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A GENERALIZATION OF BERNOULLI'S INEQUALITY

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We prove the following generalization of Bernoulli's inequality

$$\left(\sum_{k\leq K} c_k \prod_{j=1}^{J}(1+a_{jk})\right)^s \leq \sum_{k\leq K} c_k \prod_{j=1}^{J}(1+sa_{jk})$$

where $0 \leq s \leq 1$, under suitable conditions on the a_{jk} and the c_k. We also prove the opposite inequality when $s \geq 1$. These inequalities can be applied to Weierstrass product inequalities.

1. Introduction

The classical Bernoulli inequality is

$$(1+x)^s \leq 1+sx \tag{1}$$

for $x > -1$ and $0 \leq s \leq 1$. For $s > 1$, the inequality reverses. This has been generalized in a number of ways. See Mitrinović and Pećarić [5] for a survey. The version in this paper, Theorem 1.1 below, can be expressed in terms of matrices. It is more general than the versions in [7] and [8]. See also [3].

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Theorem 1.1. *Let $A = (a_{jk})$ be a real $J \times K$ matrix, with $a_{jk} > -1$ for all $1 \leq j \leq J$ and $1 \leq k \leq K$. Let $c_k > 0$ for all k, and let $S = \sum_{k=1}^{K} c_k$. Assume $m \leq \prod_{j=1}^{J}(1+a_{jk})^{\frac{1}{J}} - 1 \leq M$. Consider the following conditions on s and A:*

(C1) $0 \leq s \leq 1$, *and*

$$S^{\frac{s-1}{J}}(1+M)^s \leq 1+sm, \tag{2}$$

(C2) $0 \leq s \leq 1$, $c_1 \geq 1$, *and for all* j, k, $a_{jk} \leq a_{j1}$.

(C3) $s > 1$ *and* $a_{jk} > -\frac{1}{s}$ *for all* j, k, *and*

$$S^{\frac{s-1}{J}}(1+m)^s \geq 1+sM, \tag{3}$$

(C4) $s > 1$, $c_1 \geq 1$, *and for all* j, k, $-\frac{1}{s} \leq a_{jk} \leq a_{j1}$.

Define

$$L(A) = \left(\sum_{k \leq K} c_k \prod_{j=1}^{J}(1+a_{jk})\right)^s \quad \text{and} \quad R(A) = \sum_{k \leq K} c_k \prod_{j=1}^{J}(1+sa_{jk}).$$

Then if (C1) or (C2) hold, $L(A) \leq R(A)$ while if (C3) or (C4) hold, $L(A) \geq R(A)$.

We require J and K to be finite for simplicity; the theorem can easily be extended to the infinite case by taking limits. A special case worth mentioning is when $K = 1$, $c_1 = 1$ and $0 \leq s \leq 1$. The inequality $L(A) \leq R(A)$ reduces to

$$\left(\prod_{j=1}^{J}(1+a_{j1})\right)^s \leq \prod_{j=1}^{J}(1+sa_{j1}) \tag{4}$$

which is a straightforward consequence of (1).

Observe that when $s > 1$, the condition $a_{j1} > -\frac{1}{s}$ is necessary for $L(A) \geq R(A)$ (or even the reverse of (4)). Without this condition, the product on the right hand side might include an even number of large negative factors.

Here is an example of how our results can be useful in a perhaps surprising manner.

Example. If $0 < s < 1$, the following inequality holds

$$\left(1+\int_0^1 \frac{\sin x}{x}dx\right)^s \leq 1+\int_0^1 \frac{\sin(\sqrt{s}x)}{\sqrt{s}x}dx. \tag{5}$$

