GRADED BETTI NUMBERS OF IDEALS WITH LINEAR QUOTIENTS

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In this paper we show that every ideal with linear quotients is componentwise linear. We also generalize the Eliahou-Kervaire formula for graded Betti numbers of stable ideals to homogeneous ideals with linear quotients.

1. Introduction

Let $K$ be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in $n$ variables, and $I \subset S$ a homogeneous ideal. Let $\{f_1, \ldots, f_m\}$ be a system of homogeneous generators for $I$. We say that $I$ has linear quotients with respect to the elements $f_1, \ldots, f_m$ if the ideal $\langle f_1, \ldots, f_{i-1} \rangle : f_i$ is generated by linear forms for all $i = 2, \ldots, m$ (notice that this property depends on the order of the generators). Monomial ideals with linear quotients were introduced in [4] by Herzog and Takayama and have strong combinatorial implication, see the paper of Soleyman Jahan and Zheng [5].

In this paper we study the minimal free resolution of a graded ideal $I$ with linear quotients with respect to a minimal system of homogeneous generators $\{f_1, \ldots, f_m\}$ (in this case we simply say that $I$ has linear quotients). It is known

Entrato in redazione: 2 gennaio 2009

AMS 2000 Subject Classification: 13D02
Keywords: graded Betti numbers, componentwise linear ideals, ideals with linear quotients
Partly sponsored by Pragmatic
that $I$ is componentwise linear provided that $\deg(f_1) \leq \ldots \leq \deg(f_m)$, see the book of Herzog and Hibi [3]. We prove that $I$ is componentwise linear without the additional assumption on the degrees, giving an affirmative answer to a question of Herzog (see Corollary 2.4). Our result is a generalization of [5, Corollary 2.8], where the authors prove componentwise linearity in case $I$ is a monomial ideal and $f_1, \ldots, f_m$ are monomials.

A large class of monomial ideals with linear quotients is the class of stable ideals. A minimal free resolution of stable ideals was constructed by Eliahou and Kervaire in [1], who also give an explicit formula for their graded Betti numbers. For an ideal with linear quotients we give a formula for its Betti numbers that generalizes the formula by Eliahou and Kervaire. We express the graded Betti numbers in terms of the numbers $\deg(f_i)$ and the minimal number of generators of $\langle f_1, \ldots, f_{i-1} \rangle : f_i$, for $i = 1, \ldots, m$. As a consequence we will also obtain the Castelnuovo-Mumford regularity and the projective dimension of $I$.

2. Graded Ideals With Linear Quotients

Let $I \subset S$ be a graded ideal and $\{f_1, \ldots, f_m\}$ be a minimal system of homogeneous generators for $I$ such that $I$ has linear quotients with respect to $f_1, \ldots, f_m$. Notice that if an ideal has linear quotients with respect to a minimal system of homogeneous generators then it need not have linear quotients with respect to all minimal homogeneous system of generators (see Example 2.1). It is easy to see that $\deg(f_i) \geq \min\{\deg(f_1), \ldots, \deg(f_{i-1})\}$ for $i = 2, \ldots, m$. Particularly, $\deg(f_1) \leq \deg(f_i)$ for every $i = 1, \ldots, m$. But in general, the sequence $\deg(f_1), \ldots, \deg(f_{i-1})$ of degrees need not be increasing.

For example, the ideal $\langle xy, xy^3z + y^4z - y^3z^2, x^3 + x^2y - x^2z, x^2z^3 \rangle$ has linear quotients in the given order, but $\deg(xy^3z + y^4z - y^3z^2) > \deg(x^3 + x^2y - x^2z)$.

**Example 2.1.** Set $m = \langle x_1, \ldots, x_n \rangle \subset S$ the maximal irrelevant ideal. It is easy to see that $m^k$ has linear quotients with respect to its unique system of monomial generators for every natural number $k$ (for example ordering the monomials lexicographically).

If $K$ is large enough, we can choose a minimal system of homogeneous generators $\{g_1, \ldots, g_t\}$ for $m^k$ such that $\text{GCD}(g_i, g_j) = 1$ for each $i \neq j$. Hence it is clear that, if $k \geq 2$, $m^k$ does not have linear quotients with respect to $\{g_1, \ldots, g_t\}$, for any order, otherwise the unique factorization would be contradicted.

