

## SOME INFINITE CLASSES OF ASYMMETRIC NEARLY HAMILTONIAN SNARKS

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We determine the full automorphism group of each member of three infinite families of connected cubic graphs which are snarks. A graph is said to be nearly hamiltonian if it has a cycle which contains all vertices but one. We prove, in particular, that for every possible order  $n \geq 28$  there exists a nearly hamiltonian snark of order  $n$  with trivial automorphism group.

### 1. Introduction

Snarks are non-trivial connected cubic graphs which do not admit a 3-edge-coloring (a precise definition will be given below). The term snark owes its origin to Lewis Carroll's famous nonsense poem "The Hunting of the Snark". It was introduced as a graph theoretical term by Gardner in [13] when snarks were thought to be very rare and unusual "creatures". Tait initiated the study of snarks in 1880 when he proved that the Four Color Theorem is equivalent to the statement that no snark is planar. *Asymmetric graphs* are graphs possessing a single graph automorphism -the identity- and for that reason they are also called identity graphs. Twenty-seven examples of asymmetric graphs are illustrated in [27]. Two of them are the snarks  $S_{n8}$  and  $S_{n9}$  of order 20 listed in [21] p. 276. Asymmetric graphs have been the subject of many studies, see,

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for example, [4], [9], and [17]. Erdős and Rényi proved in [9] that almost all graphs are asymmetric. This property remains true also for cubic graphs, see [3]. Determining the full automorphism group of a given graph may require some non-trivial work, especially if the graph belongs to an infinite family and the task is that of determining the automorphism group of each member of the family. In this paper we are interested in the computation of the full automorphism group of each member of three infinite classes of snarks. We prove, in particular, that for every possible order  $\geq 28$  there exists an asymmetric (nearly hamiltonian) snark of that order.

Throughout the paper,  $G = (V(G), E(G))$  will be a finite connected simple graph with vertex-set  $V(G)$  and edge-set  $E(G)$ . The *chromatic index*  $\chi'(G)$  of a graph  $G$  is the minimum number of colors needed to color the edges of  $G$  in such a way that no two adjacent edges are assigned one and the same color. If  $\Delta(G)$  denotes the maximum degree of  $G$  then, since edges sharing a vertex require different colors, we have  $\chi'(G) \geq \Delta(G)$ . Vizing [23] proved that  $\Delta(G) + 1$  colors suffice: if  $G$  is a simple connected graph with maximum degree  $\Delta(G)$ , then the chromatic index  $\chi'(G)$  satisfies the inequalities  $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ . This result divides simple graphs into two classes: a simple graph  $G$  is *Class 1* if  $\chi'(G) = \Delta(G)$ , otherwise  $G$  is *Class 2*. Erdős and Wilson in [10] proved that, almost all graphs are Class 1. A *snark* is a Class 2 cubic graph and girth  $\geq 5$  which is cyclically 4-edge-connected (see Section 2). Some authors use a slightly different notion of a snark (see, for examples, [18] and [22]). The importance of snarks partially arises from the fact that some conjectures about graphs would have snarks as minimal counter-examples, see for example [16]: (a) (Tutte's 5-Flow Conjecture) every bridgeless graph has a nowhere-zero 5-flow; (b) (The 1-Factor Double Cover Conjecture) every bridgeless cubic graph can be covered exactly twice with 1-factors; (c) (The Cycle Double Cover Conjecture) every bridgeless graph can be covered exactly twice with cycles.

The first graph which was shown to be a snark is the Petersen graph discovered in 1898. Up until 1975 only four examples of snarks were known. In 1975 Isaacs [15] produced the first two infinite families of snarks. In [1], [2] and [5] a catalogue of snarks of order smaller than 30 is generated. For survey papers on snarks we refer the reader to [6], [5], [8] and [24]. If a connected graph  $G$  admits a cycle containing all vertices but one, then we shall say that  $G$  is *nearly hamiltonian*. In the paper [6] graphs with this property were referred to as almost-hamiltonian graphs, but we prefer to avoid this terminology here because the term almost-hamiltonian has been used with different meanings elsewhere, see [19], [20], [25]. In this paper we are interested in nearly hamiltonian snarks. In [6] several classical snarks are shown to be nearly hamiltonian: the Celmins snark [7] of order 26, the Flower snark [15] of order  $4k$ ,

the Double Star snark [12], the Goldberg snark [14] of order  $8k$ , the Szekeres snark [24], the Watkins snarks [24] of order 42 and 50. Moreover, a catalogue of all non-isomorphic nearly hamiltonian snarks of order smaller than 30 is produced in [6]. In particular the following results holds [6, Thm. 1.1]: (a) All snarks of order less than 28 are nearly hamiltonian; (b) There are exactly 2897 non-isomorphic nearly hamiltonian snarks of order 28; (c) up to isomorphism, there is a unique nearly hamiltonian snark, of order 28 and girth  $\geq 6$ , that is, the flower snark of order 28; and (d) there are exactly three snarks of order 28 which are not nearly hamiltonian.

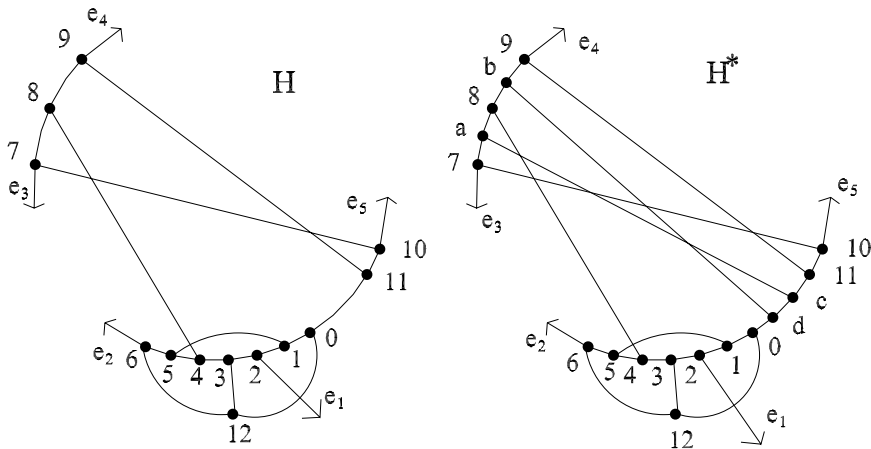
Finally, in [6], a general method to construct some infinite families of nearly hamiltonian snarks is described. In Section 2 we recall this construction in detail and in Section 3 we focus our attention on three infinite families  $\mathcal{O}$ ,  $\mathcal{F}$ ,  $\mathcal{I}$  obtained by applying the above mentioned construction. The first and the third family have been introduced in [6], while the second family is introduced here. In Section 4 we analyse the behavior of graph automorphisms on subgraphs which arise from the construction. In Section 5, by using the results of Section 4, we show that for every possible order  $n \geq 28$  there exists an asymmetric nearly hamiltonian snark of order  $n$  belonging to the set  $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$ .

## 2. Preliminaries

A path of length  $r$  in a graph  $G$  is a sequence of distinct edges of type  $[v_0, v_1]$ ,  $[v_1, v_2]$ ,  $\dots$ ,  $[v_{r-1}, v_r]$ . If all the vertices of the path are distinct, except for  $v_0$  and  $v_r$  which coincide, then the path is a *cycle of length  $r$  or  $r$ -cycle* and we denote it as  $(v_0, v_1, \dots, v_r)$ . The *girth* of  $G$  is the length of the shortest cycle of  $G$ . The graph  $G$  is *cyclically  $k$ -edge-connected* if deleting fewer than  $k$  edges from  $G$  does not disconnect  $G$  into components, each of which contains a cycle. According to our definition in Section 1 a *snark* is a Class 2 cubic graph and girth  $\geq 5$  which is cyclically 4-edge-connected.

We recall in detail a construction of infinite classes of nearly hamiltonian snarks described in [6]. Let  $H$  be the cubic graph of order 13, with five semi-edges  $e_1, e_2, \dots, e_5$ , constructed as follows. Order the first twelve vertices in a circular way and assign a number to each one of them in the clockwise order, starting from the vertex 0. The thirteenth vertex is labelled 12. The edges of  $H$  are given by the pairs:

- (a)  $[i, i+1]$  (indices mod 12), for any  $i = 0, 1, \dots, 11, i \neq 6, 9$ ;
- (b)  $[3j, 12]$ , for any  $j = 0, 1, 2$ ; and
- (c)  $[1, 5]$ ,  $[4, 8]$ ,  $[7, 10]$ , and  $[9, 11]$ .

