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THE b-CHROMATIC NUMBER OF STAR GRAPH FAMILIES

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In this paper, we investigate the b-chromatic number of central graph, middle graph and total graph of star graph, denoted by $C(K_{1,n})$, $M(K_{1,n})$ and $T(K_{1,n})$ respectively.We discuss the relationship between b-chromatic number with some other types of chromatic numbers such as achromatic number, star chromatic number and equitable chromatic number.

1. Introduction

This paper considers the b-chromatic number of graphs derived by several different constrctions from a star graph.

The b-chromatic number $\varphi(G)$ [9, 12] of a graph G is the largest positive integer k such that G admits a proper k-coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-coloring. This concept of b-chromatic number was introduced in 1999 by Irving and Manlove [9], who proved that determining $\varphi(G)$ is NP-hard in general and polynomial for trees.

Effantin and Kheddouci studied [4–6] the b-chromatic number for the powers of paths, cycles, complete binary trees, and complete caterpillars.

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It has been proved in [10] by showing that if G is a d-regular graph with girth 5 and without cycles of length 6, then $\varphi(G) = d+1$.

Recently, motivated by the works of Sandi Klavžar and Marko Jakovac [12], who proved that the b-chromatic number of cubic graphs is 4 expect for the Petersen graph, $K_{3,3}$, the prism over K_3, and one more sporadic example with 10 vertices.

The proof techinque pattern that are followed in this paper is similar to that of [14, 15].

2. Preliminaries

The notion of star chromatic number was introduced by Branko Grünbaum in 1973. A star coloring [2] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraphs formed by the vertices of any two colors has connected components that are star graphs. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.

The achromatic number was introduced by Harary, Hedetniemi and Prins [8]. An achromatic coloring [8] of a graph G is a proper vertex coloring of G in which every pair of colors appears on at least one pair of adjacent vertices. The achromatic number of G denoted $\chi_c(G)$, is the greatest number of colors in an achromatic coloring of G.

The notion of equitable coloring [11], was introduced by Meyer in 1973. If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \cdots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j), then G is said to be *equitably k-colorable*. The smallest integer k for which G is equitable k-colorable is known as the *equitable chromatic number* [11] of G and denoted by $\chi_=(G)$.

For a given graph $G = (V, E)$ we do an operation on G, by subdividing each edge exactly once and joining all the non adjacent vertices of G. The graph obtained by this process is called central graph [13, 14] of G denoted by $C(G)$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [3] of G, denoted by $M(G)$ is defined as follows.The vertex set of $M(G)$ is $V(G) \cup E(G)$.Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

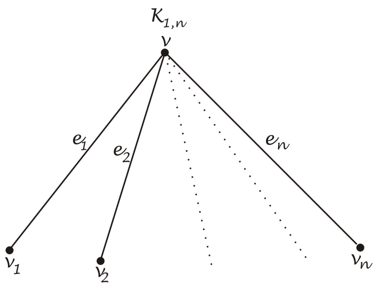
Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [3, 7] of G, denoted by $T(G)$ is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G.

(ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

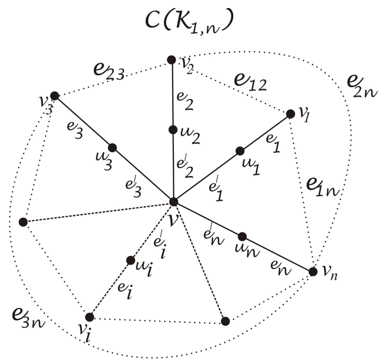
3. *b*-chromatic number on Central graph, Middle graph and Total graph of Star graph

