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A NOTE ON QUASI-POLARIZED SURFACES OF GENERAL TYPE WHOSE SECTIONAL GENUS IS EQUAL TO THE IRREGULARITY

YOSHIAKI FUKUMA

Let (X,L) be a quasi-polarized surface. In our previous papers, we studied (X,L) with $\kappa(X)=2$, $h^0(L)>0$ and $g(X,L)=h^1(\mathcal{O}_X)$. Here $g(X,L)$ denotes the sectional genus of (X,L). In this note, we give the classification of quasi-polarized surfaces (X,L) of this type completely.

1. Introduction

Let X be a smooth projective surface over the field of complex numbers $\mathbb{C}$ and let L be a nef and big (resp. an ample) divisor on X. Then the pair (X,L) is called a quasi-polarized (resp. polarized) surface. Then $g(X,L)\geq q(X)$ can be proved if (a) $\kappa(X)\leq 1$, or (b) $\kappa(X)=2$ and $h^0(L)>0$ (see [4]). (Here $g(X,L)=1+1/2(K_X+L)L$ denotes the *sectional genus* of (X,L) and $q(X)=h^1(\mathcal{O}_X)$ is the irregularity of X.) If $g(X,L)\geq q(X)$ holds, then it is natural and interesting to study (X,L) with $g(X,L)=q(X)$. In our paper [4] we classified quasi-polarized surfaces (X,L) with $g(X,L)=q(X)$ and $\kappa(X)\leq 1$ (see also [7]). Moreover in [3] and [5], we studied (X,L) with $g(X,L)=q(X)$, $\kappa(X)=2$ and $h^0(L)>0$. In the latter case we were able to characterize types of the divisor

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$D \in |L|$, but we could not characterize X completely (in particular for the case where D is irreducible).

In this short note, by using results of [1], [6] and [9], we consider the case in which $g(X,L) = q(X)$, $\kappa(X) = 2$ and $h^0(L) > 0$, and we will give a characterization of (X,L) with $g(X,L) = q(X)$, $\kappa(X) = 2$ and $h^0(L) > 0$.

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2. Main results

First we recall the following definitions.

Definition 2.1. ([4, Definition 1.9 (2)]) Let (X,L) be a quasi-polarized surface. Then (X,L) is called *L-minimal* if $LE > 0$ for any (-1)-curve E on X.

Definition 2.2. Let (X,L) and (Y,A) be quasi-polarized surfaces. Then (X,L) is called *a simple blowing up of* (Y,A) if X is the blowing up of Y at one point on Y, and $L = \mu^*(A) - E$, where $\mu : X \to Y$ is the birational morphism and E is the (-1)-curve.

Definition 2.3. (1) Let C be a smooth projective curve. Then $S^2(C)$ denotes the 2-fold symmetric product of C, $\pi : C \times C \to S^2(C)$ is the natural map and let $C_x := \pi(C \times \{x\}) = \pi(\{x\} \times C)$ for $x \in C$.

(2) Let B_1 and B_2 be smooth projective curves. Then $B_1(x)$ (resp. $B_2(y)$) denotes the divisor $B_1 \times \{x\}$ (resp. $\{y\} \times B_2$) on $B_1 \times B_2$, where $x \in B_2$ (resp. $y \in B_1$).

Theorem 2.4. *Let (X,L) be an L-minimal quasi-polarized surface with $\kappa(X) = 2$ and $h^0(L) > 0$. Assume that $g(X,L) = q(X)$. Then $h^0(L) = 1$ and the effective divisor $D \in |L|$ is reduced. Moreover (X,D) is one of the following types.*

(1) *$(X,D) \cong (S^2(C), C_x)$, where C is a smooth projective curve with $g(C) \geq 3$ and $x \in C$. In this case $q(X) \geq 3$.*

(2) *$(X,D) \cong (C_0 \times C_1, C_0(x) + C_1(y))$, where C_0 and C_1 are smooth projective curves with $g(C_0) \geq 2$ and $g(C_1) \geq 2$, $x \in C_1$ and $y \in C_0$. In this case $q(X) \geq 4$.*