Figure 1: Graphs  $H$  and  $H^*$ 

The remaining five edges  $e_1, e_2, \dots, e_5$ , considered in the ordering induced by that of the vertices, are assumed to be semi-edges to make  $H$  a cubic graph. The girth of  $H$  is 5 (see Figure 1).

Starting from  $H$  we construct another cubic graph  $H^*$  of order 17, which has the same five semi-edges  $e_1, e_2, \dots, e_5$ . We insert four new vertices, labelled  $a, b, c$ , and  $d$ , on the edges  $[7, 8]$ ,  $[8, 9]$ , and  $[0, 11]$  of  $H$  so that the pairs  $[7, a]$ ,  $[a, 8]$ ,  $[8, b]$ ,  $[b, 9]$ ,  $[11, c]$ ,  $[c, d]$ ,  $[d, 0]$ ,  $[a, c]$  and  $[b, d]$  become edges of  $H^*$  (see Figure 1). The graph  $H^*$  has girth 5. The following result (see [6] p. 68) holds:

**Theorem 2.1.** *Let  $G$  be a snark of order  $n$  with a cutset of five edges whose removal leaves components  $H$  (as defined above) and  $F$  with semi-edges  $\{e_1, e_2, e_3, e_4, e_5\}$  and  $\{f_1, f_2, f_3, f_4, f_5\}$ , respectively. Let  $G^*$  be the cubic graph obtained from  $G$  by replacing  $H$  with  $H^*$  and attaching the semi-edges of  $H^*$  to those of  $H$  according to the ordering induced by indices. Then  $G^*$  is a snark of order  $n + 4$ .*

The graph  $G^*$  contains  $H$  as a subgraph. Therefore we can repeat the construction an arbitrary number of times to obtain an infinite family of snarks. In particular we get the following corollary (see [6] p. 69):

**Corollary 2.2.** *Let  $G$  be a nearly hamiltonian snark of order  $n$  which contains  $H$  as subgraph. Let  $G_m$  be the snark obtained from  $G$  by applying the construction described in Theorem 2.1  $m$  times. Then  $G_m$  is a nearly hamiltonian snark of order  $n + 4m$ .*

The infinite family  $\{G_m\}_{m \geq 1}$  from Corollary 2.2 is said to be *generated by* the graph  $G$ .

By relabelling as in Figure 2 the vertices of the graph obtained from  $H^*$  by deleting the semi-edges we obtain the graph  $G_X = (X, E(G_X))$ , where  $X = \{x_1, x_2, \dots, x_{17}\}$  and  $E(G_X) = \{[x_i, x_{i+1}] : i = 1, 2, \dots, 16\} \cup \{[x_{10}, x_{17}], [x_{13}, x_{17}], [x_{11}, x_{15}], [x_3, x_{14}], [x_2, x_9], [x_1, x_7], [x_4, x_8]\}$  (the vertex  $x_{17}$  of  $G_X$  corresponds to the vertex 12 of  $H^*$ ).

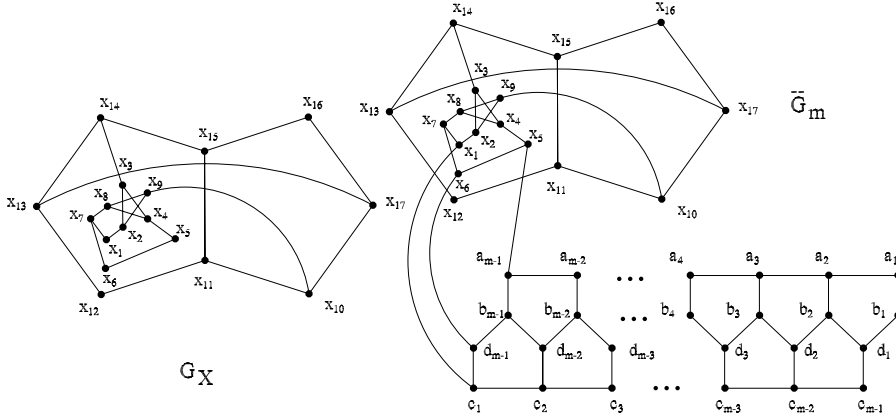


Figure 2: Graph  $\bar{G}_m$  with  $m \geq 1$

Consider the sets  $A_m = \{a_i : i = 1, 2, \dots, m-1\}$ ,  $B_m = \{b_i : i = 1, 2, \dots, m-1\}$ ,  $C_m = \{c_i : i = 1, 2, \dots, m-1\}$ ,  $D_m = \{d_i : i = 1, 2, \dots, m-1\}$  and let  $\{\bar{G}_m\}_{m \geq 1}$  be the family of the graphs defined as follows:

- (1)  $\bar{G}_m = G_X$  if  $m = 1$ ;
- (2)  $\bar{G}_m = (V(\bar{G}_m), E(\bar{G}_m))$  if  $m \geq 2$  with  $V(\bar{G}_m) = X \cup A_m \cup B_m \cup C_m \cup D_m$  and  $E(\bar{G}_m) = E(G_X) \cup \{[a_i, b_i] : i = 1, 2, \dots, m-1\} \cup \{[b_i, d_i] : i = 1, 2, \dots, m-1\} \cup \{[c_i, d_{m-i}] : i = 1, 2, \dots, m-1\} \cup \{[x_5, a_{m-1}], [x_1, c_1], [x_6, d_{m-1}]\} \cup \{[a_i, a_{i+1}] : i = 1, 2, \dots, m-2\} \cup \{[c_i, c_{i+1}] : i = 1, 2, \dots, m-2\} \cup \{[b_{i+1}, d_i] : i = 1, 2, \dots, m-2\}$  where the last three sets are empty if  $m = 2$ .

The graph  $G_X$  is a subgraph of  $\bar{G}_m$  for any  $m \geq 1$  and the graph  $\bar{G}_m$  is a subgraph of  $G_m$  (see Corollary 2.2) for any  $m \geq 1$ . Figure 2 illustrates the construction of the graph  $\bar{G}_m$  with  $m \geq 1$ .

### 3. Three families $\mathcal{O}, \mathcal{F}, \mathcal{I}$ of nearly hamiltonian snarks

In this section we apply Corollary 2.2 to construct three infinite families of nearly hamiltonian snarks. Let  $O$  and  $F$  be the nearly hamiltonian snarks of order 24 shown in Figure 3, and let  $I$  be the nearly hamiltonian snark of order 26 shown in Figure 4. Dotted lines identify the cycle missing one vertex.

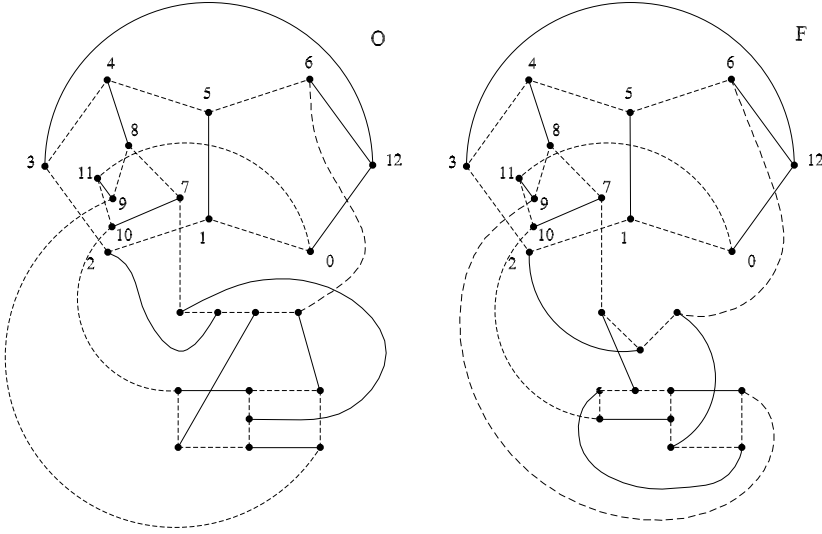
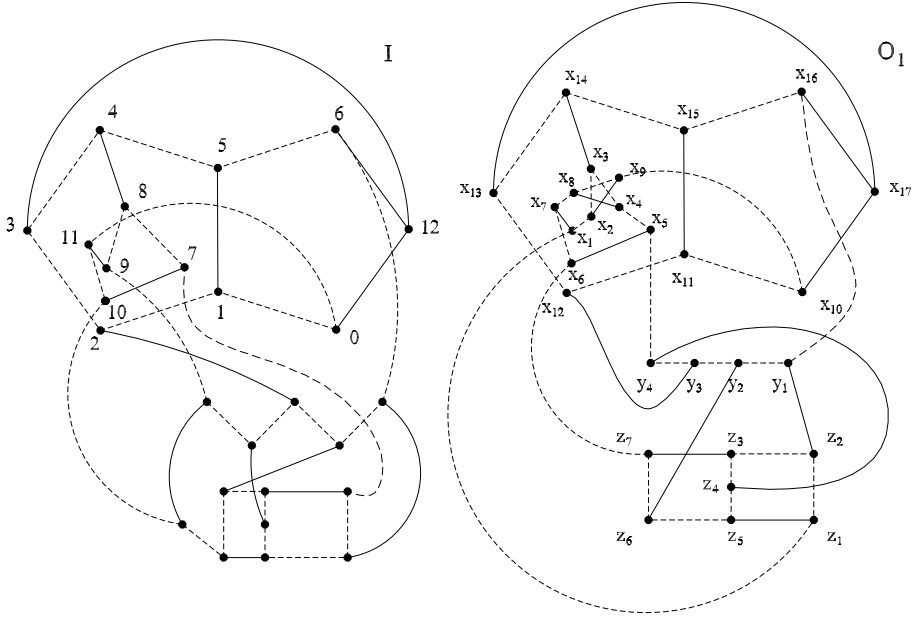


