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WEAK CONVERGENCE OF JACOBIAN DETERMINANTS UNDER ASYMMETRIC ASSUMPTIONS

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Let Ω be an open subset of $\mathbb{R}^2$ and assume that $f_k = (u_k, v_k)$, $k = 1, 2, \ldots$, and $f = (u, v)$ are mappings in the Sobolev space $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$. We prove that if one allows different assumptions on the two components of f_k and f, e.g.

$$u_k \rightharpoonup u \text{ weakly in } W^{1,2}_{\mathrm{loc}}(\Omega) \qquad v_k \rightharpoonup v \text{ weakly in } W^{1,q}_{\mathrm{loc}}(\Omega)$$

for some $q \in (1, 2)$, then

$$J_{f_k} \overset{*}{\rightharpoonup} J_f \quad \text{in } \mathcal{M}(\Omega), \tag{1}$$

i.e.

$$\int_\Omega J_{f_k} \varphi \, dz \to \int_\Omega J_f \varphi \, dz, \quad \forall \varphi \in C^0_0(\Omega).$$

Moreover, we show that this result is optimal in the sense that conclusion fails for $q = 1$.

On the other hand, we prove that (1) remains valid also if one considers the case $q = 1$, but it is necessary to require that u_k weakly converges to u in a Zygmund-Sobolev space with a slightly higher degree of regularity than $W^{1,2}_{\mathrm{loc}}(\Omega)$ and precisely

$$u_k \rightharpoonup u \text{ weakly in } W^{1, L^2 \log^\alpha L}_{\mathrm{loc}}(\Omega)$$

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for some $\alpha > 1$.

Finally we present an extension to Orlicz-Sobolev setting of the previous results.

1. Introduction and statement of the results

In [7] a general weak continuity result has been established for Jacobian determinants of $W^{1,N}(\Omega,\mathbb{R}^N)$-Sobolev mappings. We will state it here in the particular case $N = 2$ and in the local form.

Theorem 1.1 ([7]). *Let Ω be an open subset of $\mathbb{R}^2$. If*

$$\{f_k\} \subset W^{1,2}_{loc}(\Omega,\mathbb{R}^2),$$

$$f_k \rightharpoonup f \ \textit{ weakly in } \ W^{1,1}_{loc}(\Omega,\mathbb{R}^2) \tag{2}$$

and

$$J_{f_k} \overset{*}{\rightharpoonup} \mu \quad \textit{in} \ \mathcal{M}(\Omega),$$

then

$$d\mu = J_f \, dz + d\mu^s,$$

where μ^s is a singular measure with respect to the Lebesgue measure on Ω.

This is a generalization of the classical result (Reshetnyak [16], Ball [2]) that tells us that if

$$\{f_k\} \subset W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^2),\, f \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^2), \tag{3}$$

then the stronger assumption than (2)

$$f_k \rightharpoonup f \ \text{ weakly in } \ W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^2) \ \text{ for some } p > 4/3$$

implies the stronger conclusion

$$J_{f_k} \overset{*}{\rightharpoonup} J_f \quad \text{in} \ \mathcal{M}(\Omega). \tag{4}$$

Moreover, Dacorogna-Murat (see [6]) show that this weak continuity result is optimal in sense that it does not hold true when $p = 4/3$.

Our aim here is to prove that (3) together with an asymmetric assumption on the two components of f_k and f, guarantee that (4) holds true.

The first result in this direction is the following theorem, which may be deduced from a very recent div-curl result contained in [4].

Proposition 1.2. *Let Ω be an open subset of $\mathbb{R}^2$. If*

$$f_k = (u_k, v_k) \rightharpoonup f = (u, v) \ \text{weakly in}\ W^{1,2}_{loc}(\Omega) \times W^{1,q}_{loc}(\Omega) \tag{5}$$

for some $q \in (1,2)$, then

$$\operatorname{Det} Df_k \rightharpoonup \operatorname{Det} Df \quad \text{in}\ \mathcal{D}'(\Omega), \tag{6}$$

i.e.

$$\langle \operatorname{Det} Df_k, \varphi \rangle \to \langle \operatorname{Det} Df, \varphi \rangle \quad \forall \varphi \in C_0^\infty(\Omega).$$

In particular, under the assumptions (3) *and* (5) *we have*

$$J_{f_k} \overset{*}{\rightharpoonup} J_f \quad \text{in}\ \mathcal{M}(\Omega). \tag{7}$$

Moreover, this result is optimal in sense that the conclusion fails for $q = 1$.

In Section 3 we give a simpler proof of the previous proposition.

Remark 1.3. We also consider the case $q = 1$, but to do this we need that u_k weakly converge to u in a Zygmund-Sobolev space with a slightly higher degree of regularity than $W^{1,2}_{\text{loc}}(\Omega)$. Precisely we have the following result.

