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GRADED BETTI NUMBERS OF IDEALS WITH LINEAR QUOTIENTS

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In this paper we show that every ideal with linear quotients is componentwise linear. We also generalize the Eliahou-Kervaire formula for graded Betti numbers of stable ideals to homogeneous ideals with linear quotients.

1. Introduction

Let K be a field, $S = K[x_1,...,x_n]$ the polynomial ring in n variables, and $I \subset S$ a homogeneous ideal. Let $\{f_1,...,f_m\}$ be a system of homogeneous generators for I. We say that I has *linear quotients with respect to the elements* $f_1,...,f_m$ if the ideal $\langle f_1,...,f_{i-1}\rangle : f_i$ is generated by linear forms for all $i = 2,\ldots,m$ (notice that this property depends on the order of the generators). Monomial ideals with linear quotients were introduced in [4] by Herzog and Takayama and have strong combinatorial implication, see the paper of Soleyman Jahan and Zheng [5].

In this paper we study the minimal free resolution of a graded ideal I with linear quotients with respect to a *minimal* system of homogeneous generators $\{f_1,\ldots,f_m\}$ (in this case we simply say that *I has linear quotients*). It is known

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that I is componentwise linear provided that $\deg(f_1) \leq ... \leq \deg(f_m)$, see the book of Herzog and Hibi [3] . We prove that I is componentwise linear without the additional assumption on the degrees, giving an affirmative answer to a question of Herzog (see Corollary 2.4). Our result is a generalization of [5, Corollary 2.8], where the authors prove componentwise linearity in case I is a monomial ideal and $f_1,\dots,f_m$ are monomials.

A large class of monomial ideals with linear quotients is the class of stable ideals. A minimal free resolution of stable ideals was constructed by Eliahou and Kervaire in [1], who also give an explicit formula for their graded Betti numbers. For an ideal with linear quotients we give a formula for its Betti numbers that generalizes the formula by Eliahou and Kervaire. We express the graded Betti numbers in terms of the numbers $\deg(f_i)$ and the minimal number of generators of $\langle f_1,\dots,f_{i-1}\rangle : f_i$, for $i = 1,\dots,m$. As a consequence we will also obtain the Castelnuovo-Mumford regularity and the projective dimension of I.

2. Graded Ideals With Linear Quotients

Let $I \subset S$ be a graded ideal and $\{f_1,...,f_m\}$ be a minimal system of homogeneous generators for I such that I has linear quotients with respect to $f_1,...,f_m$. Notice that if an ideal has linear quotients with respect to a minimal system of homogeneous generators then it need not have linear quotients with respect to all minimal homogeneous system of generators (see Example 2.1).
It is easy to see that $\deg(f_i) \geq \min\{\deg(f_1),...,\deg(f_{i-1})\}$ for $i = 2,...,m$. Particularly, $\deg(f_1) \leq \deg(f_i)$ for every $i = 1,\dots,m$. But in general, the sequence $\deg(f_1),...,\deg(f_{i-1})$ of degrees need not be increasing.
For example, the ideal $\langle xy, xy^3z + y^4z - y^3z^2, x^3 + x^2y - x^2z, x^2z^3\rangle$ has linear quotients in the given order, but $\deg(xy^3z + y^4z - y^3z^2) > \deg(x^3 + x^2y - x^2z)$.

Example 2.1. Set $\mathfrak{m} = \langle x_1,\dots,x_n\rangle \subset S$ the maximal irrelevant ideal. It is easy to see that $\mathfrak{m}^k$ has linear quotients with respect to its unique system of monomial generators for every natural number k (for example ordering the monomials lexicographically).
If K is large enough, we can choose a minimal system of homogeneous generators $\{g_1,\dots,g_t\}$ for $\mathfrak{m}^k$ such that $\mathrm{GCD}(g_i,g_j) = 1$ for each $i \neq j$. Hence it is clear that, if $k \geq 2$, $\mathfrak{m}^k$ does not have linear quotients with respect to $\{g_1,\dots,g_t\}$, for any order, otherwise the unique factorization would be contradicted.

If J is a graded ideal of S, then we write $J_{\langle j\rangle}$ for the ideal generated by all homogeneous polynomials of degree j belonging to J. Moreover, we write $J_{\leq k}$ for the ideal generated by all homogeneous polynomials in J whose degree is less than or equal to k. We will say that J is componentwise linear if $J_{\langle j\rangle}$ has a j-linear resolution for all j.