(3) *There exist smooth projective curves A_1 and A_2 such that $g(A_1) \geq 2$ and $g(A_2) \geq 2$, (X,D) is a simple blowing up of $(A_1 \times A_2, A_1(x) + A_2(y))$ for $x \in A_2$ and $y \in A_1$, and the center of the simple blowing up is $A_1(x) \cap A_2(y)$. In this case $q(X) \geq 4$.*

(4) *There exist smooth projective curves* A_1 *and* A_2 *such that* $g(A_1) \geq 2$ *and* $g(A_2) \geq 2$, (X,D) *is a simple blowing up of* $(A_1 \times A_2, A_1(x) + A_2(y))$ *for* $x \in A_2$ *and* $y \in A_1$, *and the center of the simple blowing up is contained in* $(A_1(x) \cup A_2(y)) \setminus (A_1(x) \cap A_2(y))$. *In this case* $q(X) \geq 4$.

Proof. First we note that $h^0(L) = 1$ in this case by [5, Theorem 2.1] and let $D \in |L|$ be the effective divisor which is linearly equivalent to L. Then D is 1-connected (see e.g. [8, Lemma 2.6 (i)]), and we see from [5, Theorem 2.1] that D is reduced by [3, Proposition 3.2] and D is one of the following two types.

(A) D is an irreducible smooth projective curve.

(B) Assume that D is not irreducible. Let $D = \sum_{k=0}^{l} C_k$. Then each irreducible component C_i of D is smooth and D is one of the following.

(B-I) $C_0C_j = 1$ and $C_j^2 = -1$ for every j with $1 \leq j \leq l$, $C_0^2 = -l+1$, and $C_iC_j = 0$ for every integers i and j with $1 \leq i \leq l$, $1 \leq j \leq l$ and $i \neq j$.

(B-II) $l = 1$, $C_0^2 = C_1^2 = 0$ and $C_0C_1 = 1$.

(I) First we consider the type (B). We assume that D satisfies the type (B-II). Then by [5, Theorem 3.4 (4)] we see that X is minimal. In this case we can prove the following. (Here we use notation in Definition 2.3 (2).)

Claim 2.1. $X \cong C_0 \times C_1$ *and* $D = C_0(x) + C_1(y)$ *for* $x \in C_1$ *and* $y \in C_0$.

Proof. Since $g(C_0) \geq 2$, $g(C_1) \geq 2$, $C_0C_1 = 1$ and $C_0^2 = C_1^2 = 0$, we see from the proof of [9, Proposition 3.1] that there exists a fiber space $f_1 : X \to C_0$ (resp. $f_2 : X \to C_1$) such that C_1 (resp. C_0) is a fiber of f_1 (resp. f_2). Then there exists a morphism $h : X \to C_0 \times C_1$ such that $f_1 = p_1 \circ h$ (resp. $f_2 = p_2 \circ h$), where $p_1 : C_0 \times C_1 \to C_0$ (resp. $p_2 : C_0 \times C_1 \to C_1$) denotes the first (resp. the second) projection. Since $C_0C_1 = 1$, we see that h is birational. Hence h is an isomorphism because X is minimal. Therefore we get the assertion. □

In this case $g(C_0) \geq 2$ and $g(C_1) \geq 2$ hold. So we get $q(X) = g(C_0) + g(C_1) \geq 4$.

Next we consider the case where D satisfies the type (B-I).