If $J$ is a graded ideal of $S$, then we write $J_{(ji)}$ for the ideal generated by all homogeneous polynomials of degree $j$ belonging to $J$. Moreover, we write $J_{\leq k}$ for the ideal generated by all homogeneous polynomials in $J$ whose degree is
less than or equal to \( k \). We will say that \( J \) is componentwise linear if \( J_{(j)} \) has a \( j \)-linear resolution for all \( j \).

In [5, Lemma 2.1] the authors show that for any monomial ideal \( J \) with linear quotients with respect to the unique minimal system \( G(J) \) of monomial generators of \( J \), there exists a degree increasing order of the elements \( u_1, \ldots, u_m \) of \( G(J) \) such that \( J \) has linear quotients with respect to this order. As a corollary they are able to prove that \( J \) is componentwise linear. It is worth remarking that if \( J \) has linear quotients with respect to a non-minimal system of generators, then \( J \) is not necessarily componentwise linear (see Example 2.5). We prove the result of [5] for an arbitrary graded ideal with linear quotients using completely different methods.

In the following for a \( \mathbb{Z} \)-graded module \( M \) we write \( M(a) \) for the module obtained by shifting degrees by \( a \), for any \( a \in \mathbb{Z} \): i.e. \( M(a) \) is the module \( M \) with the grading defined as \( M(a)_b = M_{a+b} \).

For our purpose we need the next lemma.

**Lemma 2.2.** Let \( J = \langle f_1, \ldots, f_m \rangle \) be a componentwise linear ideal and let \( I = J + \langle f \rangle \) where \( f \) is a homogenous form of degree \( d \). If the ideal \( J : f \) is generated by linear forms then \( I_{(j)} \) has a \( j \)-linear resolution for each \( j \leq d \) provided that \( f_1, \ldots, f_m, f \) is a minimal system of generators for \( I \).

**Proof.** \( I_{(j)} = J_{(j)} \) for each \( j < d \) so it is enough to prove that \( I_{(d)} \) has \( d \)-linear resolution. First we show that \( J : f = J_{(d)} : f \). It is clear that \( J_{(d)} \subseteq J_{\leq d} \subseteq J \). Therefore \( J_{(d)} : f \subseteq J_{\leq d} : f \subseteq J : f \). Set \( L = J : f = \langle l_1, \ldots, l_r \rangle \) where \( l_i \) are linear forms. So there exist non-zero homogeneous polynomials \( \lambda_{t_1}, \ldots, \lambda_{t_r} \) such that \( l_i f = \lambda_{t_i} f_{t_i} \) and \( \deg(\lambda_{t_i}) + \deg(f_{t_i}) = \deg(l_i f) = d + 1 \). Since \( f_1, \ldots, f_m, f \) is a minimal system of generators for \( I \), \( \deg(\lambda_{t_i}) \geq 1 \) and so \( \deg(f_{t_i}) \leq d \). Then \( l_i f \in J_{\leq d} \) but it is clear that \( (J_{(d)})_{d+1} = (J_{\leq d})_{d+1} \). So \( l_i f \in J_{(d)} \) and \( J : f = J_{(d)} : f \).

Trivially \( J_{(d)} + \langle f \rangle = I_{(d)} \), so by the above discussion

\[
I_{(d)}/J_{(d)} \cong (S/(J_{(d)} : f))(-d) = (S/(J : f))(-d) = S/L(-d).
\]

Hence we can consider the short exact sequence

\[
0 \to J_{(d)} \to I_{(d)} \to S/L(-d) \to 0
\]

which yields the long exact sequence of \( \text{Tor} \)

\[
\ldots \to \text{Tor}^S_i(K, J_{(d)}) \to \text{Tor}^S_i(K, I_{(d)}) \to \text{Tor}^S_i(K, S/L(-d)) \to \ldots
\]

The \( \text{Tor} \)-groups on the right and on the left end of this sequence vanish for \( j \neq d \) since the corresponding modules have \( d \)-linear resolution. This proves that \( I_{(d)} \) has \( d \)-linear resolution. \( \square \)
We are ready to prove the main theorem of the paper.

**Theorem 2.3.** Let $J = \langle f_1, \ldots, f_m \rangle$ be a componentwise linear ideal and let $I = J + \langle f \rangle$ where $f$ is a homogeneous form of degree $d$. If the ideal $J : f$ is generated by linear forms then $I$ is componentwise linear provided that $f_1, \ldots, f_m, f$ is a minimal system of generators for $I$.