Proposition 1.4. *Let Ω be an open subset of $\mathbb{R}^2$. If*

$$f_k = (u_k, v_k) \rightharpoonup f = (u, v) \ \text{weakly in}\ W^{1,L^2 \log^\alpha L}_{loc}(\Omega) \times W^{1,1}_{loc}(\Omega) \tag{8}$$

for some $\alpha > 1$, then (6) *holds true. In particular, under the assumptions* (3) *and* (8) *we get* (7).

Our main result is the next theorem, which represents an extension to Orlicz-Sobolev setting.

Theorem 1.5. *Let Ω be an open subset of $\mathbb{R}^2$. If*

$$\{f_k\} \subset W^{1,1}_{loc}(\Omega, \mathbb{R}^2),\ f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2) \tag{9}$$

and

$$f_k = (u_k, v_k) \rightharpoonup f = (u, v) \ \text{weakly in}\ W^{1,\Phi}_{loc}(\Omega) \times W^{1,\Psi}_{loc}(\Omega), \tag{10}$$

where Φ and Ψ are Young functions such that

$$\widetilde{\Phi} \prec\prec \widehat{\Psi} \quad \text{near infinity} \quad \text{and} \quad \widetilde{\Psi} \preceq \widehat{\Phi} \quad \text{near infinity}, \tag{11}$$

then (6) *holds true. In particular, under the assumptions* (3), (10) *and* (11) *we obtain* (7).

(Here $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are the Young conjugate functios of Φ and Ψ respectively, while $\widehat{\Phi}$ and $\widehat{\Psi}$ are suitable Young functions defined in terms of $\widetilde{\Phi}$ and $\widetilde{\Psi}$, that have been introduced by Cianchi ([5]).)

We will define the Orlicz-Sobolev spaces $W^{1,\Phi}$, the particular Young functions $\widetilde{\Phi}$, $\widehat{\Phi}$, the symbols $\preceq$, $\prec\prec$ and the distributional Jacobian determinant $\operatorname{Det} Df$ in Section 2. We will prove our results in Section 3.

2. Notations, definitions and preliminary results

For the reader's convenience we recall that a function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if it has the form

$$\Phi(t) = \int_0^t \phi(\tau)\,d\tau \qquad \text{for } t \geq 0,$$

where $\phi : [0,\infty) \to [0,\infty]$ is an increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0,\infty)$. In particular, a Young function is convex and vanishes at 0.

The Young conjugate $\widetilde{\Phi}(t)$ of Φ is the Young function defined by

$$\widetilde{\Phi}(t) = \sup\{st - \Phi(s) : s \geq 0\} \qquad \text{for } t \geq 0$$

and satisfies

$$\widetilde{\Phi}(t) = \int_0^t \phi^{-1}(\sigma)\,d\sigma \qquad \text{for } t \geq 0,$$

where ϕ^{-1} is the (generalized) left-continuous inverse of ϕ. Notice that $\widetilde{\widetilde{\Phi}} = \Phi$.

Let Ω be a measurable subset of $\mathbb{R}^2$ and let Φ be a Young function. The Orlicz space $L^\Phi(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi\left(\frac{|u(z)|}{\lambda}\right) dz < \infty,$$

for some $\lambda = \lambda(u) > 0$ and it is equipped with the Luxemburg norm

$$\|u\|_{L^\Phi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|u(z)|}{\lambda}\right) dz \leq 1\right\}.$$

Note that, if $\Phi(t) = t^p$ and $1 \leq p < \infty$, then $L^\Phi(\Omega) = L^p(\Omega)$, the classical Lebesgue space, and $\|u\|_{L^\Phi(\Omega)} = p^{-1/p}\|u\|_{L^p(\Omega)}$; if $\Phi(t) \equiv 0$ for $0 \leq t \leq 1$ and $\Phi(t) \equiv \infty$ for $t > 1$, then $L^\Phi(\Omega) = L^\infty(\Omega)$ and $\|u\|_{L^\Phi(\Omega)} = \|u\|_{L^\infty(\Omega)}$.

The following generalized version of Hölder's inequality holds:

$$\int_\Omega u(z)v(z)\,dz \leq 2\,\|u\|_{L^\Phi(\Omega)}\|v\|_{L^{\widetilde{\Phi}}(\Omega)}, \tag{12}$$

for $u \in L^\Phi(\Omega)$ and $v \in L^{\widetilde{\Phi}}(\Omega)$.