Claim 2.2. *If* (X,D) *is the type* (B-I)*, then* X *is not minimal.*

Proof. Assume that X is minimal. Then we note that $(C_0 + C_1 + \cdots + C_{l-1})^2 = 0$ and $g(C_0 + C_1 + \cdots + C_{l-1}) \geq 1$. We also note that $g(C_l) \geq 1$. (If $g(C_l) = 0$, then C_l is a (-1)-curve and $DC_l = 0$. But this is impossible because $D \in |L|$ and (X,L) is L-minimal.) Hence by the same argument as in [9, Proposition 3.1], we see that there exist smooth projective curves B and F such that X is birationally equivalent to $B\times F$. But since X is minimal, we have $X\cong B\times F$. Moreover by the proof of [9, Proposition 3.1] we infer that $C_0+C_1+\cdots+C_{l-1}$ is a fiber of either the first projection $X\to B$ or the second projection $X\to F$. But this is impossible if $l\geq 2$. Therefore $l=1$ and $D=C_0+C_1$. Here we note that $g(C_0)\geq 1$ because $C_0^2=0$, and $g(C_1)\geq 1$ since $DC_1=0$, $C_1^2=-1$, $D\in|L|$ and (X,L) is L-minimal. Furthermore, applying the proof of [9, Proposition 3.1] again, we see that there exists a smooth projective curve G such that $X\cong C_0\times G$ and C_0 is a fiber of the second projection $X\to G$ because X is minimal by assumption. In particular, the intersection number $K_XC_1=(2g(C_0)-2)GC_1+(2g(G)-2)C_0C_1$ is even. But this is impossible because $C_1^2=-1$ and $(K_X+C_1)C_1$ is even. Hence we get the assertion of Claim 2.2. □

By [5, Theorem 3.4] and Claims 2.1 and 2.2, we see that $l\leq 2$ and there exist smooth projective curves A_1 and A_2 such that the type (B-I) is one of the following types.

(i) If $l=2$, then (X,D) is a simple blowing up of $(A_1\times A_2, A_1(x)+A_2(y))$ for $x\in A_2$ and $y\in A_1$, and the center of the simple blowing up is $A_1(x)\cap A_2(y)$.

(ii) If $l=1$, then (X,D) is a simple blowing up of $(A_1\times A_2, A_1(x)+A_2(y))$ for $x\in A_2$ and $y\in A_1$, and the center of the simple blowing up is contained in $(A_1(x)\cup A_2(y))\setminus(A_1(x)\cap A_2(y))$.

Here we note that $g(A_1)\geq 2$ and $g(A_2)\geq 2$ because $\kappa(X)=2$. Thus $q(X)=g(A_1)+g(A_2)\geq 4$.

(II) Next we consider the type (A). Here we carry on the proof by an argument similar to that in the proof of [9, Theorem 1.1]. First we prove the following claim.

Claim 2.3. *X is not birationally equivalent to $B_1\times B_2$, where B_1 and B_2 are smooth projective curves.*

Proof. Assume that X is birationally equivalent to $B_1\times B_2$. Here we note that $q(X)=g(B_1)+g(B_2)\geq 4$ since $\kappa(X)=2$. Let $\mu: X\to B_1\times B_2$ be the birational morphism, let $f_i: X\to B_1\times B_2\to B_i$, and let F_i be a general fiber of f_i for $i=1,2$. If $DF_1=1$ or $DF_2=1$, then D is isomorphic to B_1 or B_2. In particular $g(X,D)=g(B_1)$ or $g(X,D)=g(B_2)$. But since $g(X,D)=q(X)=g(B_1)+g(B_2)$, $g(B_1)\geq 2$ and $g(B_2)\geq 2$ by assumption, this is impossible. Hence $DF_i\geq 2$ for $i=1,2$.

Therefore

$$\begin{aligned} K_X D &\geq (K_{B_1\times B_2})\mu_*(D) \\ &\geq 2(2g(B_1)-2)+2(2g(B_2)-2) \\ &= 4(q(X)-2), \end{aligned}$$

and we have

$$\begin{aligned} g(X,D) &= 1+\frac{1}{2}(K_X D+D^2) \\ &\geq 1+\frac{1}{2}D^2+2(q(X)-2) \\ &= 2q(X)-3+\frac{1}{2}D^2 \\ &= q(X)+q(X)-3+\frac{1}{2}D^2 \\ &> q(X). \end{aligned}$$

But this contradicts the assumption. □

Here we put $d := D^2$. Then we prove the following.