**Proof.** By Lemma 2.2, $I_{(j)}$ has $j$-linear resolution for each $j \leq d$. Set $m = \langle x_1, \ldots, x_n \rangle$ and $p = \max \{ \deg(f_i) \}$. If $d > p$ then $I_{(d+j)} = m^j I_{(d)}$ for each $j \geq 1$. So the exact sequence

$$0 \to I_{(d+i)} \to I_{(d+i-1)} \to I_{(d+i-1)} / I_{(d+i)} \to 0$$

and induction on $i$ shows that $I_{(d+i)}$ has $d+i$-linear resolution. Therefore, $I$ is componentwise linear ideal.

If $d \leq p$, let $s = p - d$. In this case we use induction on $s$ to prove the result. If $s = 0$, again by Lemma 2.2, $I$ is componentwise linear arguing as above. Now suppose that $s \geq 1$ and the result is true for $s - 1$. Let $L = J : f = \langle l_1, \ldots, l_r \rangle$ minimally generated by $r$ linear forms. If $r = n$ then $m^f \subseteq J$. So $J_{(d+i)} = I_{(d+i)}$ for each $i \geq 1$. Now another application of Lemma 2.2 shows that $I$ is a componentwise linear ideal. If $r < n$ we complete $\{ l_1, \ldots, l_r \}$ to a $K$-basis of $S_1$ by new linear forms $e_1, \ldots, e_{n-r}$. Set $g_i = e_i f$ for $i = 1, \ldots, n-r$. One can easily check that $(J + \langle g_1, \ldots, g_{n-r} \rangle)_{(d+i)} = I_{(d+i)}$ for each $i \geq 1$. Therefore it is enough to show that $J + \langle g_1, \ldots, g_{n-r} \rangle$ is componentwise linear.

**Claim 1:** For every $i = 1, \ldots, n-r$ we have $\langle J + \langle g_1, \ldots, g_{i-1} \rangle \rangle : g_i = L + \langle e_1, \ldots, e_{i-1} \rangle$.

**Proof of Claim 1:** Let $h \in (J + \langle g_1, \ldots, g_{i-1} \rangle) : g_i$. So $he_i f \in J + \langle g_1, \ldots, g_{i-1} \rangle$. Therefore there exist suitable coefficients $\gamma_j$ and $\lambda_j$ such that $he_i f = \sum_{j=1}^m \gamma_j f_j + \sum_{j=1}^{i-1} \lambda_j e_j f$. Then $he_i - \sum_{j=1}^{i-1} \lambda_j e_j f \in J : f$. So $he_i \in \langle l_1, \ldots, l_r, e_1, \ldots, e_{i-1} \rangle$. Since $l_1, \ldots, l_r, e_1, \ldots, e_{n-r}$ is a regular sequence $h \in L + \langle e_1, \ldots, e_{i-1} \rangle$. The other inclusion is clear.

**Claim 2:** For every $i = 1, \ldots, n-r$ the ideal $J + \langle g_1, \ldots, g_i \rangle$ is minimally generated by $\{ f_1, \ldots, f_m, g_1, \ldots, g_i \}$.

**Proof of Claim 2:** Suppose that for some $t$, $f_t \in \langle f_1, \ldots, \hat{f}_t, \ldots, f_m, g_1, \ldots, g_i \rangle$. Then there exist suitable coefficients $\gamma_j$ and $\lambda_j$ such that $f_t = \sum_{j=1}^m \gamma_j f_j + \sum_{j=1}^i \lambda_j e_j f$. So $\{ f_1, \ldots, f_m, f \}$ is not a minimal system of generators for $I$ which is a contradiction.

If for some $t$, $g_t \in \langle f_1, \ldots, f_m, g_1, \ldots, \hat{g}_t, \ldots, g_i \rangle$ then again one can find coefficients
γ_j and λ_j such that g_t = ∑_{j=1}^{m} γ_j f_j + ∑_{j \neq t}^{i} λ_j g_j. So (e_t - ∑_{j \neq t}^{i} λ_j e_j) f ∈ J and therefore e_t - ∑_{j \neq t}^{i} λ_j e_j ∈ L, which is a contradiction.

Since deg(g_i) = d + 1, Claim 1 and Claim 2 together with the inductive hypothesis show that J + ⟨g_1, ..., g_i⟩ is componentwise linear for each i = 1, ..., n − r. This completes the proof.