Recalling that, for $x \in [-\pi, \pi]$, $\frac{\sin x}{x} = \prod_{j=1}^{\infty}\left(1 - \frac{x^2}{(\pi j)^2}\right)$ (see e.g. [1]) we can prove (5) by applying the theorem to the Riemann sums of these integrals. That is, given $0 \le s \le 1$ and a regular partition $\{x_k\}_{k=1,\dots,K}$ of $[0,1]$, let $a_{jk} = -\frac{x_k^2}{(\pi j)^2}$, for $j,k \ge 1$ and let $a_{j0} = 0$ (for simplicity, we may start with $k = 0$ instead of $k = 1$). Let $c_k = \frac{1}{K}$ for $k \ge 1$ and let $c_0 = 1$, so (C2) holds. Then

$$\left(1 + \sum_{k \le K} \frac{1}{K} \prod_{j=1}^{J}\left(1 - \frac{x_k^2}{(\pi j)^2}\right)\right)^s = L(A) \le R(A) = 1 + \sum_{k \le K} \frac{1}{K} \prod_{j=1}^{J}\left(1 - \frac{s x_k^2}{(\pi j)^2}\right),$$

and the inequality (5) follows by letting J and K go to infinity. We could prove a slightly more general version of (5) by replacing $[0,1]$ with $[a,b]$, where $-\pi < a < b < \pi$.

In section 2, we show that conditions like the (Ci) above are necessary for universal comparability of $L(A)$ and $R(A)$, and then we prove our main results. In section 3, we apply our generalized Bernoulli inequalities to prove new Weierstrass product inequalities.

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2. Remarks on the (Ci) and the proof of Theorem 1.1

We cannot suggest any simple necessary and sufficient conditions for the inequality $L(A) \le R(A)$ in Theorem 1.1, but will show that with weaker conditions than the given (Ci) it can fail. For example, consider

(C5) $0 \le s \le 1$, and $-1 < a_{jk}$ for all j, k.

(C6) $s \ge 1$, and $-\frac{1}{s} < a_{jk}$ for all j, k.

The following example shows (C5) cannot replace (C1) or (C2) in the theorem. Let $s = 1/2$ and let $c_k = 1/K$ for all k. Let $a_{j1} = 1$ for all j and let $a_{jk} = 0$ whenever $k \ge 2$. So, (C5) holds, but neither (C1) nor (C2) do. The product in $L(A)$ is 2^J when $k = 1$, and otherwise is 1. The product in $R(A)$ is $(3/2)^J$ when $k = 1$, and otherwise is 1. So,

$$L(A) = K^{-1/2}[2^J + (K-1)]^{1/2},$$

$$R(A) = K^{-1}[(3/2)^J + (K-1)].$$

After multiplying both sides by $K = (4/3)^J$ (or the nearest integer)

$$K \cdot L(A) \ge K^{\frac{1}{2}} 2^{J/2} = (8/3)^{J/2},$$

while

$$K \cdot R(A) \leq (3/2)^J + (4/3)^J \leq 2(3/2)^J.$$

Since $(8/3)^{1/2} > 3/2$, $L(A) > R(A)$ for large enough J, so we have an example that shows that (C5) is insufficient for $L(A) \leq R(A)$.

We now show that if (C3) and (C4) are replaced by the weaker (C6), then $L(A) \geq R(A)$ is not necessarily true. Set $s = 2$, $K = 2$, and $a_{j1} = 0$, and $a_{j2} = 1$ for all j's. We also let $c_2 = 2^{-J}$ and $c_1 = 1 - 2^{-J}$ with $J \geq 4$. Then, $L(A) = (2 - 2^{-J})^2 < 4$, and $R(A) = 2^{-J}(1+2)^J + (1 - 2^{-J}) > (3/2)^J > 4 > L(A)$.

The a_{jk} in the examples above "spike" as a function of k; that is, there is a value of k for which a_{jk} is "much larger" than the average. M is much bigger than m. The conditions (C1)-(C4) can be viewed as anti-spiking conditions. For example, with spiking (2) in (C1) is a fairly strong condition on s and S. But when $S = 1$ and $m = M$, 2 is just the standard Bernoulli inequality; and (3) is similar.

