LE MATEMATICHE
Vol. LXVI (2011) – Fasc. I, pp. 29–36
doi: 10.4418/2011.66.1.3

LABELING CACTI WITH A CONDITION AT DISTANCE TWO

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An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$. The $L(2,1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-labeling with $\max\{f(v) : v \in V(G)\} = k$. In 1992, it has been proved by Griggs and Yeh [3] that the λ-number of tree is $\Delta+1$ or $\Delta+2$. In this paper we present a graph family other than tree whose λ-number is $\Delta+1$ or $\Delta+2$.

1. Introduction

The channel assignment problem is the problem to assign a channel (non negative integer) to each TV or radio transmitters located at various places such that communication does not interfere. This problem was first formulated as a graph coloring problem by Hale [4] who introduced the notion of T-coloring of a graph.

In a private communication with Griggs during 1988 Roberts proposed a variation of the channel assignment problem in which *close* transmitters must receive different channels and *very close* transmitters must receive channels that are at least two apart. In a graph model of this problem, the transmitters are

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Entrato in redazione: 13 luglio 2010

AMS 2010 Subject Classification: 05C78.

Keywords: $L(2,1)$-labeling, λ-number, Cactus graph, Block cut point graph, One point union of cycles.

represented by the vertices of a graph; two vertices are *very close* if they are adjacent in the graph and *close* if they are at distance two apart in the graph. Motivated by this problem Griggs and Yeh [3] introduced $L(2,1)$-labeling which is defined as follows.

Definition 1.1. An $L(2,1)$-labeling (or distance two labeling) of a graph $G = (V(G), E(G))$ is a function f from the set $V(G)$ of vertices to the set of all nonnegative integers such that the following conditions are satisfied:
(1) $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$
(2) $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$

A k-$L(2,1)$-labeling is an $L(2,1)$-labeling such that no label is greater than k. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$ or λ, is the smallest number k such that G has a k-$L(2,1)$-labeling. The $L(2,1)$-labeling has been extensively studied in recent past by many researchers like Yeh [12, 13], Georges and Mauro [2], Sakai [7], Chang and Kuo [1], Kuo and Yan [5], Lu et al. [6], Shao and Yeh [8], Wang [10] and Vaidya et al. [9].

We begin with a finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. In the present work Δ denotes the maximum degree of the graph. For standard terminology and notations we refer to West [11]. We give a brief summary of definitions and information which are prerequisites for the present work.

Proposition 1.2. [1] *$\lambda(H) \leq \lambda(G)$, for any subgraph H of a graph G.*

Proposition 1.3. [12] *The λ-number of a star $K_{1,\Delta}$ is $\Delta + 1$, where Δ is the maximum degree.*

Proposition 1.4. [12] *The λ-number of a complete graph K_n is $2n - 2$.*

Proposition 1.5. [1] *If $\lambda(G) = \Delta + 1$ then $f(v) = 0$ or $\Delta + 1$ for any $\lambda(G)$-$L(2,1)$-labeling f and any vertex v of maximum degree Δ. In this case, $N[v]$ contains at most two vertices of degree Δ, for any vertex $v \in V(G)$.*

2. Main Results

In the discussion of the λ-number of graphs, much attention of researchers has been attracted by a connected graph without cycles, that is, a tree T. In fact, the maximum degree determines the labeling number of trees. Griggs and Yeh [3] proved that the λ-number of any tree is $\Delta+1$ or $\Delta+2$. Consequently, as time has passed, the classification of trees have been done based on the λ-number. The trees T with λ-number $\Delta + 1$ are classified as type 1 while the trees T with λ-number $\Delta + 2$ are classified as type 2. This concept has been the focus of many research papers. We present here a graph family whose λ-number is $\Delta+1$ or $\Delta+2$ which is not a tree.

A block of a graph G is a maximal connected subgraph of G that has no cut-vertex. The block-cutpoint graph of a graph G is a bipartite graph H in which one partite set consists of the cut-vertices of G, and the other has a vertex b_i for each block B_i of G. We include vb_i as an edge of H if and only if $v \in B_i$. The block which contains only one cut vertex is called leaf block and that cut vertex is known as leaf block cut vertex. In a block cutpoint graph the vertices corresponding to leaf blocks are pendent vertices. An n-ary cactus is a connected graph whose blocks are all isomorphic to C_n. If $n = 3$ then it is known as a triangular cactus. An n-ary k-regular cactus is a connected graph whose blocks are all isomorphic to C_n and a block cutpoint graph is a tree having each block vertex b_i of degree n except leaf blocks and each cut vertex of degree $\frac{k}{2}$. We will denote it by $C_n(k)$.

