LE MATEMATICHE
Vol. LXIII (2008) – Fasc. II, pp. 165–171

SOME REMARKS ON THE STANLEY DEPTH FOR MULTIGRADED MODULES

MIRCEA CIMPOEAS

We show that Stanley's conjecture holds for any multigraded module M over S, with $\operatorname{sdepth}(M) = 0$, where $S = K[x_1,\dots,x_n]$. Also, we give some bounds for the Stanley depth of the powers of the maximal irrelevant ideal in S.

Keywords: Stanley depth, monomial ideal.

2000 Mathematics Subject Classification: Primary: 13H10, Secondary: 13P10.

Introduction

Let K be a field and $S = K[x_1,\dots,x_n]$ the polynomial ring over K. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathscr{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i]$ as K-vector space, where $m_i \in M$, $Z_i \subset \{x_1,\dots,x_n\}$ such that $m_i K[Z_i]$ is a free $K[Z_i]$-module. The latter condition is needed, since the module M can have torsion. We define $\operatorname{sdepth}(\mathscr{D}) = min_{i=1}^{r}|Z_i|$ and $\operatorname{sdepth}(M) = max\{\operatorname{sdepth}(M)|\ \mathscr{D}$ is a Stanley decomposition of $M\}$. The

Entrato in redazione: 2 gennaio 2009

AMS 2000 Subject Classification: Primary: 13H10, Secondary: 13P10
Keywords: Stanley depth, monomial ideal, multigraded module
This paper was supported by CNCSIS, ID-PCE, 51/2007
The author would like to express his gratitude to the organizers of PRAGMATIC 2008, Catania, Italy and especially to Professor Jurgen Herzog. Also, the author would like to thank Professor Dorin Popescu for valuables discussions regarding this paper.

number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M. Herzog, Vladoiu and Zheng show in [9] that this invariant can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. A computer implementation of this algorithm, with some improvements, is given by Rinaldo in [14].

Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Stanley's conjecture says that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$. The Stanley conjecture for S/I was proved for $n \leq 5$ and in other special cases, but it remains open in the general case. See for instance, [4], [8], [10], [3] and [12]. Another interesting problem is to explicitly compute the sdepth. This is difficult, even in the case of monomial ideals! Some small progresses were made in [13], [9], [6], [7] and [15].

In the first section, we prove that the Stanley conjecture holds for modules with $\operatorname{sdepth}(M) = 0$, see Theorem 1.4. As a consequence, it follows that any torsion free module M has $\operatorname{sdepth}(M) \geq 1$. In the second section, we give an upper bound for the Stanley depth of the powers of the maximal ideal $\mathbf{m} = (x_1, \ldots, x_n) \subset S$, see Theorem 2.2. We conjecture that $\operatorname{sdepth}(\mathbf{m}^k) = \lceil \frac{n}{k+1} \rceil$, for any positive integer k.

1. Stanley's conjecture for modules with sdepth zero.

Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. We use an idea of Herzog, in order to obtain a decomposition of M, similar to the Janet decomposition given in [2]. For any $j \geq 1$, we have a natural surjective map $\varphi_j : M \to x_n^j M$ given by the multiplication with x_n^j. Obviously, $\varphi_j(x_n M) \subset x_n^{j+1} M$ and therefore φ_j induces a natural surjection $\bar{\varphi}_j : M/x_n M \to x_n^j M / x_n^{j+1} M$. We write $L_j = Ker(\bar{\varphi}_j)$.

Note that $L_j \subset L_{j+1}$ for any j, since we have a natural surjection

$$x_n^j M / x_n^{j+1} M \to x_n^{j+1} M / x_n^{j+2} M$$

given by multiplication with x_n. As $M/x_n M$ is finitely generated, it follows that there exists a nonnegative integer q such that $L_q = L_{q+1} = \cdots$ and moreover $x_n^j M / x_n^{j+1} M \cong x_n^{j+1} M / x_n^{j+2} M$ for any $j \geq q$. Now, we can prove the following Lemma.