Figure 3: Nearly hamiltonian snarks  $O$  and  $F$

By applying Corollary 2.2 to  $O, F$  and  $I$  we obtain three infinite families of nearly hamiltonian snarks  $\mathcal{O} = \{O_m : m \geq 1\}$ ,  $\mathcal{F} = \{F_m : m \geq 1\}$  and  $\mathcal{I} = \{I_m : m \geq 1\}$  (generated by the snarks  $O, F$  and  $I$ , respectively).

The first and the third family have been introduced in [6] while the second family is new. Now we give an explicit description of the classes  $\mathcal{O}, \mathcal{F}, \mathcal{I}$ . Let us consider the following sets:  $Y = \{y_1, y_2, y_3, y_4\}$ ,  $T = \{t_1, t_2, t_3\}$ ,  $Z = \{z_1, z_2, \dots, z_7\}$ ,  $S = \{s_1, s_2, s_3, s_4, s_5\}$ ,  $R = \{r_1, r_2, \dots, r_8\}$ . Figures 4 and 5 show the graphs  $O_1, F_1, I_1$ . Also in this cases dotted lines identify the cycle missing one vertex. For  $m \geq 2$  we get

- $O_m = (V(O_m), E(O_m))$  with  $V(O_m) = V(\bar{G}_m) \cup Y \cup Z$  and  $E(O_m) = E(\bar{G}_m) \cup \{[y_i, y_{i+1}] : i = 1, 2, 3\} \cup \{[z_i, z_{i+1}] : i = 1, 2, \dots, 6\} \cup \{[x_{16}, y_1], [y_1, z_2], [y_2, z_6], [y_3, x_{12}], [y_4, a_1], [y_4, z_4], [z_1, z_5], [z_3, z_7], [z_1, c_{m-1}], [z_7, b_1]\}$ ;
- $F_m = (V(F_m), E(F_m))$  with  $V(F_m) = V(\bar{G}_m) \cup T \cup R$  and  $E(F_m) = E(\bar{G}_m) \cup \{[t_i, t_{i+1}] : i = 1, 2\} \cup \{[r_i, r_{i+1}] : i = 1, 2, \dots, 7\} \cup \{[x_{16}, t_1], [x_{12}, t_2], [t_1, r_3], [t_3, r_6], [t_3, a_1], [r_2, r_7], [r_1, r_5], [r_4, r_8], [r_1, c_{m-1}], [r_8, b_1]\}$ .


 Figure 4: Nearly hamiltonian snarks  $I$  and  $O_1$ 

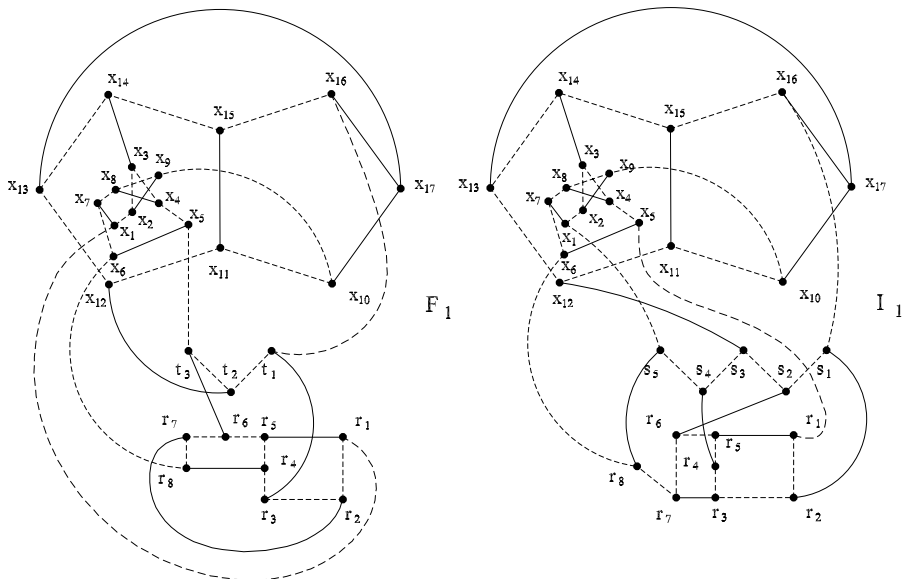
- $I_m = (V(I_m), E(I_m))$  with  $V(I_m) = V(\tilde{G}_m) \cup S \cup R$  and  $E(I_m) = E(\tilde{G}_m) \cup \{[s_i, s_{i+1}] : i = 1, 2, 3, 4\} \cup \{[r_i, r_{i+1}] : i = 1, 2, \dots, 7\} \cup \{[x_{16}, s_1], [x_{12}, s_3], [s_1, r_2], [s_2, r_6], [s_4, r_4], [s_5, r_8], [s_5, c_{m-1}], [r_1, r_5], [r_3, r_7], [r_1, a_1], [r_8, b_1]\}$ .

The graph  $\tilde{G}_m$  is a subgraph of each of  $O_m, F_m$  and  $I_m$ , with  $m \geq 1$  and in particular the graph  $\tilde{G}_X$  is a subgraph of each  $G \in \mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$ . Figure 6, 7 and 8 show the graphs  $O_m, F_m, I_m$  with  $m = 4$ . Also in these cases dotted lines identify the cycle missing one vertex.

#### 4. Automorphisms of cubic graphs with $\tilde{G}_m$ as a subgraph

In this section we describe the behavior of particular graph automorphisms of cubic graphs which act on the subgraphs  $\tilde{G}_m$ . This will be useful in Section 5 for the computation of the full automorphism group of each graph of  $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$ . We will be using the functional notation for mappings, in other words  $\alpha(x)$  denotes the image of the element  $x$  under mapping  $\alpha$  and  $\alpha|_A$  denotes the restriction of  $\alpha$  to  $A$ .

In what follows we shall make repeated use of the following four elementary properties of an automorphism of a cubic graph with girth at least five.

Figure 5: Nearly hamiltonian snarks  $F_1$  and  $I_1$ 

**Elementary Properties** Let  $G$  be a cubic graph with girth at least 5 and let  $\alpha$  be an automorphism of  $G$ . Then,

EP1) the number of  $r$ -cycles passing through a vertex  $u$  of  $G$  coincides with the number of  $r$ -cycles passing through the vertex  $\alpha(u)$ ;

EP2) if  $u$  and  $v$  are vertices fixed by  $\alpha$  with the property of being adjacent to a vertex  $w$ , then the vertex  $w$  is also fixed by  $\alpha$ ;

EP3) if  $\alpha$  fixes the vertices  $u, v, w$  with  $[u, v]$ ,  $[w, v]$  and  $[v, t]$  edges of  $G$ , then the vertex  $t$  is fixed by  $\alpha$ ;

EP4) if  $\alpha$  fixes the vertices  $v, t$  and if  $u, w$  and  $t$  are different vertices adjacent to the vertex  $v$ , then  $\alpha(\{u, w\}) = \{u, w\}$ .