A function Φ is said to dominate another function Ψ near infinity, and we write

$$\Psi \preceq \Phi \quad \text{near infinity},$$

if

$$\exists c > 0 \quad \exists t_\infty > 0 \;:\; \Psi(t) \leq \Phi(ct) \quad \forall t \geq t_\infty.$$

Two functions Φ and Ψ are called equivalent near infinity, and we write

$$\Phi \approx \Psi \quad \text{near infinity},$$

if

$$\Psi \preceq \Phi \quad \text{near infinity} \quad \text{and} \quad \Phi \preceq \Psi \quad \text{near infinity}.$$

A function Ψ is said to increase essentially more slowly than a function Φ, and we write

$$\Psi \prec\prec \Phi \quad \text{near infinity}$$

if

$$\forall c > 0 \quad \exists t_c \geq 0 \, : \, \Psi(t) \leq \Phi(ct) \quad \forall t \geq t_c.$$

Assume that $|\Omega| < +\infty$. Then the continuous embedding

$$L^{\Phi}(\Omega) \hookrightarrow L^{\Psi}(\Omega) \tag{13}$$

holds if and only if

$$\Psi \preceq \Phi \quad \text{near infinity}.$$

In particular

$$L^{\Phi}(\Omega) = L^{\Psi}(\Omega)$$

if and only if

$$\Phi \approx \Psi \quad \text{near infinity}.$$

The local Orlicz space $L^{\Phi}_{loc}(\Omega)$ (Ω any measurable subset of $\mathbb{R}^2$, Φ any Young function) is defined as the set of all measurable functions $u : \Omega \longrightarrow \mathbb{R}$ that belong to $L^{\Phi}(K)$ for every compact set $K \subseteq \Omega$. It is a locally convex topological vector space with the family of seminorms

$$\{|| \quad ||_{L^{\Phi}(K)} : \qquad K \subseteq \Omega, \quad K \text{compact}\}.$$

It follows from the above embeddings between L^{Φ} spaces that the continuous embedding

$$L^{\Phi}_{loc}(\Omega) \hookrightarrow L^{\Psi}_{loc}(\Omega)$$

holds if and only if

$$\Psi \preceq \Psi \quad \text{near infinity},$$

so, in particular,

$$L^{\Phi}_{loc}(\Omega) = L^{\Psi}_{loc}(\Omega)$$

if and only if

$$\Phi \approx \Psi \quad \text{near infinity}.$$

For more details and proofs of results about Young functions and Orlicz spaces, we refer to [1], [3], [11] and [15].

Let Ω be an open subset of $\mathbb{R}^2$ and let Φ be a Young function. The Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is defined as the space of functions u weakly differentiable on Ω such that

$$u, |\nabla u| \in L^{\Phi}(\Omega)$$

and it is equipped with the norm

$$\|u\|_{W^{1,\Phi}(\Omega)} = \|u\|_{L^{\Phi}(\Omega)} + \||\nabla u|\|_{L^{\Phi}(\Omega)}.$$

Clearly, if $\Phi(t) = t^p$ and $1 \le p < \infty$, then $W^{1,\Phi}(\Omega) = W^{1,p}(\Omega)$, the standard Sobolev space.
The space $W^{1,\Phi}_{\mathrm{loc}}(\Omega)$ is defined as the space of functions belonging to $W^{1,\Phi}(\Omega')$ for every $\Omega' \subset\subset \Omega$, that is for every open set Ω' such that $\overline{\Omega'} \subseteq \Omega$ and $\overline{\Omega'}$ is compact. It is a locally convex topological vector space with the family of seminorms

$$\{\|\quad\|_{W^{1,\Phi}(\Omega')} : \Omega' \subset\subset \Omega\}.$$

Properties of Orlicz-Sobolev spaces are presented in [1].

For any Young function Φ such that $\int_0 \widetilde{\Phi}(t)/t^3\,dt < \infty$, we denote by $\widehat{\Phi}$ the Young function defined by

$$\widehat{\Phi}(t) = \int_0^t (a^{-1}(\sigma^2))^2 \sigma\, d\sigma, \tag{14}$$

where a^{-1} is the (generalized right-continuous) inverse of

$$a(\tau) = \int_0^\tau \frac{\widetilde{\Phi}(t)}{t^3}\,dt.$$

Now, we are in a position to recall the following embedding results for Orlicz-Sobolev spaces $W^{1,\Phi}$ due to Cianchi (see [5], Theorem 2 and Theorem 3).

We state them here with regard to the local spaces $W^{1,\Phi}_{\mathrm{loc}}$. These "local" statements readily follow from the original ones by Cianchi by standard arguments. In particular, we use the fact that for every open set $\Omega \subseteq \mathbb{R}^2$ the topology of $W^{1,\Phi}_{\mathrm{loc}}(\Omega)$ [resp. $L^{\Phi}_{\mathrm{loc}}(\Omega)$] is determined by any sequence of seminorms

$$\|\quad\|_{W^{1,\Phi}(\Omega_k)} \qquad k = 1,2,\dots$$
$$[\text{resp. } \|\quad\|_{L^{\Phi}(\Omega_k)} \qquad k = 1,2,\dots],$$

where $\qquad \Omega_k \subset\subset \Omega_{k+1} \subset\subset \Omega \qquad k = 1,2,\dots$

and

$$\bigcup_{k=1}^{\infty} \Omega_k = \Omega;$$

moreover, the open sets Ω_k, $k = 1, 2, \ldots$ can be chosen to satisfy all smoothness assumptions required by Cianchi's theorems. This allows us to use a diagonal process of taking subsequences in order to get the compact embedding result.