From the above theorem the answer to a question by Herzog follows.

**Corollary 2.4.** Let I be a homogeneous ideal and {f_1, ..., f_m} be a minimal system of generators for I. If I has linear quotients with respect to f_1, ..., f_m then I is a componentwise linear ideal.

**Proof.** Set I_i = ⟨f_1, ..., f_i⟩ for i = 1, ..., m. It is clear that I_1 has a linear resolution. In particular I_1 is a componentwise linear ideal. Therefore, one can apply Theorem 2.3 and induction on i to see that each I_i is componentwise linear for i = 1, ..., m. □

**Example 2.5.** In Corollary 2.4 one cannot remove the hypothesis that I is minimally generated by {f_1, ..., f_m}. For instance, let I be the monomial ideal I = ⟨x^2, y^2⟩ ⊂ K[x, y]. Trivially I = I_{⟨2⟩} does not have a 2-linear resolution, but it has linear quotients with respect to the system of generators {x^2, xy^2, y^2}.

In general we can observe that every m-primary ideal I ⊂ S (m = ⟨x_1, ..., x_n⟩), has linear quotients with respect to a homogeneous system of generators. In fact let h be such that m^h ⊂ I. Then clearly by ordering the elements of m^h appropriately (for example lexicographically), I has linear quotients with respect to {m^h, I_{h−1}, ..., I_s}, where s is minimal such that I_s ≠ 0.

In the following we compute the graded Betti numbers of ideals with linear quotients. For this, we use the formula (1) of Herzog and Hibi (see [2, Proposition 1.3]) for the graded Betti numbers of an arbitrary componentwise linear ideal J.

\[ \beta_{i,i+j}(J) = \beta_{i}(J_{⟨j⟩}) - \beta_{i}(mJ_{⟨j−1⟩}) \quad \text{for all } i, j \]  

**Theorem 2.6.** Let J = ⟨f_1, ..., f_m⟩ be a componentwise linear ideal and let I = J + ⟨f⟩ where f is a homogeneous form of degree d. If the ideal J : f is minimally generated by r linear forms then

\[ \beta_{i,i+j}(I) = \beta_{i,i+j}(J) \quad \text{if } j ≠ d, \]
\[ \beta_{i,i+d}(I) = \beta_{i,i+d}(J) + \binom{r}{i}, \]

provided that \( \{ f_1, \ldots, f_m, f \} \) is a minimal system of homogeneous generators for \( I \).

**Proof.** Consider the exact sequence

\[ 0 \rightarrow J \rightarrow I \rightarrow S/L(-d) \rightarrow 0 \]  

which yields the long exact sequence

\[ \cdots \rightarrow \text{Tor}^S_{i+1}(K, S/L(-d))_{i+j} \rightarrow \text{Tor}^S_i(K, J)_{i+j} \rightarrow \text{Tor}^S_i(K, I)_{i+j} \rightarrow \text{Tor}^S_i(K, S/L(-d))_{i+j} \rightarrow \cdots \]

The Tor-groups on the right and on the left end of this sequence vanish for \( j \neq d, d + 1 \) since the corresponding module has a \( d \)-linear resolution. This proves that \( \beta_{i,i+j}(I) = \beta_{i,i+j}(J) \) if \( j \neq d, d + 1 \).

First consider \( j = d \). As discussed in the proof of Lemma 2.2, one has the short exact sequence

\[ 0 \rightarrow J_{(d)} \rightarrow I_{(d)} \rightarrow S/L(-d) \rightarrow 0 \]

which yields the long exact sequence

\[ \cdots \rightarrow \text{Tor}^S_i(K, J_{(d)})_{i+j} \rightarrow \text{Tor}^S_i(K, I_{(d)})_{i+j} \rightarrow \text{Tor}^S_i(K, S/L(-d))_{i+j} \rightarrow \cdots \]

Since both \( J_{(d)} \) and \( S/L(-d) \) have a \( d \)-linear resolution, the above long exact sequence gives

\[ \beta_i(I_{(d)}) = \beta_i(J_{(d)}) + \beta_i(S/L(-d)) \]

Moreover the minimal free resolution of \( S/L(-d) \) is given by Koszul complex, so

\[ \beta_i(S/L(-d)) = \binom{r}{i} \]