We note that EP2 follows from the observation that if the vertex  $w$  is not fixed by  $\alpha$  then  $(u, w, v, \alpha(w))$  would be a 4-cycle, contradicting the fact that the girth of  $G$  is at least 5.

**Lemma 4.1.** Let  $G$  be a cubic graph with girth at least 5 and with  $G_X$  as a subgraph. The 5-cycles constituted by vertices of  $X$  are the following:

$$\mathcal{C}_1 = (x_{10}, x_{11}, x_{12}, x_{13}, x_{17}), \quad \mathcal{C}_2 = (x_{10}, x_{11}, x_{15}, x_{16}, x_{17}),$$

$$\mathcal{C}_3 = (x_{11}, x_{12}, x_{13}, x_{14}, x_{15}), \quad \mathcal{C}_4 = (x_{13}, x_{14}, x_{15}, x_{16}, x_{17}),$$

$$\mathcal{C}_5 = (x_4, x_5, x_6, x_7, x_8), \quad \mathcal{C}_6 = (x_2, x_3, x_4, x_8, x_9),$$



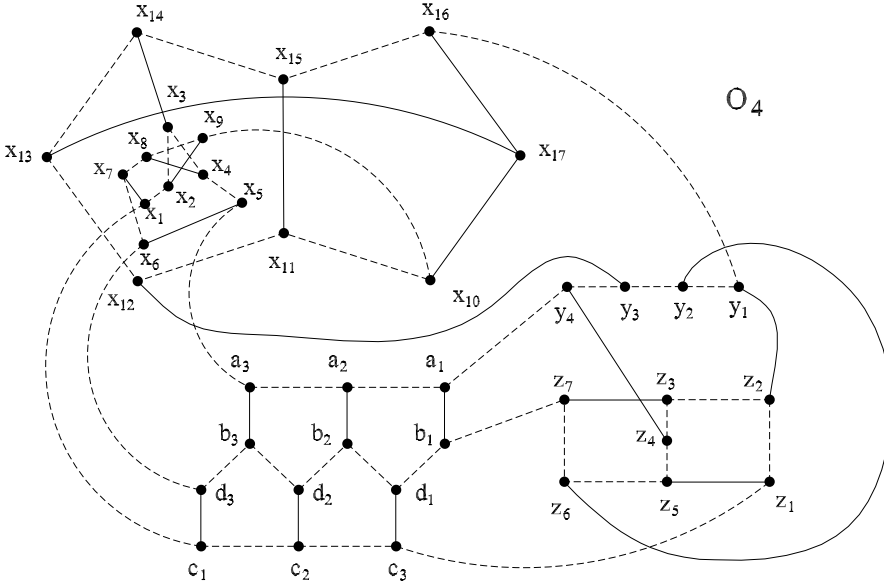


Figure 6: Nearly hamiltonian snark  $O_4$

$$\mathcal{C}_7 = (x_1, x_2, x_9, x_8, x_7).$$

Moreover, the following are the only other 5-cycles that can pass through at least one vertex of  $X$ :

$$\mathcal{C}_8 = (x_5, x_6, p_1, p_2, p_3), \mathcal{C}_9 = (x_1, x_7, x_6, q_1, q_2) \text{ with } p_i, q_j \notin X, i = 1, 2, 3, j = 1, 2,$$

$$\mathcal{C}_{10} = (x_1, x_7, x_6, x_5, p), \mathcal{C}_{11} = (x_{16}, r, x_{12}, x_{11}, x_{15}), \mathcal{C}_{12} = (x_{16}, r, x_{12}, x_{13}, x_{17}) \text{ with } p, r \notin X.$$

**Proposition 4.2.** *Let  $G$  be a cubic graph with girth at least 5 and with  $G_X$  as a subgraph. Let  $\alpha$  be an automorphism of  $G$  with  $\alpha(x_3) \in X$ , then  $\alpha(x_3) = x_1$  or  $\alpha(x_3) = x_3$ .*

*Proof.* Lemma 4.1 establishes that  $\mathcal{C}_6$  is the only 5-cycle in  $G$  passing through the vertex  $x_3$  and that there are at least two 5-cycles in  $G$  going through each vertex of the set  $X \setminus \{x_1, x_3, x_5, x_6\}$ . Therefore, by Property EP1 we get that  $\alpha(x_3) \in \{x_1, x_3, x_5, x_6\}$ . If  $\alpha(x_3) = x_5$  or  $\alpha(x_3) = x_6$ , then the 5-cycle  $\mathcal{C}_5$  has to be the only 5-cycle in  $G$  touching  $x_5$  or  $x_6$ . Hence the relation  $\alpha(\mathcal{C}_6) = \mathcal{C}_5$

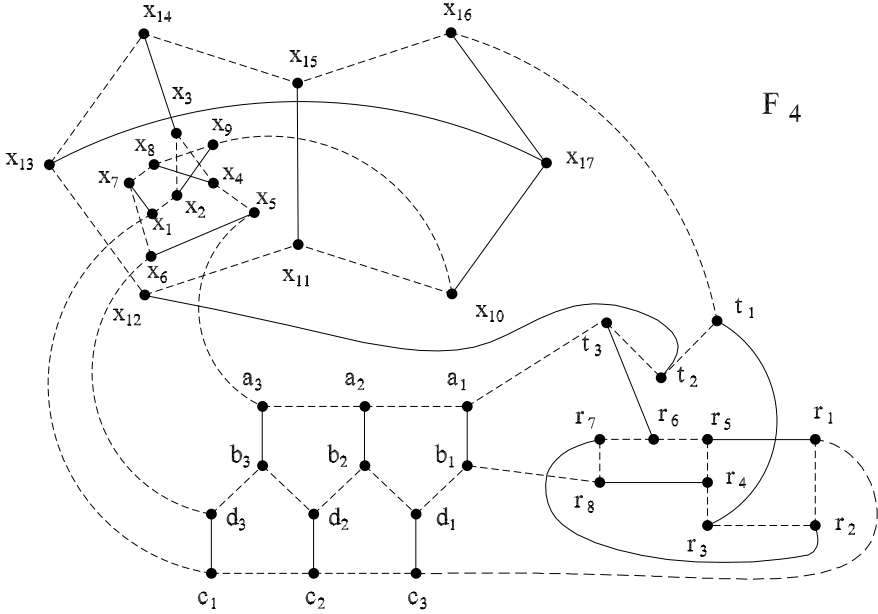


Figure 7: Nearly hamiltonian snark  $F_4$

yields one of the following cases:

$$(a) \begin{cases} \alpha(x_2) = x_4 \\ \alpha(x_3) = x_5 \\ \alpha(x_4) = x_6 \\ \alpha(x_8) = x_7 \\ \alpha(x_9) = x_8 \end{cases} \quad (b) \begin{cases} \alpha(x_2) = x_6 \\ \alpha(x_3) = x_5 \\ \alpha(x_4) = x_4 \\ \alpha(x_8) = x_8 \\ \alpha(x_9) = x_7 \end{cases}$$

$$(c) \begin{cases} \alpha(x_2) = x_5 \\ \alpha(x_3) = x_6 \\ \alpha(x_4) = x_7 \\ \alpha(x_8) = x_8 \\ \alpha(x_9) = x_4 \end{cases} \quad (d) \begin{cases} \alpha(x_2) = x_7 \\ \alpha(x_3) = x_6 \\ \alpha(x_4) = x_5 \\ \alpha(x_8) = x_4 \\ \alpha(x_9) = x_8. \end{cases}$$

Cases (a) and (d) cannot occur since only two 5-cycles pass through the vertex  $x_9$  while three 5-cycles go through the vertex  $x_8$ ; thus  $\alpha(x_9) \neq x_8$ .

