delta} \frac{x^{\mu}}{2\pi i}\int_C \chi_q(s)\left(\rho x^{\lambda}\right)^s ds$$

$$= \prod_{j=1}^{n}\left([\gamma+j]_q + xq^{\gamma+j}D_q\right)\left(I_q^{\gamma+\delta,n-\delta}\frac{x^{\mu}}{2\pi i}\int_C \chi_q(s)\left(\rho x^{\lambda}\right)^s ds\right)$$

On interchanging the order of fractional q-integral and contour integral, we have

$$\prod_{j=1}^{n}\left([\gamma+j]_q + xq^{\gamma+j}D_q\right)\left(\frac{x^{\mu}}{2\pi i}\int_C \chi_q(s)\rho^s I_q^{\gamma+\delta,n-\delta}x^{\lambda s}ds\right)$$

On using (35), it becomes

$$\prod_{j=1}^{n}\left([\gamma+j]_q + xq^{\gamma+j}D_q\right)\left(\frac{1}{2\pi i}\int_C \chi_q(s)\rho^s \frac{\Gamma_q(\gamma+\delta+\lambda s+\mu+1)}{\Gamma_q(\gamma+n+\lambda s+\mu+1)}x^{\lambda s+\mu}ds\right)$$

$$= \frac{1}{2\pi i}\prod_{j=1}^{n}[\gamma+j]_q\int_C \chi_q(s)\rho^s \frac{\Gamma_q(\gamma+\delta+\lambda s+\mu+1)}{\Gamma_q(\gamma+\lambda s+\mu+n+1)}x^{\lambda s+\mu}ds$$

$$+\frac{1}{2\pi i}\frac{q^{\gamma+j}}{1-q}\int_C \chi_q(s)\rho^s \frac{\Gamma_q(\gamma+\delta+\lambda s+\mu+1)}{\Gamma_q(\gamma+\lambda s+\mu+n+1)}\left[x^{\lambda s+\mu} - (xq)^{\lambda s+\mu}\right]ds$$

$$= \frac{1}{2\pi i}\int_C \chi_q(s)\rho^s \frac{\Gamma_q(\gamma+\delta+\lambda s+\mu+1)}{\Gamma_q(\gamma+\lambda s+\mu+n+1)}\cdot$$

$$\cdot\prod_{j=1}^{n}\left([\gamma+j]_q + q^{\gamma+j}[\lambda s+\mu]_q\right)x^{\lambda s+\mu}ds$$

$$= \frac{1}{2\pi i}\int_C \chi_q(s)\,\rho^s \frac{\Gamma_q(\gamma+\delta+\lambda s+\mu+1)}{\Gamma_q(\gamma+\lambda s+\mu+1)} x^{\lambda s+\mu} ds$$

$$= \frac{(1-q)^{-\delta} x^{\mu}}{2\pi i}\int_C \chi_q(s)\,\frac{G\left(q^{\gamma+\delta+\lambda s+\mu+1}\right)}{G\left(q^{\gamma+\lambda s+\mu+1}\right)} \left(\rho x^{\lambda}\right)^s ds$$

The above expression on being interpreted with (13) yields the right hand side of (41). The result (42) can be proved similarly on changing λ to $-\lambda$.

The other two results (43) and (44), for left hand fractional q-derivative can be established on similar lines. □

5. Special cases

(i). In Theorem 4.2, if we take $r = s = 0, \lambda = 0, a = -1$, the function ${}_r\phi_s$ reduces to $E_q(x)$ defined by (10) and we obtain the following results

$$D_q^{\gamma,\delta}\left(E_q(x)\right) = \frac{\Gamma_q(\gamma+\delta+1)}{\Gamma_q(\gamma+1)} \times {}_1\phi_1\left(\begin{array}{c} q^{\gamma+\delta+1} \\ q^{\gamma+1} \end{array} ;q,-x\right) \tag{45}$$

$$P_q^{\gamma,\delta}\left(E_q\left({}^1\!/_x\right)\right) = \frac{\Gamma_q(\gamma+\delta)}{\Gamma_q(\gamma)} \times {}_1\phi_1\left(\begin{array}{c} q^{\gamma+\delta} \\ q^{\gamma} \end{array} ;q,-{}^1\!/_{xq^{\delta}}\right) \tag{46}$$