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R}^2$ and let Φ be any Young function (which can be modified near zero, if necessary, so that $\int_0 \widetilde{\Phi}(t)/t^3\, dt < \infty$). Then we have the following continuous embedding*

$$W^{1,\Phi}_{loc}(\Omega) \hookrightarrow L^{\bar{\Phi}}_{loc}(\Omega), \tag{15}$$

where

$$\bar{\Phi}(t) = \begin{cases} \Phi(t) & \text{if } 0 \le t \le t_1 \\ \widehat{\Phi}(t) & \text{if } t \ge t_2 \end{cases} \tag{16}$$

for suitable $0 < t_1 < t_2$. Moreover, we have the following compact embedding

$$W^{1,\Phi}_{loc}(\Omega) \hookrightarrow\hookrightarrow L^{\Psi}_{loc}(\Omega), \tag{17}$$

if

$$\Psi \prec\prec \widehat{\Phi} \quad \text{near infinity}.$$

We will denote by

$$L^p \log^\gamma L(\Omega)$$

$$L^p \log^\gamma \log L(\Omega)$$

the Orlicz spaces $L^\Phi(\Omega)$ generated respectively by the following Young functions

$$\Phi(t) \approx t^p \log^\gamma(e+t) \qquad \text{near infinity}$$
$$\Phi(t) \approx t^p \log^\gamma \log(e+t) \qquad \text{near infinity}$$

where either $p = 1$ and $\gamma \ge 0$ or $p > 1$ and $\gamma \in \mathbb{R}$.

Moreover, we will denote by

$$W^{1,L^p \log^\gamma L}(\Omega)$$

$$W^{1,L^p \log^\gamma \log L}(\Omega)$$

the Orlicz-Sobolev spaces of functions u weakly differentiable on Ω such that

$$u,\, |\nabla u| \in L^p \log^\gamma L(\Omega)$$

$$u,\, |\nabla u| \in L^p \log^\gamma \log L(\Omega)$$

respectively, where either $p = 1$ and $\gamma \geq 0$ or $p > 1$ and $\gamma \in \mathbb{R}$.

The local spaces

$$L^p_{loc} log^\gamma L(\Omega), \qquad W^{1,L^p log^\gamma L}_{loc}(\Omega),$$
$$L^p_{loc} log^\gamma log L(\Omega), \qquad W^{1,L^p log^\gamma log L}_{loc}(\Omega)$$

are defined in an obvious way.

At this point, we pass to introduce the distributional Jacobian determinant. Starting with the work of Morrey [13], Reshetnyak [16] and Ball [2], it is known that one can define the distributional Jacobian determinant $\mathrm{Det}Df$ under fairly weak assumption on $f = (u,v)$ (see the subsequent Remark 2).

Actually, by Nikodym Theorem (see e.g. [12]) follows that for

$$f = (u,v) \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$$

such that

$$u\frac{\partial v}{\partial x},\; u\frac{\partial v}{\partial y},\; v\frac{\partial u}{\partial x},\; v\frac{\partial u}{\partial y} \in L^1_{\mathrm{loc}}(\Omega)$$

the two expressions

$$T_1 = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-u\frac{\partial v}{\partial x}\right) = \mathrm{div}\begin{pmatrix} u\dfrac{\partial v}{\partial y} \\ -u\dfrac{\partial v}{\partial x} \end{pmatrix} \tag{18}$$

and

$$T_2 = \frac{\partial}{\partial x}\left(-v\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right) = \mathrm{div}\begin{pmatrix} -v\dfrac{\partial u}{\partial y} \\ v\dfrac{\partial u}{\partial x} \end{pmatrix} \tag{19}$$

are well defined in the sense of distributions and they agree, so one can put

$$\mathrm{Det}Df = T_1 = T_2,$$

where the equality must be understood in the sense of distributions.

Remark 2.2. For $f = (u,v) \in W^{1,4/3}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ the two expressions (18) and (19) are well defined in the sense of distributions and they agree, because by Sobolev embedding Theorem and by Hölder's inequality we have that

$$u\frac{\partial v}{\partial x},\; u\frac{\partial v}{\partial y},\; v\frac{\partial u}{\partial x},\; v\frac{\partial u}{\partial y} \in L^1_{\mathrm{loc}}(\Omega).$$

For our purposes, we need to define $\operatorname{Det} Df$ under different assumptions on the two components u and v of f.