Theorem 3.1. *For any star graph $K_{1,n}$, the b-chromatic number is $\varphi(C(K_{1,n})) = n$, $\forall\ n \geq 2$.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the pendant vertices of $K_{1,n}$ and let v be the vertex of $K_{1,n}$ adjacent to $v_i, 1 \leq i \leq n$. Obviously, $deg(v) = n$. Let the edge vv_i be subdivided by the vertex $u_i, 1 \leq i \leq n$ in $C(K_{1,n})$ and let $V = \{v_1, v_2, \cdots v_n\}, V' = \{u_1, u_2, \cdots u_n\}$. Clearly $V(C(K_{1,n})) = V \cup V' \cup \{v\}$, $\{u_1, u_2, \cdots u_n\}$ is independent set and, also, $\{u_i : 1 \leq i \leq n\}$ adjacent with $\{v_i : 1 \leq i \leq n\}$ respectively. Note that in $C(K_{1,n})$, the induced subgraph $\langle v_1, v_2, \cdots v_n \rangle$ is complete. Therefore, by proper coloring, $\varphi(C(K_{1,n})) \geq n$.



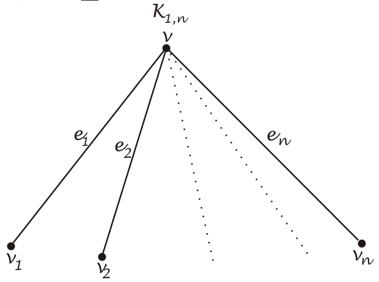
Star graph $K_{1,n}$
Figure 1(a)



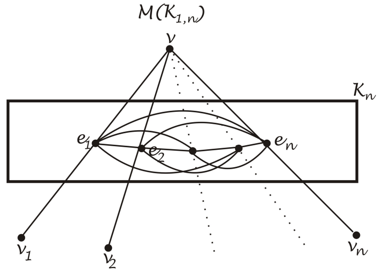
Central graph of Star graph $K_{1,n}$
Figure 1(b)

Assign the following n-coloring for $C(K_{1,n})$ as b-chromatic: For $1 \leq i \leq n$, assign the color c_i to v_i. For $2 \leq i \leq n$, assign the color c_1 to u_i and assign the color c_n to u_1. Assign the color c_2 to v. If $\varphi(C(K_{1,n})) = n+1$, $\forall\ n \geq 2$, there must be at least $n+1$ vertices of degree n in $C(K_{1,n})$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, v_1, v_2, \cdots v_n$, since these are only ones with degree at least n. If the colors of v, v_1 are c, c', respectively then it is easy to see that no vertex of color c' is adjacent to every other color (the only candidate with the right degree is v_1 itself, which cannot have a neighbour of color c. Thus, we have $\varphi(C(K_{1,n})) \leq n$. Hence, $\varphi(C(K_{1,n})) = n$, $\forall\ n \geq 2$. Note that $\varphi(C(K_{1,1})) = 2$. □

Theorem 3.2. *For any star graph $K_{1,n}$,the b-chromatic number is $\varphi(M(K_{1,n})) = n+1$, $\forall\, n \geq 2$.*



Star graph $K_{1,n}$
Figure 2(a)

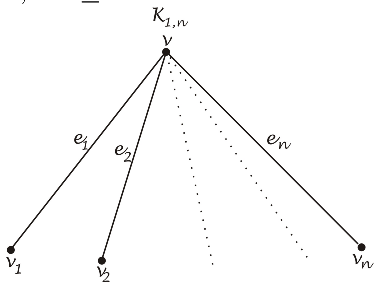


Middle graph of Star graph $K_{1,n}$
Figure 2(b)

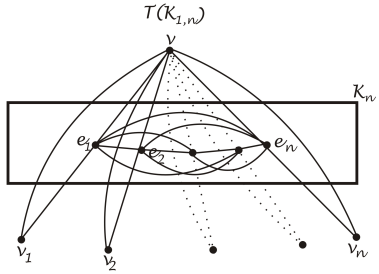
Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, \cdots, v_n\}$. By the definition of middle graph, each edge vv_i, for $1 \leq i \leq n$, of $K_{1,n}$ is subdivided by the vertex e_i in $M(K_{1,n})$ and the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $(n+1)$ in $M(K_{1,n})$. i.e.,$V(M(K_{1,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Therefore, $\varphi(M(K_{1,n})) \geq n+1$.