(ii). On taking $r = 1, s = 0, a_1 = q^{-\nu}$ in Theorem 4.2 and using the relation (9), we get

$$D_q^{\gamma,\delta}\left(x^{\lambda}\left(1-axq^{-\nu}\right)^{(\nu)}\right) =$$
$$= \frac{\Gamma_q(\gamma+\delta+\lambda+1)}{\Gamma_q(\gamma+\lambda+1)} x^{\lambda} \times {}_2\phi_1\left(\begin{array}{c} q^{\gamma+\delta+\lambda+1}, q^{-\nu} \\ q^{\gamma+\lambda+1} \end{array} ;q,ax\right) \tag{47}$$

$$P_q^{\gamma,\delta}\left(x^{-\lambda}\left(1-{}^{aq^{-\nu}}\!/_x\right)^{(\nu)}\right) =$$
$$= \frac{\Gamma_q(\gamma+\delta+\lambda)}{\Gamma_q(\gamma+\lambda)} x^{-\lambda} q^{-\delta\lambda} \times {}_2\phi_1\left(\begin{array}{c} q^{\gamma+\delta+\lambda}, q^{-\nu} \\ q^{\gamma+\lambda} \end{array} ;q,{}^{a}\!/_{xq^{\delta}}\right) \tag{48}$$

(iii). In Theorem 4.3, if we take

$$m_1 = 1, n_1 = 0, A = 0, B = 2, \rho = -{}^1\!/_q, b_1 = 0, \beta_1 = 1, b_2 = 1, \beta_2 = 1, \lambda = 1$$

the function ${}_r\phi_s$ reduces to $e_q(x)$ defined by (11) and we obtain the following results

$$D_q^{\gamma,\delta}\left(e_q(x)\right) = G(q)\,(1-q)^{-\delta} \times H_{1,3}^{1,1}\left(-\frac{x}{q};q \left| \begin{array}{c} (-\gamma-\delta,1) \\ (0,0),(1,1)(-\gamma,1) \end{array}\right.\right) \tag{49}$$

$$P_q^{\gamma,\delta}\left(e_q(x)\right)=G(q)(1-q)^{-\delta}\times H_{1,3}^{2,0}\left(-xq^{\delta-1};q\left|\begin{array}{c}(\gamma,1)\\(\gamma+\delta,1)(0,1),(1,1)\end{array}\right.\right) \tag{50}$$

(iv).In Theorem 4.3, if we take
$\mu=\nu, m_1=1, n_1=0, A=0, B=3, \rho={}^1\!/_{4q}, b_1=0, \beta_1=1, b_2=-\nu, \beta_2=1, b_3=1, \beta_3=1, \lambda=2$ the basic H-function reduce into basic analogue of Bessel function as given by (17) and we get the following results

$$D_q^{\gamma,\delta}\left(J_\nu^{(1)}(x;q)\right)=\left({}^x\!/_2\right)^\nu[G(q)]^2(1-q)^{-\delta}$$

$$\times H_{1,4}^{1,1}\left(\frac{x^2}{4q};q\left|\begin{array}{c}(-\gamma-\delta-\nu,1)\\(0,1),(-\nu,1)(1,1)(-\gamma-\nu,2)\end{array}\right.\right) \tag{51}$$

$$P_q^{\gamma,\delta}\left(J_\nu^{(1)}(x;q)\right)=\left({}^x\!/_2\right)^\nu[G(q)]^2(1-q)^{-\delta}$$

$$\times H_{1,4}^{2,0}\left(\frac{x^2q^{2\delta-1}}{4};q\left|\begin{array}{c}-,(-\gamma-\delta-\nu,1)\\(0,1)(\gamma+\delta+\nu,2),(-\nu,1)(1,1)\end{array}\right.\right) \tag{52}$$

On reducing the basic analogue of H-function occurring in Theorem 4.3 to many more q-special functions as given in [10], we can obtain the corresponding results for these functions. However, we omit the details here.

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