In [5, Lemma 2.1] the authors show that for any monomial ideal J with linear quotients with respect to the unique minimal system $G(J)$ of monomial generators of J, there exists a degree increasing order of the elements $u_1,...,u_m$ of $G(J)$ such that J has linear quotients with respect to this order. As a corollary they are able to prove that J is componentwise linear. It is worth remarking that if J has linear quotients with respect to a non-minimal system of generators, then J is not necessarily componentwise linear (see Example 2.5). We prove the result of [5] for an arbitrary graded ideal with linear quotients using completely different methods.

In the following for a $\mathbb{Z}$-graded module M we write $M(a)$ for the module obtained by *shifting degrees* by a, for any $a\in\mathbb{Z}$: i.e. $M(a)$ is the module M with the grading defined as $M(a)_b = M_{a+b}$.

For our purpose we need the next lemma.

Lemma 2.2. *Let $J=\langle f_1,...,f_m\rangle$ be a componentwise linear ideal and let $I = J+\langle f\rangle$ where f is a homogenous form of degree d. If the ideal $J:f$ is generated by linear forms then $I_{\langle j\rangle}$ has a j-linear resolution for each $j\le d$ provided that $f_1,...,f_m,f$ is a minimal system of generators for I.*

Proof. $I_{\langle j\rangle}=J_{\langle j\rangle}$ for each $j<d$ so it is enough to prove that $I_{\langle d\rangle}$ has d-linear resolution. First we show that $J:f=J_{\langle d\rangle}:f$. It is clear that $J_{\langle d\rangle}\subseteq J_{\le d}\subseteq J$. Therefore $J_{\langle d\rangle}:f\subseteq J_{\le d}:f\subseteq J:f$. Set $L=J:f=\langle l_1,...,l_r\rangle$ where l_i are linear forms. So there exist non-zero homogeneous polynomials $\lambda_{t_{1i}},...,\lambda_{t_{s_i i}}$ such that $l_i f=\sum_{k=1}^{s_i}\lambda_{t_{ki}}f_{t_{ki}}$ and $\deg(\lambda_{t_{ki}})+\deg(f_{t_{ki}})=\deg(l_i f)=d+1$. Since $f_1,...,f_m,f$ is a minimal system of generators for I, $\deg(\lambda_{t_{ki}})\ge 1$ and so $\deg(f_{t_{ki}})\le d$. Then $l_i f\in J_{\le d}$ but it is clear that $(J_{\langle d\rangle})_{d+1}=(J_{\le d})_{d+1}$. So $l_i f\in J_{\langle d\rangle}$ and $J:f=J_{\langle d\rangle}:f$.

Trivially $J_{\langle d\rangle}+\langle f\rangle = I_{\langle d\rangle}$, so by the above discussion

$$I_{\langle d\rangle}/J_{\langle d\rangle}\cong (S/(J_{\langle d\rangle}:f))(-d)=(S/(J:f))(-d)=S/L(-d).$$

Hence we can consider the short exact sequence

$$0\to J_{\langle d\rangle}\to I_{\langle d\rangle}\to S/L(-d)\to 0$$

which yields the long exact sequence of Tor

$$...\to \mathrm{Tor}_i^S(K,J_{\langle d\rangle})_{i+j}\to \mathrm{Tor}_i^S(K,I_{\langle d\rangle})_{i+j}\to \mathrm{Tor}_i^S(K,S/L(-d))_{i+j}\to ....$$

The Tor-groups on the right and on the left end of this sequence vanish for $j\ne d$ since the corresponding modules have d-linear resolution. This proves that $I_{\langle d\rangle}$ has d-linear resolution. □

We are ready to prove the main theorem of the paper.