Note 2.1. For an n-ary k-regular cactus we notice that

- $n \geq 3$.
- $k \geq 4$ and k = even.
- The maximum degree of a vertex is k.

Example 2.2. In the following *figure 1* the 4-ary 4-regular cactus is presented while *figure 2* shows its block cutpoint graph.

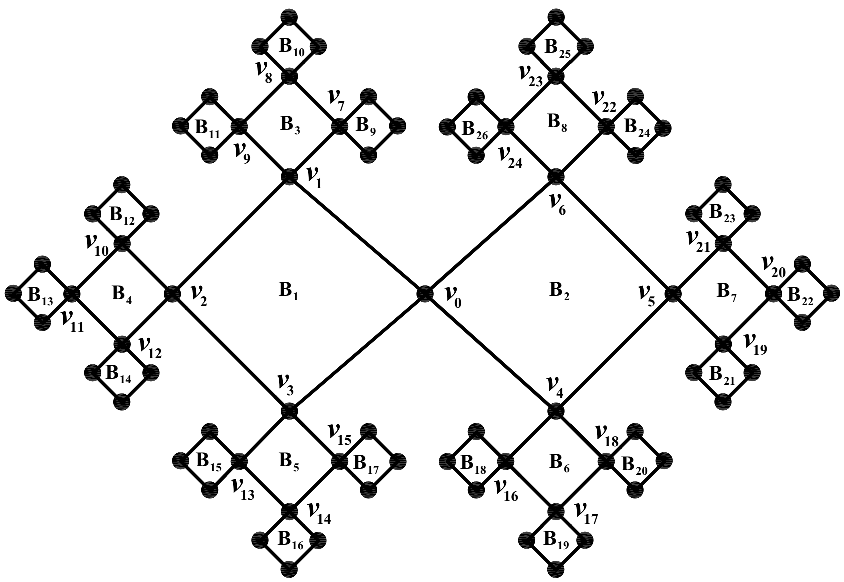

Figure 1

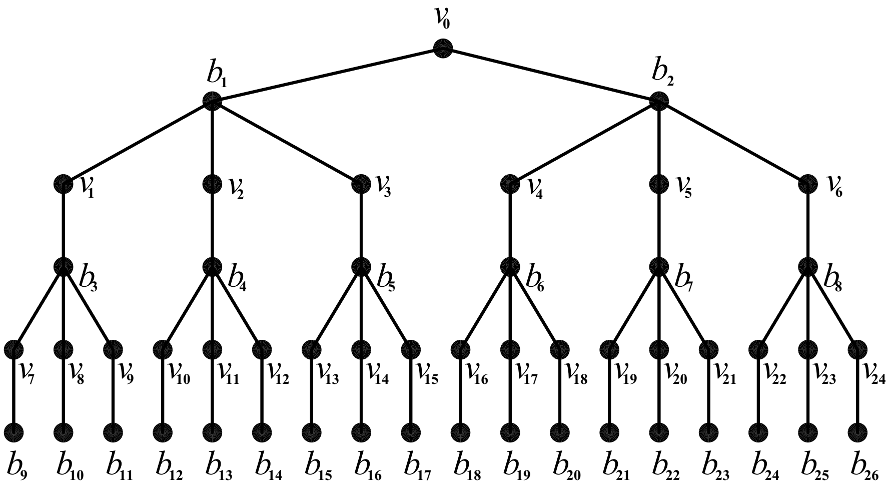

Figure 2

Theorem 2.3. *The λ-number of an n-ary Δ-regular cactus is $\Delta+1$ or $\Delta+2$.*

Proof. Let $C_n(\Delta)$ be an n-ary Δ-regular cactus. The graph $K_{1,\Delta}$ is a subgraph of G and as $\lambda(K_{1,\Delta}) = \Delta+1$ by Propositions 1.2 and 1.3 it follows that $\lambda(C_n(\Delta)) \geq \Delta+1$. We now show that there exists an $L(2,1)$-labeling of $C_n(\Delta)$ with labels from the set $S = \{0,1,...,\Delta+2\}$.