Thus by (1),

\[
\begin{align*}
\beta_{i,i+d}(I) & = \beta_i(I_{(d)}) - \beta_i(mI_{(d-1)}) \\
& = \beta_i(I_{(d)}) - \beta_i(mJ_{(d-1)}) \\
& = \beta_i(J_{(d)}) + \binom{r}{i} - \beta_i(mJ_{(d-1)}) \\
& = \beta_{i,i+d}(J) + \binom{r}{i}.
\end{align*}
\]
Now consider $j = d + 1$. By (2) we have the following exact sequence
\[
0 \to \text{Tor}^S_{i+1}(K,J)_{i+d+1} \to \text{Tor}^S_{i+1}(K,I)_{i+d+1} \to \text{Tor}^S_{i+1}(K,S/L(-d))_{i+d+1} \\
\to \text{Tor}^S_{i}(K,J)_{i+d+1} \to \text{Tor}^S_{i}(K,I)_{i+d+1} \to 0.
\]

So by the first part $\beta_{i,i+d+1}(I) =$
\[
\beta_{i,i+d+1}(J) - \binom{r}{i+1} + \beta_{i+1,i+d+1}(J) + \binom{r}{i+1} - \beta_{i+1,i+d+1}(J) \\
= \beta_{i,i+d+1}(J).
\]

**Corollary 2.7.** Let $I$ be a homogeneous ideal with linear quotients with respect to $f_1, \ldots, f_m$ where $\{f_1, \ldots, f_m\}$ is a minimal system of homogeneous generators for $I$. Let $n_p$ be the minimal number of homogeneous generators of $\langle f_1, \ldots, f_{p-1} \rangle : f_p$ for $p = 1, \ldots, m$. Then
\[
\text{reg}(I) = \max\{\deg(f_p) : p = 1, \ldots, m\}
\]
\[
\text{projdim}(I) = \max\{n_p | 1 \leq p \leq m\}
\]
\[
\beta_{i,i+j}(I) = \sum_{1 \leq p \leq m, \deg(f_p) = j} \binom{n_p}{i}
\]
\[
\beta_i(I) = \sum_{p=1}^{m} \binom{n_p}{i}.
\]

**Proof.** It suffices to compute the graded Betti numbers of $I$. Let $I_p = \langle f_1, \ldots, f_p \rangle$. Using Corollary 2.4 and Theorem 2.6 it is easy to find the Betti numbers of each $I_p$ with inductive process on $p$. \qed

A stable ideal is a monomial ideal $J \subset S = K[x_1, \ldots, x_n]$ with the following property: if $u$ is a monomial belonging to $J$, $m(u) := \max\{k : x_k | u\}$ and $i < m(u)$, then $x_i(u/x_{m(u)}) \in J$.

Suppose that $J$ is a stable ideal and the minimal monomial generators $u_1, \ldots, u_m$ of $J$ are ordered so that $i < q$ if and only if either the degree of $u_i$ is less than the degree of $u_q$, or the degrees are equal and $u_i >_{\text{revlex}} u_q$. One can easily see that for $p \geq 1$ we have
\[
\langle u_1, \ldots, u_{p-1} \rangle : u_p = \langle x_1, \ldots, x_{m(u_p)-1} \rangle.
\]

In particular $J$ has linear quotients.

The minimal free resolution of a stable ideal is given by the Eliahou-Kervaire
resolution [1]. From this resolution one can easily find the regularity, the projective dimension and the Betti numbers of a stable ideal. The Eliahou-Kervaire formula for the graded Betti numbers of a stable ideal $J$ as above is the following:

$$\beta_{i,i+j}(J) = \sum_{1 \leq p \leq m, \deg(u_p) = j \atop \deg(u_p) = j} \binom{m(u_p) - 1}{i}$$

From the above discussion it is clear that Corollary 2.7 generalizes the Eliahou-Kervaire formula to all the homogenous ideals with linear quotients.

Acknowledgements

The heart of this paper was born during the PRAGMATIC school which took place in Catania during the summer of 2008. So we would like to thank our teachers Jürgen Herzog and Volkmar Welker, who let us know beautiful aspects of mathematics and many interesting open problems.

We are grateful to organizers of PRAGMATIC, especially to Alfio Ragusa and Giuseppe Zappalà for organizing the school. Finally we also thank Aldo Conca for valuable discussions about the topics of the paper.

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