Lemma 1.1. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module and q such that $L_q = L_{q+1} = \cdots$. Then we have the following decomposition of M, as K-vector space:*

$$M \cong M/x_n M \oplus \cdots \oplus x_n^{q-1} M / x_n^q M \oplus x_n^q M / x_n^{q+1} M[x_n].$$

Proof. Note that, since M is graded, $\bigcap x_n^j M = 0$. Therefore, we have

$$M = M/x_n M \oplus x_n M = M/x_n M \oplus x_n M / x_n^2 M \oplus x_n^2 M = \cdots = \bigoplus_{j \geq 0} x_n^j M / x_n^{j+1} M.$$

Since $x_n^jM/x_n^{j+1}M \cong x_n^{j+1}M/x_n^{j+2}M$ for any $j \geq q$, the proof of Lemma is complete. □

Note that each factor $x_n^jM/x_n^{j+1}M$ naturally carries the structure of a multigraded S'-module, where $S' = K[x_1,\dots,x_{n-1}]$. Also, if $M = S/I$, where $I \subset S$ is a monomial ideal, the above decomposition is exactly the Janet decomposition of S/I, with respect to the variable x_n.

Lemma 1.2. *Let M be a multigraded S-module. Then* $\operatorname{sdepth}(M) = n$ *if and only if M is free.*

Proof. If M is free, it follows that $M \cong \bigoplus_{i=1}^r S(-a_i)$, where $a_i \in \mathbb{Z}^n$ are some multidegrees. Therefore, M has a basis $\{e_1,\dots,e_n\}$ where e_i correspond to $1 \in S(-a_i)$. Therefore $M = \bigoplus e_iS$ is a Stanley decomposition of M and thus $\operatorname{sdepth}(M) = n$. Conversely, given a Stanley decomposition $M = \bigoplus e_iS$, it follows that $M \cong \bigoplus_{i=1}^r S(-a_i)$, where $deg(e_i) = a_i$. □

Lemma 1.3. *Let M be a graded $K[x]$-module. Then, the following are equivalent:*

(1) M is free.
(2) M is torsion free.
(3) $\operatorname{depth}(M) = 1$.
(4) $\operatorname{sdepth}(M) = 1$.

Proof. The equivalences $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ are well known. $(4) \Leftrightarrow (1)$ is the case $n = 1$ of the previous Lemma. □

Let $\mathbf{m} = (x_1,\dots,x_n) \subset S$ be the maximal irrelevant ideal. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. We denote $sat(M) = (0 :_M \mathbf{m}^\infty) = \bigcup_{k\geq 1}(0 :_M \mathbf{m}^k)$ the *saturation* of M. It is well known, that $\operatorname{depth}(M) = 0$ if and only if $\mathbf{m} \in Ass(M)$ if and only if $sat(M) \neq 0$. On the other hand, $sat(M/sat(M)) = 0$. Note that if $I \subset S$ is a monomial ideal, then $sat(S/I) = I^{sat}/I$, where $I^{sat} = (I : \mathbf{m}^\infty)$ is the saturation of the ideal I. We prove the following generalization of [7, Theorem 1.5].

Theorem 1.4. *Let M be a multigraded S-modules. If* $\operatorname{sdepth}(M) = 0$ *then* $\operatorname{depth}(M) = 0$. *Conversely, if* $\operatorname{depth}(M) = 0$ *and $dim_K(M_a) \leq 1$ for any $a \in \mathbb{Z}^n$, then* $\operatorname{sdepth}(M) = 0$.

Proof. We use induction on n. If $n = 1$, then we are done by Lemma 1.3. Suppose $n > 1$. We consider the decomposition

$$(*)\ \ M \cong M/x_nM \oplus \cdots \oplus x_n^{q-1}M/x_n^qM \oplus x_n^qM/x_n^{q+1}M[x_n],$$

given by Lemma 1.2. We define $M_j := x_n^j M/x_n^{j+1}M$ for $j \in [q]$. As $\operatorname{sdepth}(M) = 0$, it follows that $\operatorname{sdepth}(M_j) = 0$ for some $j < q$. We have $M_j = sat(M_j) \oplus M/sat(M_j)$, where $sat(M_j)$ is the saturation of M_j as a S'-module. If there exists some nonzero element $m \in sat(M_j)$ such that $x_n^j m = 0$, it follows that $m \in sat(M)$ and thus $sat(M) \neq 0$.