Proposition 2.3. *The two expressions* (18) *and* (19) *are well defined in the sense of distributions and they agree if*

$$f=(u,v)\in W^{1,2}_{loc}(\Omega)\times W^{1,q}_{loc}(\Omega) \quad \textit{for some } \ q\in(1,2)$$

or if

$$f=(u,v)\in W^{1,L^2\log^\alpha L}_{loc}(\Omega)\times W^{1,1}_{loc}(\Omega) \quad \textit{for some } \ \alpha>1.$$

(Hence we define $\operatorname{Det} Df = T_1 = T_2$, where the equality must be understood in the sense of distributions).

Proof. First of all, we suppose that $f=(u,v)\in W^{1,2}_{\mathrm{loc}}(\Omega)\times W^{1,q}_{\mathrm{loc}}(\Omega)$ for some $q\in(1,2)$. By Sobolev embedding Theorem we have that

$$u\in L^{q'}_{\mathrm{loc}}(\Omega), \quad v\in L^2_{\mathrm{loc}}(\Omega),$$

where $q'=q/(q-1)$, and since

$$\frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y}\in L^q_{\mathrm{loc}}(\Omega), \quad \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}\in L^2_{\mathrm{loc}}(\Omega),$$

by Holder's inequality we deduce that

$$u\frac{\partial v}{\partial x},\ u\frac{\partial v}{\partial y},\ v\frac{\partial u}{\partial x},\ v\frac{\partial u}{\partial y}\in L^1_{\mathrm{loc}}(\Omega).$$

Now, we assume that $f=(u,v)\in W^{1,L^2\log^\alpha L}_{\mathrm{loc}}(\Omega)\times W^{1,1}_{\mathrm{loc}}(\Omega)$ for some $\alpha>1$. By Cianchi embedding Theorem (see [5], Example 1) we have that

$$W^{1,L^2\log^\alpha L}_{\mathrm{loc}}(\Omega)\hookrightarrow L^\infty_{\mathrm{loc}}(\Omega)$$

and by (13) it follows that

$$W^{1,1}_{\mathrm{loc}}(\Omega)\hookrightarrow L^2_{\mathrm{loc}}\log^{-\alpha}L(\Omega),$$

therefore

$$u\in L^\infty_{\mathrm{loc}}(\Omega), \quad v\in L^2_{\mathrm{loc}}\log^{-\alpha}L(\Omega),$$

thus, since

$$\frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y}\in L^1_{\mathrm{loc}}(\Omega), \quad \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}\in L^2_{\mathrm{loc}}\log^\alpha L(\Omega),$$

by Holder's inequality and by generalized Holder's inequality (12) we get that

$$u\frac{\partial v}{\partial x},\ u\frac{\partial v}{\partial y},\ v\frac{\partial u}{\partial x},\ v\frac{\partial u}{\partial y}\in L^1_{\mathrm{loc}}(\Omega).$$

The assertion follows by Nikodym Theorem. □

Remark 2.4. If we assume $f = (u, v) \in W^{1,2}_{\text{loc}}(\Omega) \times W^{1,1}_{\text{loc}}(\Omega)$, then only the expression (19) is well defined as a distribution, while the expression (18) is not.

Proposition 2.5. *Let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$. Then the two expressions* (18) *and* (19) *are well defined in the sense of distributions and they agree if*

$$f = (u, v) \in W^{1,\Phi}_{loc}(\Omega) \times W^{1,\Psi}_{loc}(\Omega),$$

where Φ and Ψ are Young functions such that

$$\widetilde{\Phi} \preceq \widehat{\Psi} \quad \textit{near infinity} \quad \textit{and} \quad \widetilde{\Psi} \preceq \widehat{\Phi} \quad \textit{near infinity},$$

with $\widetilde{\Phi}$ and $\widetilde{\Psi}$ Young conjugate of Φ and Ψ respectively and $\widehat{\Psi}$ and $\widehat{\Phi}$ Young functions defined by (14).

Proof. By Cianchi embedding Theorem, see (15), we have that

$$u \in L^{\bar{\Phi}}_{\text{loc}}(\Omega), \quad v \in L^{\bar{\Psi}}_{\text{loc}}(\Omega),$$

where $\bar{\Phi}$ and $\bar{\Psi}$ are functions defined by (16), and since $\widetilde{\Psi} \preceq \widehat{\Phi}$ near infinity and $\widetilde{\Phi} \preceq \widehat{\Psi}$ near infinity, by (13) it follows that

$$u \in L^{\widetilde{\Psi}}_{\text{loc}}(\Omega), \quad v \in L^{\widetilde{\Phi}}_{\text{loc}}(\Omega),$$

whence, by the fact that

$$\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^{\Psi}_{\text{loc}}(\Omega), \quad \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^{\Phi}_{\text{loc}}(\Omega),$$

and by generalized Holder's inequality (12) we obtain that

$$u\frac{\partial v}{\partial x}, u\frac{\partial v}{\partial y}, v\frac{\partial u}{\partial x}, v\frac{\partial u}{\partial y} \in L^1_{\text{loc}}(\Omega).$$