So, (C1)-(C4) cannot be replaced by the simpler (C5) or (C6), but it is possible that they can be weakened in other ways. For example, in the special case, $J = 1$, $0 \leq s \leq 1$, and $S = \sum_{k \leq K} c_k = 1$, the inequality $L(A) \leq R(A)$ holds without any anti-spiking assumptions. Indeed,

$$\left(\sum_{k \leq K} c_k(1 + a_{1k})\right)^s = \left(1 + \sum_{k \leq K} c_k a_{1k}\right)^s \leq 1 + s \sum_{k \leq K} c_k a_{1k} = \sum_{k \leq K} c_k(1 + s a_{1k}). \tag{6}$$

Lemma 2.1 below reduces the proof of Theorem 1.1 to the special case in which the a_{jk} do not depend on j. Assume the s and $A = (a_{jk})$ satisfy (C5) or (C6). Define $p_k = \prod_{j=1}^J (1 + a_{jk})^{1/J} - 1$. Recall that in Theorem 1.1 we have assumed $m \leq p_k \leq M$. Without loss of generality we can assume $M = \max_k p_k$ and $m = \min_k p_k$. Replacing each a_{jk} by p_k defines a new matrix P with $L(P) = L(A)$. Also, if A satisfies any of the conditions Ci $(1 \leq i \leq 4)$ in Theorem 1.1, then P does too.

Lemma 2.1. *With P as above:*

a) If A satisfies (C5), then $R(P) \leq R(A)$.

b) If A satisfies (C6), then $R(P) \geq R(A)$.

Proof. We consider only case a), since the proof in case b) is similar. So, $0 < s \leq 1$ and $-1 < a_{jk}$. Consider the problem of finding a $J \times K$ matrix $B = (b_{jk})$ such that

$$\prod_{j=1}^J (1 + b_{jk}) = \prod_{j=1}^J (1 + a_{jk})$$

for all k, which minimizes $R(B)$. For compactness, we also require that

$$\min a_{jk} \le b_{jk} \le \max a_{jk}$$

for all j,k,which insures that a solution B exists, with $R(B) \le R(A)$. We will show that $b_{ik} = b_{jk}$ for all i,j,k, which implies $B = P$ and proves the Lemma. If not, then without loss of generality, i and j are 1 and 2, $k = 1$ and $b_{11} < b_{21}$.

Define β by $(1+\beta)^2 = (1+b_{11})(1+b_{21})$. Define a new matrix $\overline{B}$ from B by replacing both b_{11} and b_{21} with β. This substitution does not change $\prod_{j=1}^{J}(1+b_{j1})$, so $\overline{B}$ is admissible for the minimization problem above. Also, the variables b_{11} and b_{21} only occur in one term of $R(B)$:

$$(1+sb_{11})(1+sb_{21}) = ([1-s]+s[1+b_{11}])([1-s]+s[1+b_{21}]).$$

So, for some positive constant c, $R(B) - R(\overline{B}) = c([1+b_{11})] + [1+b_{21}] - 2[1+\beta]) = c([1+b_{11}]^{1/2} - [1+b_{21}]^{1/2})^2 > 0$. This contradicts our assumption that B minimizes R and completes the proof of the Lemma. □

We now proceed with the proof of Theorem 1.1.

Proof that (C1) implies $L(A) \le R(A)$. With P defined as above, $L(A) = L(P)$, and by Lemma 2.1, $R(P) \le R(A)$. Then,

$$\begin{aligned} L(P) &= \left(\sum_{k\le K} c_k(1+p_k)^J\right)^s \\ &\le \left(\sum_{k\le K} c_k(1+M)^J\right)^s = S^s(1+M)^{sJ} \le S(1+ms)^J \\ &\le \sum_{k\le K} c_k \prod_{j=1}^{J}(1+sp_k) = R(P), \end{aligned}$$

proving $L(A) \le R(A)$. The proof that (C3) implies $L(A) \ge R(A)$ is similar.