For the converse, we assume $depth(M) > 0$. It follows that $x_n sat(M_j) \subset sat(M_{j+1})$ for any $j < q$. Since $sat(M_j/sat(M_j)) = 0$, by induction hypothesis, it follows that $\operatorname{sdepth}(M_j/sat(M_j)) \geq 1$. Therefore, $(*)$ implies

$$(**) M \cong \bigoplus_{j=0}^{q-1} M_j/sat(M_j) \oplus M_q/sat(M_q)[x_n] \oplus \bigoplus_{j=0}^{q-1} sat(M_j) \oplus sat(M_q)[x_n].$$

Also, $\bigoplus_{j=0}^{q-1} sat(M_j) \oplus sat(M_q)[x_n] = \bigoplus_{j=0}^{q} \bigoplus_{\bar{m}\in sat(M_j)/sat(M_{j-1})} mK[x_n]$ since $dim_K(M_a) \leq 1$, and therefore, by $(**)$, we obtain a Stanley decomposition of M with it's sdepth ≥ 1! □

Corollary 1.5. *If M is torsion free, then* $\operatorname{sdepth}(M) \geq 1$.

Proof. Obviously, since M is torsion free, we have $\operatorname{depth}(M) \geq 1$. □

Example 1.6. (Dorin Popescu, [12]) The condition $dim_K(M_a) \leq 1$ is essential in the second part of Theorem 1.4. Let $S = K[x_1, x_2]$ and consider the module $M := (Se_1 \oplus Se_2)/(x_1 z, x_2 z$, where $z = x_1 e_2 - x_2 e_1$. M is multigraded with $deg(e_1) = deg(x_1) = (1,0)$ and $deg(e_2) = deg(x_2) = (0,1)$. Note that $dim_K(M_a) = 1$ for any $a \in \mathbb{Z}^2 \setminus \{(1,1)\}$ and $dim_K(M_{(1,1)}) = 2$. Since $z \in Soc(M)$, it follows that $\operatorname{depth}(M) = 0$. We have a Stanley decomposition of M,

$$M = \bar{e}_1 K[x_2] \oplus \bar{e}_1 x_1 K[x_1] \oplus \bar{e}_2 K[x_1] \oplus \bar{e}_2 x_2 K[x_2] \oplus \bar{e}_1 x_1 x_2 K[x_1, x_2],$$

where $\bar{e_1}, \bar{e_2}$ are the images of e_1 and e_2 in M. It follows that $\operatorname{sdepth}(M) \geq 1$ and thus $\operatorname{sdepth}(M) = 1$, since M is not free.

Remark 1.7. Let M be a torsion free finitely generated $\mathbb{Z}^n$-graded S-module. Then we have an inclusion $0 \to M \to F$, where F is a free module with the same rank as M. Let $Q := F/M$. Is it true that $\operatorname{sdepth}(M) \geq \operatorname{sdepth}(Q) + 1$? In particular, if $I \subset S$ is a monomial ideal, is it true that $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(S/I) + 1$?

If this result were true, then by $\operatorname{depth}(M) = \operatorname{depth}(Q) + 1$, if Q satisfy Stanley's conjecture, then M also satisfy Stanley's conjecture. Note that, in general we cannot expect that $\operatorname{sdepth}(M) = \operatorname{sdepth}(Q) + 1$. Take for instance $M = \mathbf{m} = (x_1, \dots, x_n) \subset S$ and $Q = k = S/\mathbf{m}$. It is known from [9] and [5] that $\operatorname{sdepth}(\mathbf{m}) = \lceil \frac{n}{2} \rceil$, but $\operatorname{sdepth}(k) = 0$. It would be interesting to characterize those modules M with $\operatorname{sdepth}(M) = \operatorname{sdepth}(Q) + 1$. Or, at least, the monomials ideals $I \subset S$ with $\operatorname{sdepth}(I) = \operatorname{sdepth}(S/I) + 1$.