Case (b) implies  $\alpha(\mathcal{C}_7) = \alpha((x_1, x_2, x_9, x_8, x_7)) = (\alpha(x_1), \alpha(x_2), \alpha(x_9), \alpha(x_8), \alpha(x_7)) = (\alpha(x_1), x_6, x_7, x_8, \alpha(x_7))$ . Lemma 4.1 establishes that  $\mathcal{C}_5 =$

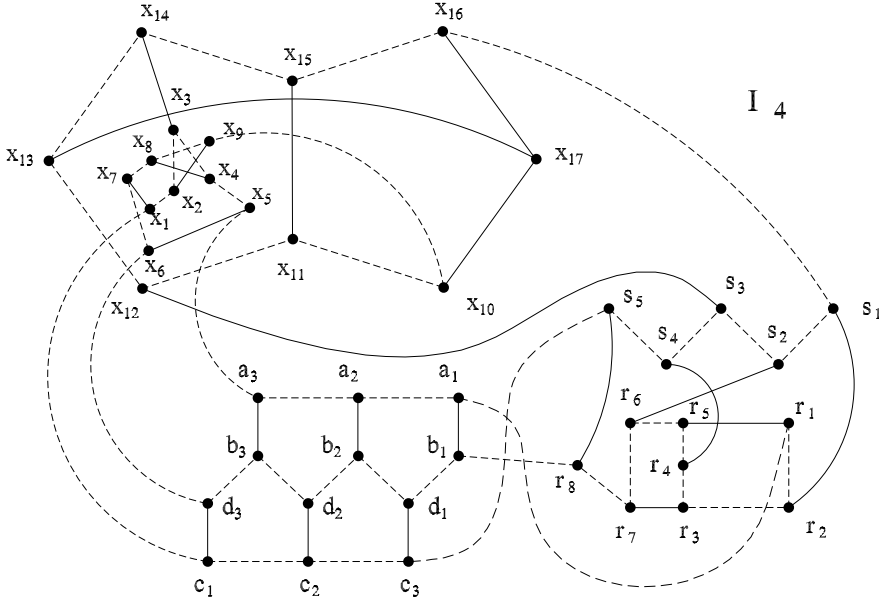


Figure 8: Nearly hamiltonian snark  $I_4$

$(x_5, x_6, x_7, x_8, x_4)$  is the only 5-cycle touching the three vertices  $x_6, x_7, x_8$ , whereby  $\alpha(x_7) = x_4$ ; a contradiction, since from (b) we have  $\alpha(x_4) = x_4$ .

Case (c) does not occur either. In  $G$  two 5-cycles pass through  $x_2$  and  $\alpha(x_2) = x_5$ . Thus, there must be two 5-cycles going through the vertex  $x_5$ , either  $\mathcal{C}_5$  and  $\mathcal{C}_8$  or  $\mathcal{C}_5$  and  $\mathcal{C}_{10}$  (see Lemma 4.1). Therefore two 5-cycles touch the vertex  $x_6$  (either  $\mathcal{C}_5$  and  $\mathcal{C}_8$  or  $\mathcal{C}_5$  and  $\mathcal{C}_{10}$ ), whereas only one 5-cycle pass through the vertex  $x_3$ . Thus  $\alpha(x_3) = x_6$  is a contradiction.

Therefore, if  $\alpha(x_3) \in X$ , then we have either  $\alpha(x_3) = x_1$  or  $\alpha(x_3) = x_3$ .  $\square$

**Proposition 4.3.** *Let  $G$  be a cubic graph with girth at least 5 and with  $G_X$  as a subgraph. Let  $\alpha$  be an automorphism of  $G$  that fixes  $x_3$ . Then  $\alpha$  fixes  $X$  set-wise and the restriction  $\alpha|_X$  is either the identity permutation or the involution  $(x_{11} x_{17})(x_{12} x_{16})(x_{13} x_{15})$ .*

*Proof.* Let  $\alpha(x_3) = x_3$ . By Property EP1 and Lemma 4.1 we get  $\alpha(\mathcal{C}_6) = \mathcal{C}_6$ , hence  $\alpha(x_8) \in \{x_8, x_9\}$ ; therefore, by Property EP1 and Lemma 4.1 we have  $\alpha(x_8) = x_8$ . From  $\alpha(\mathcal{C}_6) = \mathcal{C}_6$ ,  $\alpha(x_3) = x_3$ ,  $\alpha(x_8) = x_8$  we obtain  $\alpha(x_2) = x_2$ ,  $\alpha(x_4) = x_4$ ,  $\alpha(x_9) = x_9$ . Property EP3 implies  $\alpha(x_{10}) = x_{10}$ ,  $\alpha(x_{14}) = x_{14}$ ,  $\alpha(x_7) = x_7$ . By Lemma 4.1 and  $\alpha(x_4) = x_4$ ,  $\alpha(x_8) = x_8$ ,  $\alpha(x_7) = x_7$  we have  $\alpha(\mathcal{C}_5) = \mathcal{C}_5$ , thus  $\alpha(x_6) = x_6$ ,  $\alpha(x_5) = x_5$  and by Property EP2 we finally get  $\alpha(x_1) = x_1$ .

Since the automorphism  $\alpha$  fixes  $x_3$  and  $x_{14}$ , then Property EP4 implies that  $\alpha(\{x_{13}, x_{15}\}) = \{x_{13}, x_{15}\}$ .

Case I: if  $\alpha(x_{13}) = x_{13}$  and  $\alpha(x_{15}) = x_{15}$ , then by Property EP2 we get  $\alpha(x_{11}) = x_{11}$ ,  $\alpha(x_{12}) = x_{12}$ ,  $\alpha(x_{17}) = x_{17}$  and  $\alpha(x_{16}) = x_{16}$ . Therefore  $\alpha(X) = X$  and the restriction of  $\alpha$  to  $X$  is the identity permutation.

Case II: if  $\alpha(x_{13}) = x_{15}$  and  $\alpha(x_{15}) = x_{13}$ , then  $\alpha(\mathcal{C}_2) = \alpha((x_{10}, x_{17}, x_{16}, x_{15}, x_{11})) = (x_{10}, \alpha(x_{17}), \alpha(x_{16}), x_{13}, \alpha(x_{11}))$ , hence  $\alpha(x_{11})$  is adjacent to the vertices  $x_{13}$  and  $x_{10}$ , and so  $\alpha(x_{11}) = x_{17}$ . Moreover,  $\alpha(\mathcal{C}_3) = \alpha((x_{11}, x_{12}, x_{13}, x_{14}, x_{15})) = (x_{17}, \alpha(x_{12}), x_{15}, x_{14}, x_{13})$ , hence  $\alpha(x_{12})$  is adjacent to the vertices  $x_{15}$ ,  $x_{17}$ , and  $\alpha(x_{12}) = x_{16}$ ; therefore  $\alpha(x_{17}) = x_{11}$  and  $\alpha(x_{16}) = x_{12}$ . We have proved that  $\alpha(X) = X$  and  $\alpha|_X = (x_{11} \ x_{17})(x_{12} \ x_{16})(x_{13} \ x_{15})$ .  $\square$

**Proposition 4.4.** *Let  $G$  be a cubic graph with girth at least 5 and with  $\bar{G}_m$  as a subgraph for some  $m \geq 2$ . Let  $\alpha$  be an automorphism of  $G$  that fixes  $x_3$ . Then  $\alpha$  fixes  $A_m \cup B_m \cup C_m \cup D_m$  pointwise.*

*Proof.* Let  $\alpha(x_3) = x_3$ . By Proposition 4.3 the vertex  $x_i$ , with  $i = 1, 2, 4, 5, 6, 7$ , is fixed by  $\alpha$  and by Property EP3 we have  $\alpha(c_1) = c_1$ ,  $\alpha(a_{m-1}) = a_{m-1}$  and  $\alpha(d_{m-1}) = d_{m-1}$ ; hence by Property EP3 we also get  $\alpha(b_{m-1}) = b_{m-1}$ . If  $m = 2$  the statement is proved.

If  $m \geq 3$  we prove

- 1)  $\alpha(a_i) = a_i$  with  $i = m - 2, m - 3, \dots, 1$ ;
- 2)  $\alpha(b_i) = b_i$  with  $i = m - 2, m - 3, \dots, 1$ ;
- 3)  $\alpha(c_j) = c_j$  with  $j = 2, 3, \dots, m - 1$ ;
- 4)  $\alpha(d_i) = d_i$  with  $i = m - 2, m - 3, \dots, 1$ .