Claim 2.4. *$q(X) \geq d$.*

Proof. Assume that $q(X) < d$. Let $\lambda : X \to S$ be the minimalization of X and let $D_S = \lambda_*(D)$. Then we note that D_S is nef and big, $d \leq D_S^2$, and $K_X D \geq K_S D_S$. Since $q(S) = q(X) \neq 0$, we see from [2, Théorème 6.1 and Addendum] that $K_S^2 \geq 2h^0(K_S) \geq 2q(S)$. Hence by Hodge index theorem we have $(K_S D_S)^2 \geq (K_S)^2(D_S)^2 \geq 2q(S)d > q(X)^2$. Therefore $K_X D \geq K_S D_S > q(X)$. On the other hand, since $g(X,D) = q(X)$ we have $d+q(X) < d+K_X D = 2q(X)-2$. Namely we get $d < q(X)-2$. But this contradicts the assumption that $q(X) < d$. Hence we get the assertion. □

By [9, Proposition 4.3] and Claim 2.4, there exists a d-dimensional system $\mathscr{C}$ of effective divisors numerically equivalent to D. For any member D' of $\mathscr{C}$, we see that D' is nef and big, and $g(X,D') = q(X)$. Here we note that (X,L) is L-minimal, $D \in |L|$, and D' is numerically equivalent to D. So we infer that $D'E > 0$ for any (-1)-curve E on X. Hence by Claim 2.3 and the above argument (I), we see that D' is the type (A) above, that is, D' is irreducible and smooth. Namely any member of $\mathscr{C}$ is irreducible and smooth. Since the Jacobian of every curve of $\mathscr{C}$ is isomorphic to $\mathrm{Pic}^0(D)$, we see that any curve of $\mathscr{C}$ is isomorphic to D. If $d > 1$, then by [6, Lemma 2.2.1] X is not of general type. So we get $d = 1$. Hence by [1, Theorem 0.20], we infer that there exist a smooth projective curve C and a birational morphism $\rho : X \to S^2(C)$ such that $\rho_*(D) = C_x$. (Here we use notation in Definition 2.3 (1).) In particular $1 = D^2 \leq (C_x)^2 = 1$. Since (X,L) is L-minimal we have $X \cong S^2(C)$ and $D = C_x$. In this case $g(C) \geq 3$ holds because $\kappa(X) = 2$. So we have $q(X) = g(C) \geq 3$. Therefore we get the assertion of Theorem 2.4. □

As a corollary, we can get the characterization of polarized surfaces (X,L) with $\kappa(X) = 2$, $h^0(L) > 0$ and $g(X,L) = q(X)$. If (X,D) is either the type (3) or type (4) in Theorem 2.4, then D is not ample. So we get the following corollary.

Corollary 2.5. *Let (X,L) be a polarized surface with $\kappa(X) = 2$, $h^0(L) > 0$ and $g(X,L) = q(X)$. Then (X,L) is one of the following types.*

(a) $(X,L) \cong (S^2(C), \mathscr{O}(C_x))$, *where C is a smooth projective curve and $x \in C$. In this case $g(C) \geq 3$.*

(b) $(X,L) \cong (C_0 \times C_1, \mathscr{O}(C_0(x) + C_1(y)))$, *where C_0 and C_1 are smooth projective curves with $g(C_0) \geq 2$ and $g(C_1) \geq 2$, $x \in C_1$ and $y \in C_0$.*

We also note that the following corollary is the answer to [3, Problem 5.2].

Corollary 2.6. *Let (X,L) be a polarized surface with $\kappa(X) = 2$, $h^0(L) > 0$ and $g(X,L) = q(X)$. Let D be the member of $|L|$. If D is irreducible, then $D^2 = 1$.*

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YOSHIAKI FUKUMA
Department of Mathematics
Faculty of Science
Kochi University
Akebono-cho, Kochi 780-8520
Japan
e-mail: fukuma@kochi-u.ac.jp