We end this section with the following example.

Example 1.8. Let $M_i := syz_i(K)$ the i-th syzygy module of K. It is known that $\operatorname{depth}(M_i) = i$ for all $0 \leq i \leq n$. The problem of computing $\operatorname{sdepth}(M_i)$ is a chellenging problem. Obviously, $\operatorname{sdepth}(M_0) = \operatorname{sdepth}(K) = 0$. On the other hand, $\operatorname{sdepth}(M_1) = \operatorname{sdepth}(\mathbf{m}) = \lceil \frac{n}{2} \rceil$. Also, $\operatorname{sdepth}(M_n) = \operatorname{sdepth}(S) = n$. We claim that $\operatorname{sdepth}(M_{n-1}) = n-1$.

Indeed, $M_{n-1} = Coker(S \xrightarrow{\psi} S^n)$, where we define $S^n = \bigoplus_{i=1}^n Se_i$ and we set $\psi(1) := x_1e_1 + \cdots + x_ne_n$. Therefore, $M_{n-1} := S\bar{e}_1 + \cdots + S\bar{e}_n$, where $\bar{e}_i$ are the class of e_i in M_{n-1} for all $i \in [n]$. Note that $\bar{e}_1, \ldots, \bar{e}_{n-1}$ are linearly independent in M_{n-1}, since the only relation in M_{n-1} is $x_1\bar{e}_1 + \cdots + x_{n-1}\bar{e}_n = -x_n\bar{e}_n$. It follows that, $M_{n-1} = S\bar{e}_1 \oplus \cdots \oplus S\bar{e}_{n-1} \oplus K[x_1, \ldots, x_{n-1}]\bar{e}_n$, and therefore $\operatorname{sdepth}(M_{n-1}) \geq n-1$. On the other hand, $\operatorname{sdepth}(M_{n-1}) \leq n-1$, since M is not free. Thus $\operatorname{sdepth}(M_{n-1}) = n-1$.

2. Bounds for the sdepth of powers of the maximal irrelevant ideal

Let $\mathbf{m} = (x_1, \ldots, x_n)$ be the maximal irrelevant ideal of S. Let $k \geq 1$ be an integer. In this section, we will give some upper bounds for $\operatorname{sdepth}(\mathbf{m}^k)$. In order to do so, we consider the following poset, associated to $\mathbf{m}^k$,

$$P := \{u \in \mathbf{m}^k \ monomial \ : \ u | x_1^k x_2^k \cdots x_n^k\},$$

where $u \leq v$ if and only if $u|v$. For any $u \in P$, we denote $\rho(u) = |\{j : x_j^k | u\}|$. Note that, by [9, Theorem 2.4], there exists a partition of $P = \bigoplus_{i=1}^r [u_i, v_i]$, i.e. a disjoint sum of intervals $[u_i, v_i] = \{u \in P : \ u_i | u \text{ and } u | v_i\}$, such that $\min_{i=1}^r \{\rho(v_i)\} = \operatorname{sdepth}(\mathbf{m}^k)$.

We write $P_d = \{u \in P : deg(u) = d\}$, where $k \leq d \leq kn$, and $\alpha_d := |P_d|$. First, we want to compute the numbers α_d.

Lemma 2.1. *We the above notations, we have:*

$$\alpha_d = \sum_{i \geq 0} (-1)^i \binom{n}{i} \binom{n + d - i(k+1) - 1}{n-1}.$$

Proof. We fix $d \geq k$. For any $j \in [n]$, we write $A_j := \{u \in S : \ deg(u) = d, x_j^{k+1} | u\}$. Obviously, $P_d := S_d \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)$, where S_d is the set of all monomials of degree d in S. For any nonempty subset $I \subset [n]$, we write $A_I := \bigcap_{i \in I} A_i$. By inclusion-exclusion principle,

$$|A_1 \cup \cdots \cup A_n| = \sum_{\emptyset \neq I \subset [n]} (-1)^{|I|-1} |A_I|.$$

Note that a monomial $u \in A_I$ can be written as $u = w \cdot \prod_{i \in I} x_i^{k+1}$. Therefore, $|A_I| = \binom{n+d-i(k+1)-1}{n-1}$. Now, one can easily get the required conclusion. □

Theorem 2.2. *Let $a \leq \lceil \frac{n}{2} \rceil$ be a positive integer. Then* $\operatorname{sdepth}(\mathbf{m}^k) \leq \lceil \frac{n}{k+1} \rceil$. *In particular, if $k \geq n-1$, then* $\operatorname{sdepth}(\mathbf{m}^k) = 1$.