The vertices  $a_{m-1}, x_5, b_{m-1}$  are fixed by  $\alpha$ , hence Property EP3 implies that  $\alpha(a_{m-2}) = a_{m-2}$ . The vertices  $x_1, c_1, d_{m-1}$  are fixed by  $\alpha$ , thus by Property EP3 we obtain  $\alpha(c_2) = c_2$ . The vertex  $d_{m-2}$  is adjacent to the fixed vertices  $c_2$  and  $b_{m-1}$ , thus by Property EP2 we get  $\alpha(d_{m-2}) = d_{m-2}$ . The vertex  $b_{m-2}$  is adjacent to the fixed vertices  $a_{m-2}$  and  $d_{m-2}$ , so Property EP2 yields  $\alpha(b_{m-2}) = b_{m-2}$ . Let  $h$  be an integer  $h \geq 2$ . By induction we assume that 1), 2), 4) are true for  $i \geq m - h$  and that 3) is true for  $j \leq h$ . We prove 1), 2), 4) for  $i = m - (h + 1)$  and 3) for  $j = h + 1$ . By induction hypothesis the vertices  $b_{m-h}, a_{m-h}$  and  $a_{m-(h-1)}$  are fixed by  $\alpha$ , hence Property EP3 implies that  $\alpha(a_{m-(h+1)}) = a_{m-(h+1)}$ . The vertices  $c_h, c_{h-1}$  and  $d_{m-h}$  are fixed by  $\alpha$ , thus by Property EP3 we obtain  $\alpha(c_{h+1}) = c_{h+1}$ . The vertex  $d_{m-(h+1)}$  is adjacent to the fixed vertices  $c_{h+1}$  and  $b_{m-h}$ , hence Property EP2 implies that  $\alpha(d_{m-(h+1)}) = d_{m-(h+1)}$ . The vertex  $b_{m-(h+1)}$  is adjacent to the fixed vertices  $a_{m-(h+1)}$  and  $d_{m-(h+1)}$  and so by Property EP2 we get  $\alpha(b_{m-(h+1)}) = b_{m-(h+1)}$ . Therefore, the automorphism  $\alpha$  fixes the vertices of the set  $A_m \cup B_m \cup C_m \cup D_m$ .  $\square$

**Corollary 4.5.** *Let  $G$  be a cubic graph with girth at least 5 and with  $\bar{G}_m$  as a subgraph for some  $m \geq 1$ . Let  $\alpha$  be an automorphism of  $G$  that fixes  $x_3$ . Then  $\alpha$  leaves  $\bar{G}_m$  invariant and the restriction of  $\alpha$  to  $\bar{G}_m$  is either the involution  $(x_{11} x_{17})(x_{12} x_{16})(x_{13} x_{15})$  or the identity permutation.*

*Proof.* The statement follows from Propositions 4.3 and 4.4.  $\square$

## 5. $\mathcal{O}, \mathcal{F}, \mathcal{I}$ : asymmetric nearly hamiltonian snarks

In this section we prove that each member of  $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$  is an asymmetric graph. First of all, we consider some properties of cycles of graphs from  $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$  which will be useful for characterizing the automorphisms of the graphs.

**Lemma 5.1.** *The vertices of  $O_m$  lying in just one 5-cycle are the following:*

- (a)  $x_1, x_3, x_5, x_6, z_1, z_2, z_6, z_7$  if  $m = 1$ ;
- (b)  $x_3, a_1, b_1, z_1, z_2, z_6, z_7, c_{m-1}$  if  $m \geq 2$ .

*If  $m = 1$ , there are exactly six 8-cycles going through the vertex  $x_3$  while there are exactly three 8-cycles touching the vertex  $x_1$ .*

*Proof.* The statement follows from the definition of  $O_m$ . The following two tables show the 5-cycles and the 8-cycles passing through each vertex considered in the statement:

	$x_1$	$x_5, x_6$	$x_3$	$z_1, z_2$	$z_6, z_7$	$a_1, b_1$	$a_1, b_1$	$c_1$	$c_{m-1}$
cycle of length 5	$\mathcal{C}_7$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_{16}$	$\mathcal{C}_{17}$	$\mathcal{C}_{15}$	$\mathcal{C}_{13}$	$\mathcal{C}_{18}$	$\mathcal{C}_{14}$
	$m = 1$	$m = 1$	$m \geq 1$	$m \geq 1$	$m \geq 1$	$m = 2$	$m \geq 3$	$m = 2$	$m \geq 3$

	$x_1$	$x_3$
cycle of length 8	$\Omega_1, \Omega_2, \Omega_3$	$\Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_{12}, \Omega_{13}$
	$m = 1$	$m = 1$

where  $\mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7$  are the cycles of Lemma 4.1,  $\mathcal{C}_{13} = (a_1, a_2, b_2, d_1, b_1)$ ,  $\mathcal{C}_{14} = (c_{m-2}, c_{m-1}, d_1, b_2, d_2)$ ,  $\mathcal{C}_{15} = (a_1, x_5, x_6, d_1, b_1)$ ,  $\mathcal{C}_{16} = (z_1, z_2, z_3, z_4, z_5)$ ,  $\mathcal{C}_{17} = (z_6, z_7, z_3, z_4, z_5)$ ,  $\mathcal{C}_{18} = (x_1, c_1, d_1, x_6, x_7)$ ,  $\Omega_1 = (x_5, x_4, x_8, x_9, x_2, x_1, x_7, x_6)$ ,  $\Omega_2 = (y_4, x_5, x_6, x_7, x_1, z_1, z_5, z_4)$ ,  $\Omega_3 = (x_1, z_1, z_5, z_4, z_3, z_7, x_6, x_7)$ ,  $\Omega_4 = (x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_3, x_2, x_9)$ ,  $\Omega_5 = (x_{10}, x_9, x_8, x_4, x_3, x_{14}, x_{13}, x_{17})$ ,  $\Omega_6 = (x_{10}, x_{11}, x_{15}, x_{14}, x_3, x_4, x_8, x_9)$ ,  $\Omega_7 = (x_{10}, x_9, x_2, x_3, x_{14}, x_{15}, x_{16}, x_{17})$ ,  $\Omega_{12} = (x_{12}, x_{13}, x_{14}, x_3, x_4, x_5, y_4, y_3)$ ,  $\Omega_{13} = (x_5, x_4, x_3, x_2, x_9, x_8, x_7, x_6)$ .  $\square$

**Lemma 5.2.** *The vertices of  $F_m$  lying in just one 5-cycle are the following:*

- (a)  $x_1, x_3, x_5, x_6$  if  $m = 1$ ;

(b)  $x_3, a_1, b_1, c_{m-1}$  if  $m \geq 2$ .

If  $m = 1$  there are exactly four 8-cycles going through the vertex  $x_1$  while there are exactly six 8-cycles touching the vertex  $x_3$ .

*Proof.* The statement follows from the definition of  $F_m$ . The following two tables show the 5-cycles and the 8-cycles passing through each vertex considered in the statement:

	$x_1$	$x_5, x_6$	$x_3$	$a_1, b_1$	$a_1, b_1$	$c_1$	$c_{m-1}$
cycle of length 5	$\mathcal{C}_7$ $m = 1$	$\mathcal{C}_5$ $m = 1$	$\mathcal{C}_6$ $m \geq 1$	$\mathcal{C}_{15}$ $m = 2$	$\mathcal{C}_{13}$ $m \geq 3$	$\mathcal{C}_{18}$ $m = 2$	$\mathcal{C}_{14}$ $m \geq 3$

	$x_1$	$x_3$
cycle of length 8	$\Omega_1, \Omega_8, \Omega_9, \Omega_{10}$ $m = 1$	$\Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_{11}, \Omega_{13}$ $m = 1$

where  $\mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7$  are the cycles of Lemma 4.1,  $\mathcal{C}_{13}, \mathcal{C}_{14}, \mathcal{C}_{15}, \mathcal{C}_{18}, \Omega_1, \Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_{13}$  are cycles of Lemma 5.1;  $\Omega_8 = (x_1, r_1, r_2, r_3, r_4, r_8, x_6, x_7)$ ,  $\Omega_9 = (t_3, x_5, x_6, x_7, x_1, r_1, r_5, r_6)$ ,  $\Omega_{10} = (x_1, r_1, r_5, r_6, r_7, r_8, x_6, x_7)$ ,  $\Omega_{11} = (x_{12}, x_{13}, x_{14}, x_3, x_4, x_5, t_3, t_2)$ .  $\square$

**Lemma 5.3.** *The vertices of  $I_m$  lying in just one 5-cycle are the following:*

(a)  $s_5, x_3, x_5, r_1, r_2, r_6, r_7, r_8$  if  $m = 1$ ;

(b)  $s_5, x_3, a_1, r_1, r_2, r_6, r_7, r_8$  if  $m \geq 2$ .