Proof that (C2) implies $L(A) \le R(A)$. By Lemma 2.1, it suffices to prove that $L(P) \le R(P)$. By (C1) and the definition of p_k, $-1 < p_k \le p_1$, for all k; also, $c_1 \ge 1$. The calculation below uses the facts that $s \to a^s$ is an increasing function of s when $a > 1$ and is decreasing when $a < 1$ in the first two inequalities, and Bernoulli 1 in the last one:

$$L(P) = \left(\sum_{k\le K} c_k(1+p_k)^J\right)^s$$

$$
\begin{aligned}
&= c_1^s(1+p_1)^{sJ}\left(1+\sum_{k=2}^{K} c_1^{-1}c_k\left(\frac{1+p_k}{1+p_1}\right)^J\right)^s \\
&\leq c_1^s(1+p_1)^{sJ}\left(1+\sum_{k=2}^{K} c_1^{-1}c_k\left(\frac{1+p_k}{1+p_1}\right)^J\right) \\
&\leq c_1^s(1+p_1)^{sJ}\left(1+\sum_{k=2}^{K} c_1^{-1}c_k\left(\frac{1+p_k}{1+p_1}\right)^{sJ}\right) \\
&= c_1^s(1+p_1)^{sJ}+\sum_{k=2}^{K} c_1^{s-1}c_k\,(1+p_k)^{sJ} \\
&\leq c_1(1+sp_1)^J+\sum_{k=2}^{K} c_k\,(1+sp_k)^J = R(P).
\end{aligned}
$$

and the proof is complete. The proof that (C4) implies $L(A) \geq R(A)$ is similar.

We conclude this section with a proposition which is very similar in spirit to Lemma 2.1. That Lemma showed that $R(A)$ is minimal (subject to certain constraints) when $A = P$; that is, when the a_{jk} depend only on k. Our next proposition goes further; if $L = L(B)$ is fixed, then the minimal $R(B)$ occurs when $b_{jk} = b_k$ has only 3 distinct values. In effect, it shows we may assume $K \leq 3$. Though not used in this paper, we believe that this second reduction may have independent interest, and might be applied to other generalized Bernoulli inequalities.

Proposition 2.2. *Fix $0 \leq s \leq 1$, $\{c_k\}$, J, K, L and $-1 < m < M$. Consider the family of all matrices B with constant columns determined by the b_k, and so that $m \leq b_k \leq M$ for $k = 1, \ldots K$, and $L(B) = L$. If such B minimizes $R(B)$, then all b_k, with the possible exception of one, are equal to m or M. This conclusion is also true in the opposite case; when $s > 1$, $m > -\frac{1}{s}$ and $R(B)$ is maximal.*

Proof. Suppose first that $0 < s < 1$. We will show that one element in every pair of b_k's is either m or M. We may take the pair to be $\{b_1, b_2\}$ and can assume $b_1 \leq b_2$. We will treat these elements as variables in the interval $[m, M]$ and consider minimizing $R(B)$. Let $t_1 = (1+b_1)^J$ and $t_2 = (1+b_2)^J$, with $(1+m)^J \leq t_1 \leq t_2 \leq (1+M)^J$. Fix $c_1t_1 + c_2t_2 = T$, so that changing the t_i will not affect $L(B)$. We will show that $R(B)$ is minimal when at least one of the t_i is an endpoint, meaning that one of the b_i is an endpoint. The summand in $R(B)$ affected by t_i is equal to $(st_i^{\frac{1}{J}} + 1 - s)^J$. Thus, we minimize

$$f(t_1) = c_1(st_1^{\frac{1}{J}} + 1 - s)^J + c_2(st_2^{\frac{1}{J}} + 1 - s)^J.$$

where t_2 is a function of t_1. Note that $\frac{dt_2}{dt_1} = -\frac{c_1}{c_2}$. So,

$$
\begin{aligned}
\frac{df(t_1)}{dt_1} &= sc_1(st_1^{\frac{1}{J}}+1-s)^{J-1}t_1^{\frac{1}{J}-1} - sc_1(st_2^{\frac{1}{J}}+1-s)^{J-1}t_2^{\frac{1}{J}-1} \\
&= sc_1[(1+sb_1)^{J-1}(1+b_1)^{1-J} - (1+sb_2)^{J-1}(1+b_2)^{1-J}]
\end{aligned}
$$