Let v_0 be the vertex with degree Δ. Label the vertex v_0 by 0 and its adjacent vertices from the set $\{2,3,...,\Delta+2\}$. Let v_{0i} be the adjacent vertex to v_0 and it has label i, for some $i \in \{2,3,...,\Delta+1\}$. Now consider v_{oij} which is a vertex adjacent to v_{0i}. In $C_n(\Delta)$, the vertex v_{0i} is adjacent to at most $\Delta-1$ vertices in the graph. Hence v_{0ij} can be assigned a label that differ from those assigned to at most $\Delta-1$ vertices and differ from any label within 1 of the labels assigned to v_{oi}. Hence at most $(\Delta-1)+3 = \Delta+2$ labels cannot be used to label v_{0ij} leaving at least one available label in S to label v_{0ij} and obtain $\lambda(C_n(\Delta)) \leq \Delta+2$.
Thus, $\lambda(C_n(\Delta)) = \Delta+1$ or $\Delta+2$. □

On dropping the k regularity of cut vertices in an n-ary k-regular cactus we prove a general result as a corollary.

Corollary 2.4. *The λ-number of an n-ary cactus with maximum degree Δ is $\Delta+1$ or $\Delta+2$.*

Proof. Let G be the arbitrary an n-ary cactus with maximum degree Δ. The graph $K_{1,\Delta}$ is a subgraph of G and hence, by Propositions 1.2 and 1.3, $\lambda(G) \geq \Delta+1$. Note that G is a subgraph of $C_n(\Delta)$ and hence by Theorem 2.3 $\lambda(G) \leq \Delta+2$.
Thus, $\lambda(G) = \Delta+1$ or $\Delta+2$. □

First we present some regular cacti whose λ-number is precisely $\Delta+1$ and later we give an example of a regular cactus whose λ-number is precisely $\Delta+2$. A one point union of k-copies of cycle C_n is the graph obtained by taking v as a common vertex such that any two cycles are edge disjoint and do not have any vertex in common except v. We will denote it by $C_n^{(k)}$. Theorem 2.5 deals with one point union of two cycles in which exact label assignment is carried out while to prove Theorem 2.6 we choose an analytical approach.

Theorem 2.5. $\lambda(C_n^{(2)}) = \lambda(C_n(4)) = 5$.

Proof. Let $C_n^{(2)}$ be the one point union of two cycles C_n with n vertices respectively. Let $v_j^1, 0 \leq j \leq n-1$ and $v_j^2, 0 \leq j \leq n-1$ be the vertices of $C_n^{(2)}$. Without loss of generality assume that $v_0 = v_0^1 = v_0^2$. The graph $K_{1,4}$ is a subgraph of one point union of two cycles and hence by Propositions 1.2 and 1.3 $\lambda(C_n^{(2)}) \geq 5$.

Now we want to show that $\lambda(C_n^{(2)}) \leq 5$. Define $f : V(C_n^{(2)}) \to \{0,1,2,\ldots,5\}$ as follows:
(1) $n \equiv 0\ (mod 3)$

$$\begin{aligned} f(v_j^1) &= 0 \text{ if } j \equiv 0\ (mod 3)\\ f(v_j^1) &= 2 \text{ if } j \equiv 1\ (mod 3)\\ f(v_j^1) &= 4 \text{ if } j \equiv 2\ (mod 3)\\ f(v_j^2) &= 0 \text{ if } j \equiv 0\ (mod 3)\\ f(v_j^2) &= 3 \text{ if } j \equiv 1\ (mod 3)\\ f(v_j^2) &= 5 \text{ if } j \equiv 2\ (mod 3) \end{aligned}$$

(2) $n \equiv 1\ (mod 3)$ except for $n = 4$, redefine the above f of (1) at v_{n-3}^1, v_{n-2}^1, v_{n-1}^1, v_{n-2}^2, v_{n-1}^2 as

$$\begin{aligned} f(v_j^1) &= 3 \text{ if } j = n-3\\ f(v_j^1) &= 1 \text{ if } j = n-2\\ f(v_j^1) &= 4 \text{ if } j = n-1\\ f(v_j^2) &= 1 \text{ if } j = n-2\\ f(v_j^2) &= 5 \text{ if } j = n-1 \end{aligned}$$

For $n = 4$, f is given by $f(v_0) = 0$, $f(v_1^1) = 2$, $f(v_2^1) = 5$, $f(v_3^1) = 3$, $f(v_1^2) = 4$, $f(v_2^2) = 1$, $f(v_3^2) = 5$.
(3) $n \equiv 2\ (mod 3)$ then redefine the above f of (1) at v_{n-2}^1, v_{n-1}^1, v_{n-2}^2, v_{n-1}^2 as

$$\begin{aligned} f(v_j^1) &= 1 \text{ if } j = n-2\\ f(v_j^1) &= 5 \text{ if } j = n-1\\ f(v_j^2) &= 1 \text{ if } j = n-2\\ f(v_j^2) &= 4 \text{ if } j = n-1 \end{aligned}$$

Thus, $\lambda(C_n^{(2)}) = \lambda(C_n(4)) = 5$. □

Theorem 2.6. $\lambda(C_n^{(k)}) = \lambda(C_n(2k)) = 2k+1$.