*Proof.* The statement follows from the definition of  $I_m$ . The following table shows the 5-cycles going through each vertex considered in the statement:

	$x_5$	$x_3$	$a_1$	$a_1$	$s_5, r_8$	$s_5, r_8$	$r_1, r_2$	$r_6, r_7$
cycle of length 5	$\mathcal{C}_5$ $m = 1$	$\mathcal{C}_6$ $m \geq 1$	$\mathcal{C}_{15}$ $m = 2$	$\mathcal{C}_{13}$ $m \geq 3$	$\mathcal{C}_{19}$ $m = 1$	$\mathcal{C}_{20}$ $m \geq 2$	$\mathcal{C}_{21}$ $m \geq 1$	$\mathcal{C}_{22}$ $m \geq 1$

where  $\mathcal{C}_5, \mathcal{C}_6$  are the cycles of Lemma 4.1,  $\mathcal{C}_{13}, \mathcal{C}_{15}$  are cycles of Lemma 5.1;  $\mathcal{C}_{19} = (s_5, x_1, x_7, x_6, r_8)$ ,  $\mathcal{C}_{20} = (s_5, c_{m-1}, d_1, b_1, r_8)$ ,  $\mathcal{C}_{21} = (r_1, r_2, r_3, r_4, r_5)$ ,  $\mathcal{C}_{22} = (r_3, r_4, r_5, r_6, r_7)$ .  $\square$

By using the above Lemmas we obtain the following proposition:

**Proposition 5.4.** *Let  $G$  be any graph from  $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$ . Then every automorphism of  $G$  fixes  $x_3$ .*

*Proof.* Let  $\alpha$  be an automorphism of  $G$ . The cycle  $\mathcal{C}_6$  is the only 5-cycle in  $G$  touching the vertex  $x_3$ . By Property EP1 if  $\alpha(x_3) = v$ , the vertex  $v$  is a vertex contained just in one 5-cycle  $\mathcal{C}$  of  $G$ . Thus,  $v$  is one of the vertices of Lemmas 5.1, 5.2 and 5.3 and  $\alpha(\mathcal{C}_6) = \mathcal{C}$  where  $\mathcal{C}$  is one of the 5-cycles highlighted in the proof of Lemmas 5.1, 5.2 and 5.3. We prove that  $\alpha(x_3) \in \{x_1, x_3, x_5, x_6\}$ . The assumption  $\alpha(x_3) \notin \{x_1, x_3, x_5, x_6\}$  yields the following cases:

- (a)  $G \in \{O_2, F_2\}$  with  $\alpha(x_3) = a_1$ , then we obtain  $\alpha(\mathcal{C}_6) = \mathcal{C}_{15}$  and  $\alpha^{-1}(b_1) \in \{x_2, x_4\}$ ;
- (b)  $G \in \{O_m, F_m : m \geq 3\}$ , with  $\alpha(x_3) = a_1$ , then we get  $\alpha(\mathcal{C}_6) = \mathcal{C}_{13}$  and  $\alpha^{-1}(b_1) \in \{x_2, x_4\}$ ;
- (c)  $G \in \{O_m, F_m : m \geq 2\}$  with  $\alpha(x_3) = b_1$ , then  $\alpha^{-1}(a_1) \in \{x_2, x_4\}$ ;
- (d)  $G \in \{O_m : m \geq 1\}$  with  $\alpha(x_3) = z_i$  with  $i = 1$  or  $i = 2$  or  $i = 6$  or  $i = 7$ , then we obtain  $\alpha^{-1}(z_j) \in \{x_2, x_4\}$  with  $j = 2$  or  $j = 1$  or  $j = 7$  or  $j = 6$ , respectively;
- (e)  $G \in \{I_m : m \geq 1\}$  with  $\alpha(x_3) = s_5$  or  $\alpha(x_3) = r_1$  or  $\alpha(x_3) = r_2$  or  $\alpha(x_3) = r_6$  or  $\alpha(x_3) = r_7$  or  $\alpha(x_3) = r_8$ , then we obtain  $\alpha^{-1}(r_8) \in \{x_2, x_4\}$ , or  $\alpha^{-1}(r_2) \in \{x_2, x_4\}$ , or  $\alpha^{-1}(r_1) \in \{x_2, x_4\}$ , or  $\alpha^{-1}(r_7) \in \{x_2, x_4\}$ , or  $\alpha^{-1}(r_6) \in \{x_2, x_4\}$ , or  $\alpha^{-1}(s_5) \in \{x_2, x_4\}$ , respectively;
- (f)  $G = I_2$  with  $\alpha(x_3) = a_1$ , then we get  $\alpha(\mathcal{C}_6) = \mathcal{C}_{15}$  and  $\alpha(x_9) \in \{d_1, x_6\}$ ;
- (g)  $G \in \{I_m : m \geq 3\}$  with  $\alpha(x_3) = a_1$  implies  $\alpha(\mathcal{C}_6) = \mathcal{C}_{13}$  and  $\alpha(x_9) \in \{d_1, b_2\}$ ;
- (h)  $G \in \{O_m, F_m : m \geq 3\}$  with  $\alpha(x_3) = c_{m-1}$ , then we obtain  $\alpha^{-1}(d_1) \in \{x_2, x_4\}$ ;
- (i)  $G \in \{O_2, F_2\}$  with  $\alpha(x_3) = c_1$ , then we get  $\alpha(\mathcal{C}_6) = \mathcal{C}_{18}$  and  $\alpha(x_9) \in \{x_6, x_7\}$ .

We show that each one of these cases yields a contradiction.

Cases (a)–(e): While each of  $x_2$  or  $x_4$  is contained in precisely two 5-cycles (the cycles  $\mathcal{C}_6, \mathcal{C}_7$  or  $\mathcal{C}_5, \mathcal{C}_6$ , respectively), the number of 5-cycles touching their image  $\alpha(x_2), \alpha(x_4)$  is different from 2 (it is namely 1 by Lemmas 5.1, 5.2 and 5.3).

Case (f): Only two 5-cycles,  $\mathcal{C}_6$  and  $\mathcal{C}_7$ , go through  $x_9$  while the three cycles  $\mathcal{C}_5, \mathcal{C}_8 = (x_6, d_1, b_1, a_1, x_5)$  and  $\mathcal{C}_9 = (x_6, d_1, c_1, x_1, x_7)$  contain the vertex  $x_6$  and the three cycles  $\mathcal{C}_8, \mathcal{C}_9, \mathcal{C}_{20}$  touch the vertex  $d_1$ .

Case (g): Only two 5-cycles,  $\mathcal{C}_6$  and  $\mathcal{C}_7$ , go through  $x_9$  while the three cycles  $\mathcal{C}_{13}, \mathcal{C}_{14}$  and  $\mathcal{C}_{20}$ , go through  $d_1$  and the three cycles  $\mathcal{C}_{13}, \mathcal{C}_{14}$  and  $(b_2, d_2, b_3, a_3, a_2)$  contain the vertex  $b_2$ .

Case (h): There do not exist 6-cycles containing  $d_1$  while the 6-cycle  $(x_1, x_2, x_3, x_4, x_8, x_7)$  goes through  $x_2$  or  $x_4$ .

Case (i): While  $x_9$  is contained in precisely two 5-cycles, ( $\mathcal{C}_6$  and  $\mathcal{C}_7$ ), the three 5-cycles  $\mathcal{C}_5, \mathcal{C}_{15}$  and  $\mathcal{C}_{18}$  go through  $x_6$  and the three 5-cycles  $\mathcal{C}_5, \mathcal{C}_7, \mathcal{C}_{18}$  contain  $x_7$ .