The assertion follows by Nikodym Theorem. □

Now, we compare the Jacobian determinant $J_f = \det Df$ with the distributional Jacobian determinant $\operatorname{Det} Df$. We recall that, if

$$f \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^2)$$

or if

$$f \in W^{1,4/3}_{\text{loc}}(\Omega, \mathbb{R}^2) \text{ and } \operatorname{Det} Df \in L^1_{\text{loc}}(\Omega),$$

then

$$\operatorname{Det} Df = \det Df. \tag{20}$$

(see [14]). Furthermore, considering the grand Lebesgue space $L^{2)}(\Omega)$, introduced by Iwaniec-Sbordone in [10], defined as

$$L^{2)}(\Omega) = \left\{ \varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \mid \sup_{\varepsilon \in (0,1)} \left(\varepsilon \int_\Omega |\varphi(z)|^{2-\varepsilon} dz \right)^{\frac{1}{2-\varepsilon}} < \infty \right\}$$

and denoting by $\Sigma^2(\Omega)$ the subclass of $L^{2)}(\Omega)$ defined as

$$\Sigma^2(\Omega) = \left\{ \varphi \in L^{2)}(\Omega) \mid \lim_{\varepsilon \to 0^+} \varepsilon \int_\Omega |\varphi(z)|^{2-\varepsilon} dz = 0 \right\},$$

it is well known that (20) holds if

$$f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2),\ \det Df \geq 0 \text{ a.e. in } \Omega \text{ and } |Df| \in \Sigma^2_{\mathrm{loc}}(\Omega)$$

(see [8]). Moreover, it is interesting to note as in [8] is shown that the equality (20) remains valid if one assumes

$$\det Df \in L^1_{\mathrm{loc}}(\Omega)$$

and that the two components u and v of f satisfy an asymmetric assumption, namely

$$|\nabla u| \in L^{2)}_{\mathrm{loc}}(\Omega) \ \text{ and } \ |\nabla v| \in \Sigma^2_{\mathrm{loc}}(\Omega).$$

On the other hand, we observe that the identity (20) fails in general if one only assumes

$$f \in W^{1,p}(\Omega, \mathbb{R}^2) \ \text{ for every } \ p \in [1,2),$$

as is shown by the mapping

$$f(z) = \frac{z}{|z|} \qquad \text{for } z \in \mathbb{D}(0,1),$$

where $\mathbb{D}(0,1)$ denotes the disk of $\mathbb{R}^2$ centered at 0 with radius 1. In fact, we have

$$\det Df = 0 \quad \text{a.e.},$$

while

$$\mathrm{Det}\, Df = \pi \delta_0,$$

where δ_0 is the Dirac mass at 0.

Recently Hencl in [9] constructs a homeomorphism

$$f \in W^{1,p}(\Omega, \mathbb{R}^2) \ \text{ for every } \ p \in [1,2)$$

such that

$$\det Df = 0 \quad \text{a.e.}$$

and

$$\mathrm{Det}\, Df \text{ is a singular measure.}$$

3. Proofs of the results

Now, we are able to prove our results. We start to prove Proposition 1.2 and Proposition 1.4, next we will see the proof of Theorem 1.5.

Proofs of Propositions 1.2 and 1.4. By Proposition 2.3 we know that $\mathrm{Det}Df_k$ and $\mathrm{Det}Df$ are well defined in the sense of distributions.
Let $\varphi \in C_0^\infty(\Omega)$ and let $\Omega' \subset\subset \Omega$ be fixed in such a way that the support of φ be contained in Ω'. Integrating by parts we get

$$\langle \mathrm{Det}Df_k, \varphi\rangle = \left\langle \frac{\partial}{\partial x}\left(-v_k \frac{\partial u_k}{\partial y}\right) + \frac{\partial}{\partial y}\left(v_k \frac{\partial u_k}{\partial x}\right), \varphi \right\rangle =$$

$$\int_{\Omega'} v_k \left(\frac{\partial u_k}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial u_k}{\partial x}\frac{\partial \varphi}{\partial y}\right) dz.$$

