LE MATEMATICHE
Vol. LXV (2010) – Fasc. I, pp. 23–40
doi: 10.4418/2010.65.1.2

PROBABILITY DENSITY FUNCTIONS INVOLVING A GENERALIZED *r*–GAUSS HYPERGEOMETRIC FUNCTION

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The aim of this paper is to study r–generalized gamma functions of the form

$$ {}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right) = v^{-a}\int_0^\infty t^{u-1}\, e^{-pt}\quad {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\, dt, $$

where a, b, c and p are complex numbers with $c \neq 0,-1,-2,\ldots$, $\operatorname{Re} p$, $\operatorname{Re} u > 0$ and $|\arg v| < \pi$, and the r–generalized beta functions of the form

$$ {}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = v^{-a}\int_0^\infty t^{u-1}\ (1+t)^{-\mu-u}\quad {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\, dt, $$

where $\operatorname{Re}(a+\mu)$, $\operatorname{Re}(b+\mu) > 0$ and ${}_r\tilde{F}(a,b;c;x)$ is the r-Gauss hypergeometric function. Moreover, we define a new probability density function (p.d.f) involving these new generalized functions. Some basic functions associated with the p.d.f's, such as moment generating functions, mean residue functions and hazard rate functions are derived .

Entrato in redazione: 20 agosto 2009

AMS 2000 Subject Classification: 33C75, 33C20.
Keywords: r–generalized Gamma functions, incomplete gamma functions, r–generalized beta functions, incomplete beta functions, r–Gauss hypergeometric Function.

1. Introduction

Due to the success of the gamma function, several generalizations and modifications have been considered by various authors [3, 4, 6, 12]. Al-Musallam and Kalla [3, 4] have studied a generalization gamma function of the form

$$D\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right) \triangleq v^{-a}\int_0^\infty t^{u-1}\, e^{-pt}\ {}_2F_1\left(a,b;c;-\frac{t}{v}\right)\, dt\ . \tag{1}$$

By putting $p=1$ and $b=c$ in equation (1), we obtain the Kobayashi [6] generalized gamma function defined recently as follows,

$$D\left(\begin{array}{c} a,b;b;1 \\ u,v \end{array}\right) = \int_0^\infty \frac{t^{u-1}\, e^{-t}}{(v+t)^a}\, dt \triangleq \Gamma_a(u,v) \tag{2}$$

i.e. their work is a generalization of Kobayashi's function. Moreover, if we take $a=0$ in this equation, we get the well-known gamma function

$$\Gamma(u) = \int_0^\infty t^{u-1}\, e^{-t}\, dt.$$

Recently, Ben Nakhi and Kalla [5] have treated a generalized beta function in the form

$$\overset{\omega}{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = v^{-a}\int_0^\infty t^{u-1}\,(1+t)^{-\mu-u}\, {}_2\overset{\omega}{R}_1\left(a,b;c;-\frac{t}{v}\right)\, dt, \tag{3}$$

and

$$B\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = v^{-a}\int_0^\infty t^{u-1}\,(1+t)^{-\mu-u}\ {}_2F_1\left(a,b;c;-\frac{t}{v}\right)\, dt. \tag{4}$$

In section 2, we recall some special functions and give some basic results that will be used in latter sections. In section 3, we define our $r-$ generalized gamma function and its incomplete functions. Further, we study a probability density function (p.d.f) involving this new function ${}_r\widetilde{D}$. Moreover, we introduce and study a new probability density function (p.d.f) involving our second new $r-$generalized beta function ${}_r\tilde{B}$ in the fourth section. In each section of the last two sections we derive some basic functions associated with our new generalized density functions, namely, we compute the k-th Moment, moment generating function, the hazard rate function and the mean residue life function.