It is easy to check that, for $s<1$, $\frac{1+sx}{1+x}$ decreases with x. So, $f'>0$, which implies the minimum of f occurs when t_1 is minimal or when t_2 is maximal, and we are done. When $s>1$, $\frac{1+sx}{1+x}$ increases with x, so $f'<0$, and the maximum of f occurs when t_1 is minimal or t_2 is maximal. □

3. Weierstrass Inequalities

We can use Theorem 1.1 to prove some Weierstrass product inequalities. In order to study the convergence of infinite products, it is useful to find lower bounds for products of the form $\prod_{i=1}^{n}(1+x_i)$ and $\prod_{i=1}^{n}(1-x_i)$ which are named after K. Weierstrass, in terms of linear functions. K. Weierstrass was probably the first to prove the following inequalities:

$$1+\sum_{i=1}^{n}x_i \leq \prod_{i=1}^{n}(1+x_i), \qquad x_i>-1, \tag{7}$$

and if $0\leq x\leq 1$

$$1-\sum_{i=1}^{n}x_i \leq \prod_{i=1}^{n}(1-x_i). \tag{8}$$

These inequalities and their generalizations have attracted a lot of interest. See for example [2], [4], [9] just to cite a few. The following is a corollary of Theorem 1.1:

Theorem 3.1. *Let $c_k\geq 0$ and $0\leq a_{jk}<1$ and $0<s<1$. Then,*

$$\left(1+\sum_{k\leq K}c_k\prod_{j=1}^{J}(1-a_{jk})\right)^s \leq 1+\sum_{k\leq K}c_k\prod_{j=1}^{J}(1-sa_{jk}). \tag{9}$$

When $s>1$ and $0\leq a_{jk}\leq\frac{1}{s}$ the inequality reverses.

Proof. In Theorem 1.1, set $c_1=1$ and $a_{j1}=0$ for all j. For $k\geq 2$, replace a_{jk} by $-a_{j,k-1}$, with a similar re-indexing of the c_k. Then (C2) is satisfied and Theorem 1.1 gives immediately Theorem 3.1. For the reverse, apply (C4). □

Theorem 3.2. *Let $a_{jk}\in\left[-\frac{1}{s},1\right]$ and $c_k\geq 0$. Assume $\sum_{k\leq K}c_k=1$ and $s\geq 1$ and $m\leq\prod_{j=1}^{J}(1+a_{jk})^{1/J}-1\leq M$ and*

$$1+sM\leq(1+m)^s.$$

Then

$$1 + s \sum_{\substack{k \le K \\ j \le J}} c_k a_{jk} \le \left(\sum_{k \le K} c_k \prod_{j=1}^{J} (1 + a_{jk}) \right)^s .$$

Proof. By (C3) of Theorem 1.1, and (7),

$$\left(\sum_{k \le K} c_k \prod_{j=1}^{J} (1 + a_{jk}) \right)^s \ge \sum_{k \le K} c_k \prod_{j=1}^{J} (1 + s a_{jk})$$

$$\ge \sum_{k \le K} c_k \left(1 + s \sum_{j \le J} a_{jk} \right) = 1 + s \sum_{\substack{k \le K \\ j \le J}} c_k a_{jk}$$

as required. □

Theorem 3.3. *Let $s \ge 1$ and let $a_{jk} \in [0, 1/s]$; Let $c_k \ge 0$. Then*

$$2 - s \sum_{\substack{k \le K \\ j \le J}} c_k a_{jk} \le \left(1 + \sum_{k \le K} c_k \prod_{j=1}^{J} (1 - a_{jk}) \right)^s .$$

Proof. Follows from Theorem 3.1 and (8). Indeed,

$$\left(1 + \sum_{k \le K} c_k \prod_{j=1}^{J} (1 - a_{jk}) \right)^s \ge 1 + \sum_{k \le K} c_k \prod_{j=1}^{J} (1 - s a_{jk})$$

$$\ge 1 + \sum_{k \le K} c_k \left(1 - s \sum_{j \le J} a_{jk} \right) = 2 - s \sum_{\substack{k \le K \\ j \le J}} c_k a_{jk}.$$

□

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