Now, consider the color class $C = \{c_1, c_2, \cdots c_n, c_{n+1}\}$ and assign the b-coloring to $M(K_{1,n})$ as follows. For every $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v. For every $2 \leq i \leq n$, assign the color c_1 to v_i and assign the color c_n to v_1. If $\varphi(M(K_{1,n})) = n+2$, $\forall\, n \geq 2$, there must be at least $n+2$ vertices of degree $n+1$ in $M(K_{1,n})$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $e_1, e_2, \cdots e_n$, since these are only ones with degree at least $n+1$. So an $(n+2)-$ coloring is impossible. Thus, we have $\varphi(M(K_{1,n})) \leq n+1$. Hence, $\varphi(M(K_{1,n})) = n+1$, $\forall\, n \geq 2$. Note that $\varphi(M(K_{1,1})) = 3$. □

Theorem 3.3. *For any star graph $K_{1,n}$, the b-chromatic number is $\varphi(T(K_{1,n})) = n+1$, $\forall\, n \geq 2$.*



Star graph $K_{1,n}$
Figure 3(a)



Total graph of Star graph $K_{1,n}$
Figure 3(b)

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, \cdots, v_n\}$ and $E(K_{1,n}) = \{e_1, e_2, \cdots, e_n\}$. By the definition of total graph, we have $V(T(K_{1,n})) = \{v\} \cup \{e_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n\}$, in which the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $(n+1)$. Therefore, $\varphi(T(K_{1,n})) \ge n+1$.

Assign the following $(n+1)$-coloring to $T(K_{1,n})$ as *b*-chromatic. For every $1 \le i \le n$, assign the color c_i to e_i and assign the color c_{n+1} to v. For every $2 \le i \le n$, assign the color c_1 to v_i and assign the color c_n to v_1. If $\varphi(T(K_{1,n})) = n+2$, $\forall\, n \ge 2$, there must be at least $n+2$ vertices of degree $n+1$ in $T(K_{1,n})$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, e_1, e_2, \cdots e_n$, since these are only ones with degree at least $n+1$. So an $(n+2)$-coloring is impossible. Thus, we have $\varphi(T(K_{1,n})) \le n+1$. Hence, $\varphi(T(K_{1,n})) = n+1$, $\forall\, n \ge 2$. Note that $\varphi(T(K_{1,1})) = 3$. □

4. Main Theorems

Theorem 4.1. *For any star graph, $K_{1,n}$, $\chi_s(M(K_{1,n})) = \chi_c(M(K_{1,n})) = \chi_=(M(K_{1,n})) = \varphi(M(K_{1,n}))$, $\forall\, n \ge 2$.*

Proof. For any star graph, $K_{1,n}$, $\chi_c(M(K_{1,n})) = n+1$ [14]. For any star graph, $K_{1,n}$, $\chi_s(M(K_{1,n})) = n+1$ and $\chi_=(M(K_{1,n})) = n+1$ [16] and hence, from Theorem 3.2, $\chi_s(M(K_{1,n})) = \chi_c(M(K_{1,n})) = \chi_=(M(K_{1,n})) = \varphi(M(K_{1,n}))$, $\forall\, n \ge 2$. □

Theorem 4.2. *For any star graph, $K_{1,n}$, $\chi_=(C(K_{1,n})) = \varphi(C(K_{1,n}))$, $\forall\, n \ge 2$.*

Proof. For any star graph, $K_{1,n}$, $\chi_=(C(K_{1,n})) = n$ [1], and hence, from Theorem 3.1, $\chi_=(C(K_{1,n})) = \varphi(C(K_{1,n}))$, $\forall\, n \ge 2$. □

Theorem 4.3. *For any star graph, $K_{1,n}$, $\chi_s(T(K_{1,n})) = \varphi(T(K_{1,n}))$, $\forall\, n \ge 2$.*

Proof. For any star graph, $K_{1,n}$, $\chi_s(T(K_{1,n})) = n+1$ [16], and hence, from Theorem 3.3, $\chi_s(T(K_{1,n})) = \varphi(T(K_{1,n}))$, $\forall\, n \ge 2$. □

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