2. Definitions and Preliminaries

Throughout this sequel, we shall use the following definitions and the *Fox-Wright Psi function* [9] is defined as follows

$$ {}_p\Psi_q\left[\ (a_k,b_k)_{1,p};(\alpha_k,\beta_k)_{1,q};z\ \right]=\sum_{n=0}^{\infty}\frac{\prod\limits_{k=1}^{p}\Gamma(a_k+nb_k)}{\prod\limits_{k=1}^{q}\Gamma(\alpha_k+n\beta_k)}\frac{z^n}{n!}, \tag{5} $$

$$ b_i>0 \quad (i=1,...,p);\beta_j>0 \quad (j=1,...,q); \qquad 1+\sum_{k=1}^{q}\beta_k-\sum_{k=1}^{p}b_k\geqslant 0. \tag{6} $$

for suitably bounded values of $|z|$, its *normalization* is

$$ {}_p\overset{*}{\Psi}_q\left[\ (a_k,b_k)_{1,p};(\alpha_k,\beta_k)_{1,q};z\ \right]=\frac{\prod\limits_{k=1}^{p}\Gamma(\alpha_k)}{\prod\limits_{k=1}^{q}\Gamma(a_k)}\ {}_p\Psi_q\left[\ (a_k,b_k)_{1,p};(\alpha_k,\beta_k)_{1,q};z\ \right]. \tag{7} $$

In particular, we have the following relationship with *generalized hypergeometric function* ${}_pF_q$ (see [8]): for $p\le q+1$

$$ {}_pF_q\left[\ a_1,...,a_p;\alpha_1,...,\alpha_q;z\ \right]=\ {}_p\overset{*}{\Psi}_q\left[\ (a_k,1)_{1,p};(\alpha_k,1)_{1,q};z\ \right]. \tag{8} $$

The *generalized $r-$Gauss hypergeometric function* defined in [11] as

$$ {}_r\tilde{F}(a,b;c;z)= $$

$$ =\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{t^{b-1}\ (1-t)^{c-b-1}}{(1-zt)^a}\,{}_1\overset{*}{\Psi}_1\left[(\lambda,\tau);(\gamma,\eta);\frac{r}{t\ (t\ -1)}\right]\ dt, \tag{9} $$

where $\mathrm{Re}\,c>\mathrm{Re}\,b>0$, $r>0$; $r=0$, $|z|<1$; $\mathrm{Re}\,\gamma>\mathrm{Re}\,\lambda>0$, $\tau,\eta\subset\Re$, $\tau>0$, $\tau-\eta<1$. For $r=0$ our function reduces to Gauss hypergeometric function, i.e.

$$ {}_0\tilde{F}(a,b;c;z)\ =\ {}_2F_1(a,b;c;z). \tag{10} $$

Other special cases of this function are given in [11]. The differential relation for ${}_r\tilde{F}(a,b;c;z)$ is (see [11])

$$ \left(\frac{d}{dz}\right)^n\ {}_r\tilde{F}(a,b;c;z)=\frac{(a)_n(b)_n}{(c)_n}\ {}_r\tilde{F}(a+n,b+n;c+n;z). \tag{11} $$

The functional relations for ${}_r\tilde{F}\,(a,b;c;z)$ are (see [[11], Eqn.11-12])

$$b\;{}_r\tilde{F}\,(a,b+1;c;z)+(c-b-1)\;{}_r\tilde{F}\,(a,b;c;z)=(c-1)\;{}_r\tilde{F}\,(a,b;c-1;z) \quad (12)$$

$$c\;{}_r\tilde{F}\,(a,b;c;z)-(c-b)\;{}_r\tilde{F}\,(a,b;c+1;z)=b\;{}_r\tilde{F}\,(a,b+1;c+1;z) \quad (13)$$

As a standard notation, $F(a,b;c;x)$ stands for Gauss Hypergeometric function ${}_2F_1\,(a,b;c;x)$, but if there are other number of parameters (other than 2 and 1), like p and q then only we write ${}_pF_q$. So here we have used this convention of not writing sub indices in our definition.

3. The $r-$Generalized Gamma Function ${}_r\widetilde{D}$

We begin this section by introducing our $r-$generalized gamma function.