Theorem 2.3. *Let $J = \langle f_1,...,f_m\rangle$ be a componentwise linear ideal and let $I = J+\langle f\rangle$ where f is a homogeneous form of degree d. If the ideal $J : f$ is generated by linear forms then I is componentwise linear provided that $f_1,...,f_m, f$ is a minimal system of generators for I.*

Proof. By Lemma 2.2, $I_{\langle j\rangle}$ has j-linear resolution for each $j \le d$. Set $\mathfrak{m} = \langle x_1,...,x_n\rangle$ and $p = \max\{\deg(f_i)\}$.
if $d > p$ then $I_{\langle d+j\rangle} = \mathfrak{m}^j I_{\langle d\rangle}$ for each $j \ge 1$. So the exact sequence

$$0 \to I_{\langle d+i\rangle} \to I_{\langle d+i-1\rangle} \to I_{\langle d+i-1\rangle}/I_{\langle d+i\rangle} \to 0$$

and induction on i shows that $I_{\langle d+i\rangle}$ has $d+i$-linear resolution. Therefore, I is componentwise linear ideal.
If $d \le p$, let $s = p-d$. In this case we use induction on s to prove the result. If $s = 0$, again by Lemma 2.2, I is componentwise linear arguing as above. Now suppose that $s \ge 1$ and the result is true for $s-1$. Let $L = J : f = \langle l_1,...,l_r\rangle$ minimally generated by r linear forms. If $r = n$ then $\mathfrak{m}f \subseteq J$. So $J_{\langle d+i\rangle} = I_{\langle d+i\rangle}$ for each $i \ge 1$. Now another application of Lemma 2.2 shows that I is a componentwise linear ideal. If $r < n$ we complete $\{l_1,...,l_r\}$ to a K-basis of S_1 by new linear forms $e_1,...,e_{n-r}$. Set $g_i = e_i f$ for $i = 1,...,n-r$. One can easily check that $(J+\langle g_1,...,g_{n-r}\rangle)_{\langle d+i\rangle} = I_{\langle d+i\rangle}$ for each $i \ge 1$. Therefore it is enough to show that $J+\langle g_1,...,g_{n-r}\rangle$ is componentwise linear.

Claim 1: For every $i = 1,...,n-r$ we have $(J+\langle g_1,...,g_{i-1}\rangle) : g_i = L+\langle e_1,...,e_{i-1}\rangle$.
Proof of Claim 1: Let $h \in (J+\langle g_1,...,g_{i-1}\rangle) : g_i$. So $he_i f \in J+\langle g_1,...,g_{i-1}\rangle$. Therefore there exist suitable coefficients γ_j and λ_j such that $he_i f = \sum_{j=1}^m \gamma_j f_j + \sum_{j=1}^{i-1} \lambda_j e_j f$. Then $he_i - \sum_{j=1}^{i-1} \lambda_j e_j \in J : f$. So $he_i \in \langle l_1,...,l_r, e_1,...,e_{i-1}\rangle$. Since $l_1,...,l_r, e_1,...e_{n-r}$ is a regular sequence $h \in L+\langle e_1,...,e_{i-1}\rangle$. The other inclusion is clear.

Claim 2: For every $i = 1,...,n-r$ the ideal $J+\langle g_1,...,g_i\rangle$ is minimally generated by $\{f_1,...,f_m, g_1,...,g_i\}$.
Proof of Claim 2: Suppose that for some t, $f_t \in \langle f_1,...,\widehat{f_t},...,f_m, g_1,...,g_i\rangle$. Then there exist suitable coefficients γ_j and λ_j such that $f_t = \sum_{\substack{j=1\\ j\neq t}}^{m} \gamma_j f_j + \sum_{j=1}^{i} \lambda_j g_j = \sum_{\substack{j=1\\ j\neq t}}^{m} \gamma_j f_j + (\sum_{j=1}^{i} \lambda_j e_j) f$. So $\{f_1,...,f_m, f\}$ is not a minimal system of generators for I which is a contradiction.
If for some t, $g_t \in \langle f_1,...,f_m, g_1,...,\widehat{g}_t,...,g_i\rangle$ then again one can find coefficients

γ_j and λ_j such that $g_t = \sum_{j=1}^{m} \gamma_j f_j + \sum_{\substack{j=1 \\ j\neq t}}^{i} \lambda_j g_j$. So $(e_t - \sum_{\substack{j=1 \\ j\neq t}}^{i} \lambda_j e_j) f \in J$ and therefore $e_t - \sum_{\substack{j=1 \\ j\neq t}}^{i} \lambda_j e_j \in L$, which is a contradiction.