We can pass to the limit on the right-hand side, because if

$$f_k = (u_k, v_k) \rightharpoonup f = (u,v) \text{ weakly in } W^{1,2}_{\mathrm{loc}}(\Omega) \times W^{1,q}_{\mathrm{loc}}(\Omega),$$

for some $q \in (1,2)$, then

$$u_k \rightharpoonup u \text{ weakly in } W^{1,2}(\Omega')$$

and by Rellich-Kondrackov Theorem

$$v_k \to v \text{ strongly in } L^2(\Omega');$$

while if

$$f_k = (u_k, v_k) \rightharpoonup f = (u,v) \text{ weakly in } W^{1,L^2 \log^\alpha L}_{\mathrm{loc}}(\Omega) \times W^{1,1}_{\mathrm{loc}}(\Omega),$$

for some $\alpha > 1$, then

$$u_k \rightharpoonup u \text{ weakly in } W^{1,L^2 \log^\alpha L}(\Omega')$$

and by the fact that

$$W^{1,1}(\Omega') \hookrightarrow\hookrightarrow L^2 \log^{-\alpha} L(\Omega')$$

we have

$$v_k \to v \text{ strongly in } L^2 \log^{-\alpha} L(\Omega').$$

As result we obtain

$$\int_{\Omega'} v \left(\frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial y}\right) dz.$$

Integrating by parts we have that

$$\int_{\Omega'} v \left(\frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial y}\right) dz = \left\langle \frac{\partial}{\partial x}\left(-v \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(v \frac{\partial u}{\partial x}\right), \varphi \right\rangle =$$

$$\langle \mathrm{Det}Df, \varphi\rangle.$$

Therefore

$$\operatorname{Det} Df_k \rightharpoonup \operatorname{Det} Df \quad \text{in } \mathcal{D}'(\Omega).$$

In particular if

$$\{f_k\} \subset W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2), \quad f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2),$$

then

$$\operatorname{Det} Df_k = J_{f_k}, \quad k = 1, 2, \dots$$
$$\operatorname{Det} Df = J_f,$$

so by the density of $C_0^\infty(\Omega)$ in $C_0^0(\Omega)$ we can conclude that

$$J_{f_k} \overset{*}{\rightharpoonup} J_f \quad \text{in } \mathcal{M}(\Omega).$$

□

Proof of Theorem 1.5. By Proposition 2.5 we know that $\operatorname{Det} Df_k$ and $\operatorname{Det} Df$ are well defined in the sense of distributions.
Let $\varphi \in C_0^\infty(\Omega)$ and let $\Omega' \subset\subset \Omega$ be fixed in such a way that the support of φ be contained in Ω'. Integrating by parts we have

$$\langle \operatorname{Det} Df_k, \varphi \rangle = \left\langle \frac{\partial}{\partial x}\left(-v_k \frac{\partial u_k}{\partial y} \right) + \frac{\partial}{\partial y}\left(v_k \frac{\partial u_k}{\partial x} \right), \varphi \right\rangle =$$

$$\int_{\Omega'} v_k \left(\frac{\partial u_k}{\partial y} \frac{\partial \varphi}{\partial x} - \frac{\partial u_k}{\partial x} \frac{\partial \varphi}{\partial y} \right) dz.$$

We can pass to the limit on the right-hand side, because if

$$f_k = (u_k, v_k) \rightharpoonup f = (u, v) \text{ weakly in } W^{1,\Phi}_{\mathrm{loc}}(\Omega) \times W^{1,\Psi}_{\mathrm{loc}}(\Omega),$$

where Φ and Ψ are Young functions satisfying (11), then

$$u_k \rightharpoonup u \text{ weakly in } W^{1,\Phi}(\Omega')$$

and by Cianchi embedding Theorem, see (17),

$$v_k \to v \text{ strongly in } L^{\widetilde{\Phi}}(\Omega').$$

As result we obtain

$$\int_{\Omega'} v \left(\frac{\partial u}{\partial y} \frac{\partial \varphi}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial y} \right) dz.$$

Integrating by parts

$$\int_{\Omega'} v \left(\frac{\partial u}{\partial y} \frac{\partial \varphi}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial y} \right) dz =$$

$$= \left\langle \frac{\partial}{\partial x}\left(-v\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right), \varphi \right\rangle = \langle \mathrm{Det} Df, \varphi \rangle.$$

Therefore

$$\mathrm{Det} Df_k \rightharpoonup \mathrm{Det} Df \quad \text{in } \mathcal{D}'(\Omega).$$

In particular if

$$\{f_k\} \subset W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2), \, f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2),$$

arguing as in the proof of Proposition 1.2 and Proposition 1.4 we can conclude that

$$J_{f_k} \overset{*}{\rightharpoonup} J_f \quad \text{in } \mathcal{M}(\Omega).$$

□

Example 3.1. We give here a counterexample. In order to show that the conclusion of Proposition 1.2 fails for $q = 1$, we consider the following mappings, suggested by [4]:

$$f_k = (u_k, v_k),$$

$$u_k(z) = \phi(kz) \qquad v_k(z) = k\psi(kz), \quad k = 1, 2, \dots$$

for $z \in \Omega' = \mathbb{D}(0,1)$, the disk of $\mathbb{R}^2$ centered at 0 with radius 1, with $\phi, \psi \in C^1_0(\Omega')$ such that

$$r_0 = \int_{\Omega'} \phi \star \nabla \psi \, dz \neq 0,$$

where $\star$ denotes the Hodge star operator, i.e.