Definition 3.1. Let a, b, c, u and p be complex numbers with $\operatorname{Re} p$, $\operatorname{Re} u>0$, $c\neq 0$ and $|\arg v|<\pi$. Then we define the $r-$generalized gamma function,

$${}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right)=v^{-a}\int_0^\infty t^{u-1}\,e^{-pt}\;{}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\,dt. \quad (14)$$

For $r=0$, we get the Al-Musallam and Kalla [3, 4] generalized gamma function (1). Observing that equation (14) can be rewritten as

$${}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right)=v^{u-a}\int_0^\infty t^{u-1}\,e^{-pvt}\;{}_r\tilde{F}\,(a,b;c;-t)\,dt. \quad (15)$$

The following recurrence relations for ${}_r\widetilde{D}$ can be easily derived from its definition besides the recurrence relations of ${}_r\tilde{F}$ given in (12) and (13).

Theorem 3.2. *The following relations hold:*

$$(c-b-1)\;{}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right)=$$

$$=(c-1)\;{}_r\widetilde{D}\left(\begin{array}{c} a,b;c-1;p \\ u,v \end{array}\right)-\,b\;{}_r\widetilde{D}\left(\begin{array}{c} a,b+1;c;p \\ u,v \end{array}\right);$$

$$c\;{}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right)=$$

$$=(c-b)\;{}_r\widetilde{D}\left(\begin{array}{c} a,b;c+1;p \\ u,v \end{array}\right)+\quad b\;{}_r\widetilde{D}\left(\begin{array}{c} a,b+1;c+1;p \\ u,v \end{array}\right).$$

Moreover, using (11) and integration by parts to the integral representation of ${}_r\widetilde{D}$ we obtain the following result.

Lemma 3.3. *The following relation holds*

$$\frac{u}{p}\,{}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right) = \frac{ab}{pc}\,{}_r\widetilde{D}\left(\begin{array}{c} a+1,b;c;p \\ u+1,v \end{array}\right) + {}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u+1,v \end{array}\right). \quad (16)$$

We end this section by giving the partial derivatives of the $r-$generalized gamma function.

Lemma 3.4. *The partial derivatives of* ${}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right)$ *are*

$$\frac{\partial^n}{\partial u^n}\,{}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right) = v^{-a}\int_0^\infty t^{u-1}\, e^{-pt}\, [\ln t]^n \;\; {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right) dt, \quad (17)$$

$$\frac{\partial^n}{\partial v^n}\,{}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array}\right) = (-1)^n\, (a)_n\, {}_r\widetilde{D}\left(\begin{array}{c} a+n,b;c;p \\ u,v \end{array}\right). \quad (18)$$

Proof. The first formula is obtained by observing that

$$\frac{\partial^n}{\partial u^n}\, t^{u-1} = t^{u-1}\, [\ln t]^n\, .$$

The second formula is obtained by using the integral representation of ${}_r\tilde{F}$, that is

$${}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right) =$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{s^{b-1}\,(1-s)^{c-b-1}}{\left(1+\frac{st}{v}\right)^a}\,{}_1\overset{*}{\Psi}_1\left[(\lambda,\tau);(\gamma,\eta);\frac{r}{s\,(s\,-1)}\right]\; ds$$

to write

$${}_r\widetilde{D} = \frac{v^{-a}}{L}\int_0^\infty t^{u-1}\, e^{-pt}\,\cdot$$

$$\cdot\left(\int_0^1 \frac{s^{b-1}\,(1-s)^{c-b-1}}{\left(1+\frac{st}{v}\right)^a}\,{}_1\overset{*}{\Psi}_1\left[(\lambda,\tau);(\gamma,\eta);\frac{r}{s\,(s\,-1)}\right]\; ds\right) dt =$$

$$= \frac{1}{L}\int_0^1 s^{b-1}\,(1-s)^{c-b-1}\,{}_1\overset{*}{\Psi}_1\left[(\lambda,\tau);(\gamma,\eta);\frac{r}{s\,(s\,-1)}\right]\cdot$$

$$\cdot \left(\int_0^\infty t^{u-1}\, e^{-pt}\, (v+st)^{-a}\, dt \right) ds$$

where

$${}_r\widetilde{D} = {}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right) \text{ and } L = B(b, c-b)$$

besides noting that

$$\frac{\partial^n}{\partial v^n}\, (v+st)^{-a} = (-1)^n\, (a)_n\, (v+st)^{-a-n}.$$