Proof. Let $C_n^{(k)}$ be the one point union of k cycles C_n. If $k = 2$ then the result follows by Theorem 2.5, hence assume $k \geq 3$. Without loss of generality assume that v_0 is the common vertex of all cycles. The graph $K_{1,2k}$ is a subgraph of $C_n^{(k)}$ and hence by Propositions 1.2 and 1.3 $\lambda(C_n^{(k)}) \geq 2k+1$.

Now we want to prove that $\lambda(C_n^{(k)}) \leq 2k+1$. In a graph $C_n^{(k)}$, there is one vertex of degree $2k$ which is a common vertex of all cycles and the remaining vertices are of degree 2. Label the common vertex v_0 by 0 or $2k+1$ and its adjacent vertices from the set $\{2,3,...,2k+1\}$ or $\{0,1,...,2k-1\}$. For the remaining vertices, observe that enough number of labels are available in the set $\{0,1,...,2k+1\}$ as they have degree 2.
Thus, $\lambda(C_n^{(k)}) = \lambda(C_n(2k)) = 2k+1$. □

Corollary 2.7. *The λ-number of a Friendship graph is $F_k(= C_3(2k))$ is $2k+1$.*

Example 2.8. In the following *Figure 3* an $L(2,1)$-labeling of Friendship graph F_4 is shown in which $\lambda(F_4) = 9$.

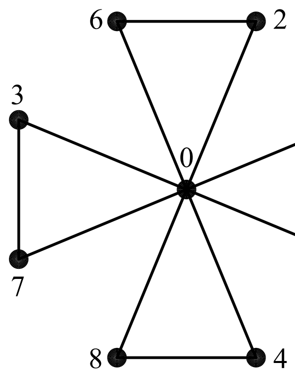

Figure 3

Thus, we have investigated a graph family whose λ-number is precisely $\Delta+1$. But there are some graphs whose λ-number is precisely $\Delta+2$. Here we give an example of such graphs.

Example 2.9. In the following *Figure 4* an $L(2,1)$-labeling of a 4-ary 4-regular cactus is shown in which $\lambda(C_4(4)) = 6$ using Proposition 1.5.

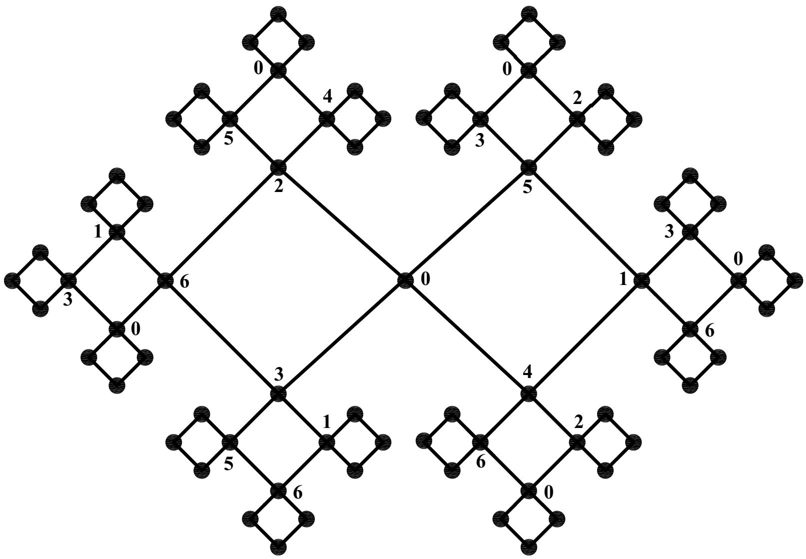

Figure 4

3. Concluding Remarks

Here we have proved that the λ number of an n-ary k-regular cactus is $\Delta+1$ or $\Delta+2$. The λ-number is also completely determined for one point union of k-cycles.

Acknowledgement

The authors are highly thankful to anonymous referees for their constructive suggestions and kind comments.

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