Proof. Let $a = \lceil \frac{n}{k+1} \rceil$ and assume, by contradiction, that $\operatorname{sdepth}(\mathbf{m}^k) \geq a+1$. Obviously, by Lemma 2.1, $\alpha_k = \binom{n+k-1}{n-1}$ and $\alpha_{k+1} = \binom{n+k}{n-1} - n$. We consider a partition of $\mathscr{P} : P_{n,k} = \bigcup_{i=1}^r [x^{c_i}, x^{d_i}]$ with $\operatorname{sdepth}(\mathscr{D}(\mathscr{P})) = a+1$. Note that $\mathbf{m}^k$ is minimally generated by all the monomials of degree k in S. We can assume that $S_k = \{x^{c_i} | i = 1, \dots, N\}$, where $N = \binom{n+k-1}{n-1}$. We consider an interval $[x^{c_i}, x^{d_i}]$. If $c_i = x_j^k$, then by $\rho(x^{d_i}) \geq a+1$, it follows that in $[x^{c_i}, x^{d_i}]$ are at least a distinct monomials of degree $k+1$. If $c_i(j) < k$ for all $j \in [n]$, then, in $[x^{c_i}, x^{d_i}]$ are at least $a+1$ distinct monomials of degree $k+1$.

We assume that $k \geq \lceil \frac{n-a}{a} \rceil$. Since $\mathscr{P} : P_{n,k} = \bigcup_{i=1}^r [x^{c_i}, x^{d_i}]$ is a partition of $P_{n,k}$, by above considerations, it follows that $\alpha_{k+1} \geq na + (\alpha_k - n)(a+1)$. Therefore, $\binom{n+k}{k-1} \geq (a+1)\binom{n+k-1}{n-1}$. This implies $n+k \geq (k+1)(a+1) \geq (k+1)(\frac{n}{k+1}+1) = n+k+1$, a contradiction. □

We conjecture that $\operatorname{sdepth}(\mathbf{m}^k) \leq \lceil \frac{n}{k+1} \rceil$. Using the computer, see [14], one can prove that this conjecture is true for small n. Also, the conjecture is true for $k = 1$, from [9], [5]. We end this section with the following proposition.

Proposition 2.3. *Let $I \subset S$ be a monomial ideal. Then* $\operatorname{sdepth}(\mathbf{m}^k I) = 1$ *for $k \gg 0$.*

Proof. We consider the K-algebra $A := \bigoplus_{i \geq 0} \mathbf{m}^i I / \mathbf{m}^{i+1} I$ and denote A_i the i^{th} graded component of A. Note that $H(A,i) := dim_K(A_i) = |G(\mathbf{m}^i I)|$, where $G(\mathbf{m}^i I)$ is the set of minimal monomial generators of $\mathbf{m}^i I$. Since A is a finitely generated K-algebra, it follows that the Hilbert function $H(A,i)$ is polynomial for $i \gg 0$.

Therefore, $lim_{i \to \infty} H(A,i)/H(A,i+1) = 1$. Note that there are exactly $H(A,i+1)$ monomials of degree $i+1$ in $\mathbf{m}^i I$. Suppose $\operatorname{sdepth}(\mathbf{m}^i I) \geq 2$. As in the proof of Theorem 2.2, it follows that $H(A,i+1) \geq 2(H(A,i) - n) + n$, which is false for $i \gg 0$, since it contradicts that $\lim_{i \to \infty} H(A,i)/H(A,i+1) = 1$. □

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MIRCEA CIMPOEAS
Institute of Mathematics of the Romanian Academy
Bucharest, Romania
E-mail: e-mail: mircea.cimpoeas@imar.ro