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# ESSENTIALLY $\alpha$-HYPONORMAL OPERATORS WITH ESSENTIAL SPECTRUM OF AREA ZERO.

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We prove that essentially $\alpha$-hyponormal operators with zero area of the essential spectrum are essentially normal .

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all linear bounded operators on $\mathcal{H}$ and let $\mathbb{K}$ denote the two-sided ideal of all compact operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *essentially normal* if $C_T := T^*T - TT^*$ is a compact operator, and $T$ is called *$\alpha$-hyponormal* (*essentially $\alpha$-hyponormal*) if $C_T^\alpha := (T^*T)^\alpha - (TT^*)^\alpha$ is a positive definite operator ( $C_T^\alpha$ is the sum of a positive definite operator and a compact operator, or equivalently, $\pi(T)$ is an $\alpha$-hyponormal operator in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathbb{K}$, where $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbb{K}$ is the canonical map, respectively). The class of 1-hyponormal and essentially 1-hyponormal operators will be simply referred as hyponormal and essentially hyponormal operators. Furthermore, let $\sigma(T)$ and $\sigma_e(T)$ denote the spectrum and the essential spectrum of $T$, and let $||T||_e$ denote the norm of $\pi(T)$ in the Calkin algebra.

In [3], B. Conway and N. Feldman proved that for a hyponormal operator $T$,

$$||C_T||_e \leq \frac{\mu_2(\sigma_e(T))}{\pi} ,$$

where $\mu_2$ denotes the area measure. Consequently, a hyponormal operator with essential spectrum of area zero is essentially normal. In [4], S. Lohaj and M.

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Lohaj proved that an essentially hyponormal operator with essential spectrum contained in a circle is essentially hyponormal. Their proof is based on the fact that hyponormal elements of a $C^*$-algebra have equal norm and spectral radius. On the other hand, Conway-Feldman's proof made use of Putnam's inequality (cf. [6]), which states that for a hyponormal operator $T$,

$$||C_T|| \leq \frac{\mu_2(\sigma(T))}{\pi}.$$

Putnam's inequality has been extended to $\alpha$-hyponormal operators. Let $\mu_\alpha$ denote the measure defined by $\mu_\alpha(E) = \frac{\alpha}{2}\int_E r^{\alpha-1}\,dr\,d\theta$, which for $\alpha = 2$ is the area measure.

Putnam's inequality has been generalized by many authors to various classes of hyponormal related operators. For instance (although the literature is quite extensive), according to [1] and [2],

$$||C_T^\alpha|| \leq \frac{\mu_{2\alpha}(\sigma(T))}{\pi}, \tag{1}$$

for an $\alpha$-hyponormal operator $T$ with $0 < \alpha \leq 1$.

It is the purpose of this note to show that the results in [3] and [4] can be extended to a larger class of operators.

**Theorem 1.** *If $T \in \mathcal{L}(\mathcal{H})$ is an essentially $\alpha$-hyponormal operator with $\alpha \in (0,1]$, then*

$$||C_T^\alpha||_e \leq \frac{\mu_{2\alpha}(\sigma_e(T))}{\pi}.$$

*Proof.* Let $T \in \mathcal{L}(\mathcal{H})$ be an essentially $\alpha$-hyponormal operator and let $\mathcal{A_T}$ be the unital $C^*$-algebra generated by $\pi(T)$. Let $\rho : \mathcal{A_T} \to \mathcal{L}(\mathcal{K})$ be a faithful representation and denote $\rho(\pi(T))$ by $Q_T$. Since $\mathcal{A_T}$ is a unital algebra, $\sigma(Q_T) = \sigma_e(T)$ (according [5], Prop. 1.3) and $Q_T$ is an $\alpha$-hyponormal operator. According to inequality (1),

$$||C_{Q_T}^\alpha|| \leq \frac{\mu_{2\alpha}(\sigma(Q_T))}{\pi},$$

and since $\rho$ is an isometric $^*$-isomorphism, $||C_{Q_T}^\alpha|| = ||C_T^\alpha||_e$ , which concludes the proof.

□

**Corollary 2.** *If $T$ is an essentially $\alpha$-hyponormal operator for $\alpha > 0$ and $\mu_2(\sigma_e(T)) = 0$, then $T$ is essentially normal.*

*Proof.* If $\alpha \in (0,1]$, the theorem implies that $C_T^\alpha \in \mathbb{K}$, and a standard argument involving approximations by polynomials implies that $C_T$ is compact. If $\alpha > 1$, then $Q_T$ is $\alpha$-hyponormal and according to Lowner inequality, $Q_T$ is hyponormal and Putnam's inequality implies tat $Q_T$ is normal, or $T$ is essentially normal. □

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