$$\star = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2.$$

Note that $\{f_k\} \subset W^{1,2}(\Omega', \mathbb{R}^2)$,

$$u_k \rightharpoonup 0 \text{ weakly in } W^{1,2}(\Omega') \qquad v_k \rightharpoonup 0 \text{ weakly in } W^{1,1}(\Omega').$$

By the fact that $\mathrm{div}(\star \nabla v_k) = 0$ and integrating by parts, we have for $\varphi \in C^\infty_0(\Omega')$,

$$\int_{\Omega'} J_{f_k} \varphi \, dz = \int_{\Omega'} \nabla u_k \cdot \star \nabla v_k \, \varphi \, dz =$$

$$- \int_{\Omega'} u_k \star \nabla v_k \cdot \nabla \varphi \, dz = -k^2 \int_{\Omega'/k} \phi(kz) \star \nabla \psi(kz) \cdot \nabla \varphi(z) \, dz.$$

Making the change of variables $w = kz$ in the last integral we obtain

$$\int_{\Omega'} J_{f_k} \varphi \, dz = - \int_{\Omega'} \phi(w) \star \nabla \psi(w) \cdot \nabla \varphi(w/k) \, dw.$$

Passing to the limit for $k \to \infty$, as result we get

$$-\int_{\Omega'} \phi \star \nabla \psi \cdot \nabla \varphi(0)\, dw = \int_{\Omega'} \operatorname{div}(r_0 \delta_0)\, \varphi\, dw.$$

By the density of $C_0^\infty(\Omega')$ in $C_0^0(\Omega')$ we conclude that

$$J_{f_k} \overset{*}{\rightharpoonup} \operatorname{div}(r_0 \delta_0) \neq 0 \quad \text{in } \mathcal{M}(\Omega').$$

4. Examples

In this section we will show suitable functions verifying the hypotheses of Theorem 1.5.

Example 4.1. Consider Young functions

$$\Phi(t) \approx t^2 \log(e+t) \text{ near infinity and } \Psi(t) \approx t \log^{1/2} \log(e+t) \text{ near infinity.}$$

We have that

$$\widetilde{\Phi}(t) \approx t^2 \log^{-1}(e+t) \text{ near infinity, } \widetilde{\Psi}(t) \approx e^{e^{t^2}} - e \text{ near infinity,}$$

$$\widehat{\Phi}(t) \approx e^{e^{t^2}} - e \text{ near infinity, } \widehat{\Psi}(t) \approx t^2 \log\log(e+t) \text{ near infinity.}$$

Then we obtain

$$\widetilde{\Phi} \prec\prec \widehat{\Psi} \text{ near infinity } \text{ and } \widetilde{\Psi} \preceq \widehat{\Phi} \text{ near infinity.}$$

Therefore by Theorem 1.5 we get that if

$$f_k = (u_k, v_k) \rightharpoonup f = (u,v) \text{ weakly in } W^{1,L^2 \log L}_{\mathrm{loc}}(\Omega) \times W^{1,L \log^{1/2} \log L}_{\mathrm{loc}}(\Omega)$$

then

$$\operatorname{Det} Df_k \rightharpoonup \operatorname{Det} Df \quad \text{in } \mathcal{D}'(\Omega).$$

Example 4.2. Let $\beta < 1$ and let us consider

$$\Phi(t) \approx t^2 \log^\beta(e+t) \text{ near infinity and } \Psi(t) \approx t \log^{(1-\beta)/2}(e+t) \text{ near infinity}$$

so that

$$\widetilde{\Phi}(t) \approx t^2 \log^{-\beta}(e+t) \text{ near infinity, } \widetilde{\Psi}(t) \approx e^{t^{2/(1-\beta)}} - 1 \text{ near infinity,}$$

$$\widehat{\Phi}(t) \approx e^{t^{2/(1-\beta)}} - 1 \text{ near infinity, } \widehat{\Psi}(t) \approx t^2 \log^{1-\beta}(e+t) \text{ near infinity.}$$

Then we have

$$\widetilde{\Phi} \prec\prec \widehat{\Psi} \text{ near infinity and } \widetilde{\Psi} \preceq \widehat{\Phi} \text{ near infinity.}$$

Therefore by Theorem 1.5 we obtain that if

$$f_k = (u_k, v_k) \rightharpoonup f = (u,v) \text{ weakly in } W^{1,L^2 \log^\beta L}_{\mathrm{loc}}(\Omega) \times W^{1,L \log^{(1-\beta)/2} L}_{\mathrm{loc}}(\Omega)$$

then

$$\mathrm{Det}\, Df_k \rightharpoonup \mathrm{Det}\, Df \quad \text{in } \mathcal{D}'(\Omega).$$

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