Therefore, we have proved that  $\alpha(x_3) \in \{x_1, x_3, x_5, x_6\}$ . Hence, Proposition 4.2 implies that  $\alpha(x_3) = x_3$  or  $\alpha(x_3) = x_1$ . The second case does not occur: if  $\alpha \in \text{Aut}(I_m)$ , with  $m \geq 1$ , there is a different number of 5-cycles going through each vertex  $x_3$  and  $x_1$ ; if  $\alpha \in \text{Aut}(G)$ , with  $G \in \{O_m, F_m : m \geq 2\}$ , there is a different number of 5-cycles containing each vertex  $x_3$  and  $x_1$ ; and finally, if  $G \in \{O_1, F_1\}$ , there is a different number of 8-cycles passing through each vertex  $x_3$  and  $x_1$  (see Lemmas 5.1 and 5.2).  $\square$

**Proposition 5.5.** *Let  $G$  be any graph from  $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$  with  $\bar{G}_m$  as subgraph of  $G$  for an integer  $m \geq 1$ . Let  $\alpha$  be an automorphism of  $G$ , then  $\alpha$  fixes  $\bar{G}_m$  setwise and the restriction  $\alpha$  to  $\bar{G}_m$  is either the involution  $(x_{11} \ x_{17}) (x_{12} \ x_{16})(x_{13} \ x_{15})$  or the identity permutation.*

*Proof.* The statement follows from Proposition 5.4 and Corollary 4.5.  $\square$

**Theorem 5.6.** *Let  $G$  be any graph from  $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$ , then the automorphism group of  $G$  is the trivial group.*

*Proof.* Let  $\alpha$  be an automorphism of  $G$  with  $G \in \mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$ . Define  $\bar{V} = X$  if  $m = 1$  and  $\bar{V} = X \cup A_m \cup B_m \cup C_m \cup D_m$  if  $m \geq 2$ . Every vertex  $v \in \bar{V}$  is adjacent to no more than one vertex  $p_v \notin \bar{V}$ . Let us consider the vertices  $x_5, x_6, x_1$  if  $m = 1$ , or the vertices  $a_1, b_1, c_{m-1}$  if  $m \geq 2$ ; Proposition 5.5 and Property EP3 imply that  $\alpha$  fixes the vertices not belonging to  $\bar{V}$  and adjacent to each of the vertices  $x_5, x_6, x_1$  if  $m = 1$ , or vertices  $a_1, b_1, c_{m-1}$  if  $m \geq 2$ . In particular we get

- $\alpha(y_4) = y_4, \alpha(z_7) = z_7$  and  $\alpha(z_1) = z_1$ , if  $G \in \mathcal{O}$ ;
- $\alpha(t_3) = t_3, \alpha(r_8) = r_8$  and  $\alpha(r_1) = r_1$ , if  $G \in \mathcal{F}$ ;
- $\alpha(r_1) = r_1, \alpha(r_8) = r_8$  and  $\alpha(s_5) = s_5$ , if  $G \in \mathcal{I}$ . Moreover, in this case Property EP3 also implies that  $\alpha(s_4) = s_4$ .

By Proposition 5.5 we have only two cases:

I) The automorphism  $\alpha$  acts on  $\bar{G}_m$  as the permutation  $(x_{11} \ x_{17})(x_{12} \ x_{16})(x_{13} \ x_{15})$ .



If  $G \in \mathcal{O}$ , then the pair  $\alpha([x_{12}, y_3]) = [\alpha(x_{12}), \alpha(y_3)] = [x_{16}, \alpha(y_3)]$  is an edge and so the vertex  $\alpha(y_3)$  is adjacent to  $x_{16}$ ; Proposition 5.5 implies that  $\alpha(X) = X$ ; thus  $\alpha(y_3) = y_1$ , hence  $\alpha([y_3, y_4]) = [\alpha(y_3), \alpha(y_4)] = [y_1, y_4]$ . A contradiction since  $[y_1, y_4]$  is not an edge.

If  $G \in \mathcal{F}$ , then the pair  $\alpha([x_{12}, t_2]) = [\alpha(x_{12}), \alpha(t_2)] = [x_{16}, \alpha(t_2)]$  is an edge with the vertex  $\alpha(t_2)$  adjacent to  $x_{16}$ . From Proposition 5.5 we obtain  $\alpha(X) = X$ ; hence  $\alpha(t_2) = t_1$ , thus  $\alpha([t_2, t_3]) = [\alpha(t_2), \alpha(t_3)] = [t_1, t_3]$ . A contradiction since  $[t_1, t_3]$  is not an edge.

If  $G \in \mathcal{I}$ , then the pair  $\alpha([x_{12}, s_3]) = [\alpha(x_{12}), \alpha(s_3)] = [x_{16}, \alpha(s_3)]$  is an edge with the vertex  $\alpha(s_3)$  adjacent to  $x_{16}$ . From Proposition 5.5 we get  $\alpha(X) = X$ , thus  $\alpha(s_3) = s_1$ , therefore  $\alpha([s_3, s_4]) = [\alpha(s_3), \alpha(s_4)] = [s_1, s_4]$ . A contradiction since  $[s_1, s_4]$  is not an edge.

Therefore, this first case does not occur.

II) The automorphism  $\alpha$  acts on  $\bar{G}_m$  as the trivial permutation.

Let  $G \in \mathcal{O}$ . By Property EP3 we have  $\alpha(y_3) = y_3$  (the vertices  $x_{12}, x_{11}, x_{13}$  are fixed by  $\alpha$ ); Property EP3 implies  $\alpha(y_2) = y_2$  and  $\alpha(y_1) = y_1$  (the vertices  $y_3, y_4, x_{12}$  and  $x_{16}, x_{17}, x_{15}$  are respectively fixed by  $\alpha$ ). By Property EP2 we also have  $\alpha(z_2) = z_2$  (the vertices  $z_1, y_1$  are fixed),  $\alpha(z_3) = z_3$  (the vertices  $z_2, z_7$  are fixed),  $\alpha(z_4) = z_4$  (the vertices  $z_3, y_4$  are fixed),  $\alpha(z_5) = z_5$  (the vertices  $z_1, z_4$  are fixed) and  $\alpha(z_6) = z_6$  (the vertices  $z_5, z_7$  are fixed). Therefore,  $\alpha$  is the identity permutation on  $G$ .

Let  $G \in \mathcal{F}$ . By Property EP3 we obtain  $\alpha(t_1) = t_1$  (the vertices  $x_{15}, x_{16}, x_{17}$  are fixed by  $\alpha$ ),  $\alpha(t_2) = t_2$  (the vertices  $x_{12}, x_{11}, x_{13}$  are fixed by  $\alpha$ ), thus  $\alpha(r_3) = r_3$  (the vertices  $x_{16}, t_1, t_2$  are fixed) and  $\alpha(r_6) = r_6$  (the vertices  $a_1, t_3, t_2$  or  $x_5, t_3, t_2$  if  $m = 1$ , are fixed). By Property EP2 we also have  $\alpha(r_2) = r_2$  (the vertices  $r_1, r_3$  are fixed),  $\alpha(r_4) = r_4$  (the vertices  $r_3, r_8$  are fixed),  $\alpha(r_5) = r_5$  (the vertices  $r_4, r_6$  are fixed) and  $\alpha(r_7) = r_7$  (the vertices  $r_6, r_8$  are fixed). Therefore,  $\alpha$  is the identity permutation on  $G$ .

Let  $G \in \mathcal{I}$ . By Property EP3 we get  $\alpha(s_3) = s_3$  (the vertices  $x_{12}, x_{11}, x_{13}$  are fixed by  $\alpha$ ), hence  $\alpha(s_2) = s_2$  (the vertices  $x_{12}, s_3, s_4$  are fixed). By Property EP2 we obtain  $\alpha(s_1) = s_1$  (the vertices  $s_2$  and  $x_{16}$  are fixed). By Property EP3 we have  $\alpha(r_2) = r_2$  (the vertices  $s_1, s_2, x_{16}$  are fixed),  $\alpha(r_4) = r_4$  (the vertices  $s_3, s_4, s_5$  are fixed) and  $\alpha(r_6) = r_6$  (the vertices  $s_1, s_2, s_3$  are fixed). Therefore, by Property EP2 the vertices  $r_3, r_5$  and  $r_7$  are also fixed. The automorphism  $\alpha$  is the identity permutation on  $G$ . The statement follows.  $\square$

**Corollary 5.7.** *For every possible order greater than 26 there exists an asymmetric nearly hamiltonian snark of that order.*

*Proof.* The nearly hamiltonian snarks  $O$  and  $F$  shown in Figure 3 have order 24 while the nearly hamiltonian snark  $I$  shown in Figure 4 has order 26. From

Corollary 2.2 the nearly hamiltonian snarks  $O_m$  and  $F_m$  have order  $24 + 4m$  while  $I_m$  has order  $26 + 4m$ . The statement follows from Theorem 5.6.  $\square$

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