Since $\deg(g_i) = d+1$, Claim 1 and Claim 2 together with the inductive hypothesis show that $J + \langle g_1, ..., g_i \rangle$ is componentwise linear for each $i = 1, ..., n-r$. This completes the proof.

□

From the above theorem the answer to a question by Herzog follows.

Corollary 2.4. *Let I be a homogeneous ideal and $\{f_1, ..., f_m\}$ be a minimal system of generators for I. If I has linear quotients with respect to $f_1, ..., f_m$ then I is a componentwise linear ideal.*

Proof. Set $I_i = \langle f_1, ..., f_i \rangle$ for $i = 1, \ldots, m$. It is clear that I_1 has a linear resolution. In particular I_1 is a componentwise linear ideal. Therefore, one can apply Theorem 2.3 and induction on i to see that each I_i is componentwise linear for $i = 1, \ldots, m$. □

Example 2.5. In Corollary 2.4 one cannot remove the hypothesis that I is minimally generated by $\{f_1, \ldots, f_m\}$. For instance, let I be the monomial ideal $I = \langle x^2, y^2 \rangle \subset K[x, y]$. Trivially $I = I_{\langle 2 \rangle}$ does not have a 2-linear resolution, but it has linear quotients with respect to the system of generators $\{x^2, xy^2, y^2\}$.
In general we can observe that every $\mathfrak{m}$-primary ideal $I \subset S$ ($\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$), has linear quotients with respect to a homogeneous system of generators. In fact let h be such that $\mathfrak{m}^h \subset I$. Then clearly by ordering the elements of $\mathfrak{m}^h$ appropriately (for example lexicographically), I has linear quotients with respect to $\{\mathfrak{m}^h, I_{h-1}, \ldots, I_s\}$, where s is minimal such that $I_s \neq 0$.

In the following we compute the graded Betti numbers of ideals with linear quotients. For this, we use the formula (1) of Herzog and Hibi (see [2, Proposition 1.3]) for the graded Betti numbers of an arbitrary componentwise linear ideal J.

$$\beta_{i,i+j}(J) = \beta_i(J_{\langle j \rangle}) - \beta_i(\mathfrak{m}J_{\langle j-1 \rangle}) \qquad \text{for all } \ i, j \tag{1}$$

Theorem 2.6. *Let $J = \langle f_1, ..., f_m \rangle$ be a componentwise linear ideal and let $I = J + \langle f \rangle$ where f is a homogeneous form of degree d. If the ideal $J : f$ is minimally generated by r linear forms then*

$$\beta_{i,i+j}(I) = \beta_{i,i+j}(J) \qquad if \ \ j \neq d,$$

$$\beta_{i,i+d}(I) = \beta_{i,i+d}(J) + \binom{r}{i},$$

provided that $\{f_1,...,f_m,f\}$ *is a minimal system of homogeneous generators for* I.

Proof. Consider the exact sequence

$$0 \to J \to I \to S/L(-d) \to 0 \tag{2}$$

which yields the long exact sequence

$$\begin{aligned} \dots \quad \to \mathrm{Tor}_{i+1}^S(K,S/L(-d))_{i+j} \quad &\to \mathrm{Tor}_i^S(K,J)_{i+j} \to \\ &\to \mathrm{Tor}_i^S(K,I)_{i+j} \to \mathrm{Tor}_i^S(K,S/L(-d))_{i+j} \to \end{aligned}$$

The Tor-groups on the right and on the left end of this sequence vanish for $j \neq d, d+1$ since the corresponding module has a d-linear resolution. This proves that $\beta_{i,i+j}(I) = \beta_{i,i+j}(J)$ if $j \neq d, d+1$.

First consider $j = d$. As discussed in the proof of Lemma 2.2, one has the short exact sequence

$$0 \to J_{\langle d\rangle} \to I_{\langle d\rangle} \to S/L(-d) \to 0$$

which yields the long exact sequence

$$... \to \mathrm{Tor}_i^S(K,J_{\langle d\rangle})_{i+j} \to \mathrm{Tor}_i^S(K,I_{\langle d\rangle})_{i+j} \to \mathrm{Tor}_i^S(K,S/L(-d))_{i+j} \to ...$$

Since both $J_{\langle d\rangle}$ and $S/L(-d)$ have a d-linear resolution, the above long exact sequence gives

$$\beta_i(I_{\langle d\rangle}) = \beta_i(J_{\langle d\rangle}) + \beta_i(S/L(-d))$$

Moreover the minimal free resolution of $S/L(-d)$ is given by Koszul complex, so

$$\beta_i(S/L(-d)) = \binom{r}{i}$$

Thus by (1),

$$\begin{aligned} \beta_{i,i+d}(I) &= \beta_i(I_{\langle d\rangle}) - \beta_i(\mathfrak{m}I_{\langle d-1\rangle}) \\ &= \beta_i(I_{\langle d\rangle}) - \beta_i(\mathfrak{m}J_{\langle d-1\rangle}) \\ &= \beta_i(J_{\langle d\rangle}) + \binom{r}{i} - \beta_i(\mathfrak{m}J_{\langle d-1\rangle}) \\ &= \beta_{i,i+d}(J) + \binom{r}{i}. \end{aligned}$$

Now consider $j = d+1$. By (2) we have the following exact sequence

$$\begin{aligned} 0 \quad &\to \mathrm{Tor}^S_{i+1}(K,J)_{i+d+1} \to \mathrm{Tor}^S_{i+1}(K,I)_{i+d+1} \to \mathrm{Tor}^S_{i+1}(K,S/L(-d))_{i+d+1} \\ &\to \mathrm{Tor}^S_i(K,J)_{i+d+1} \to \mathrm{Tor}^S_i(K,I)_{i+d+1} \to 0. \end{aligned}$$

So by the first part $\beta_{i,i+d+1}(I) =$

$$\begin{aligned} = \beta_{i,i+d+1}(J) - \binom{r}{i+1} + \beta_{i+1,i+d+1}(J) + \binom{r}{i+1} - \beta_{i+1,i+d+1}(J) \\ = \beta_{i,i+d+1}(J). \end{aligned}$$

□

Corollary 2.7. *Let I be a homogeneous ideal with linear quotients with respect to $f_1,\dots,f_m$ where $\{f_1,\dots,f_m\}$ is a minimal system of homogeneous generators for I. Let n_p be the minimal number of homogeneous generators of $\langle f_1,..,f_{p-1}\rangle : f_p$ for $p = 1,\dots,m$. Then*

$$\mathrm{reg}(I) = \max\{\deg(f_p) : p = 1,\dots,m\}$$

$$\mathrm{projdim}(I) = \max\{n_p | 1 \le p \le m\}$$

$$\beta_{i,i+j}(I) = \sum_{1\le p\le m,\ \deg(f_p)=j} \binom{n_p}{i}$$

$$\beta_i(I) = \sum_{p=1}^{m} \binom{n_p}{i}.$$

Proof. It suffices to compute the graded Betti numbers of I. Let $I_p = \langle f_1,\dots,f_p\rangle$. Using Corollary 2.4 and Theorem 2.6 it is easy to find the Betti numbers of each I_p with inductive process on p. □

A stable ideal is a monomial ideal $J \subset S = K[x_1,\dots,x_n]$ with the following property: if u is a monomial belonging to J, $m(u) := \max\{k : x_k|u\}$ and $i < m(u)$, then $x_i(u/x_{m(u)}) \in J$.
Suppose that J is a stable ideal and the minimal monomial generators $u_1,\dots,u_m$ of J are ordered so that $i < q$ if and only if either the degree of u_i is less than the degree of u_q, or the degrees are equal and $u_i >_{\mathrm{revlex}} u_q$. One can easily see that for $p \ge 1$ we have

$$\langle u_1,\dots,u_{p-1}\rangle : u_p = \langle x_1,\dots,x_{m(u_p)-1}\rangle.$$

In particular J has linear quotients.
The minimal free resolution of a stable ideal is given by the Eliahou-Kervaire

resolution [1]. From this resolution one can easily find the regularity, the projective dimension and the Betti numbers of a stable ideal. The Eliahou-Kervaire formula for the graded Betti numbers of a stable ideal J as above is the following:

$$\beta_{i,i+j}(J) = \sum_{1\leq p\leq m,\ \deg(u_p)=j} \binom{m(u_p)-1}{i}$$

From the above discussion it is clear that Corollary 2.7 generalizes the Eliahou-Kervaire formula to all the homogenous ideals with linear quotients.

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