□

3.1. The Generalized Incomplete Gamma Functions

Generalizing ${}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right)$, we define, for $x > 0$, $\mathrm{Re}\, p$, $\mathrm{Re}\, u > 0$, $|\arg v| < \pi$,

$${}_r\widetilde{D}_0^x\left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right) = v^{-a} \int_0^x t^{u-1}\, e^{-pt}\ {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\, dt\ , \tag{19}$$

and call it as a *generalized incomplete gamma function*, and its companion function

$${}_r\widetilde{D}_x^\infty\left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right) = v^{-a} \int_x^\infty t^{u-1}\, e^{-pt}\ {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\, dt\ , \tag{20}$$

and call it as a *generalized complementary incomplete gamma function*. In other words we have

$${}_r\widetilde{D}\left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right) = {}_r\widetilde{D}_0^x\left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right) + {}_r\widetilde{D}_x^\infty\left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right). \tag{21}$$

Remark 3.5. *Substituting equations (19) and (20), with* $p = 1$, $r = 0$, $b = c$, $a = 0$, *we obtain the well-known incomplete gamma functions respectively, that is*

$$\gamma(u,x) = \int_0^x t^{u-1}\, e^{-t}\, dt\ ,\ \Gamma(u,x) = \int_x^\infty t^{u-1}\, e^{-t}\, dt. \tag{22}$$

The next theorem lists some differential properties and recurrence relations of these incomplete functions. The proof is straightforward. For simplicity, we let

$${}_r\widetilde{D}_0^x \triangleq {}_r\widetilde{D}_0^x\left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right) \quad \text{and} \quad {}_r\widetilde{D}_x^\infty \triangleq {}_r\widetilde{D}_x^\infty\left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right)$$

Theorem 3.6. *Continue with the preceding notations, we have*

$$\begin{aligned} {}_r\widetilde{D}_0^x(u+1) &= \frac{u}{p}\,{}_r\widetilde{D}_0^x - \frac{a\,L_k}{p}\,{}_r\widetilde{D}_0^x\begin{pmatrix} a+1,b;c;p \\ u+1,v \end{pmatrix} \\ &\quad -\frac{v^{-a}}{p}\;x^u\,e^{-px}\;{}_rF_q\left(a,b;c;-\frac{x}{v}\right), \end{aligned} \tag{23}$$

$$\begin{aligned} {}_r\widetilde{D}_x^\infty(u+1) &= \frac{u}{p}\,{}_r\widetilde{D}_x^\infty - \frac{a\,L_k}{p}\,{}_r\widetilde{D}_x^\infty\begin{pmatrix} a+1,b;c;p \\ u+1,v \end{pmatrix} \\ &\quad +\frac{v^{-a}}{p}\;x^u\,e^{-px}\;{}_rF_q\left(a,b;c;-\frac{x}{v}\right), \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{d}{dx}\left[x^{-u}\;{}_r\widetilde{D}_x^\infty\right] &= -\,u\,x^{-u-1}\,{}_r\widetilde{D}_x^\infty + \frac{v^{-a}}{x}\;e^{-px}\;{}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right), \\ \frac{d}{dx}\left[e^{px}\,{}_r\widetilde{D}_x^\infty\right] &= p\,e^{px}\,{}_r\widetilde{D}_x^\infty - v^{-a}\,x^{-u-1}\;{}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right), \\ \frac{d}{dx}\left[x^{-u}\;{}_r\widetilde{D}_0^x\right] &= -\,u\,x^{-u-1}\,{}_r\widetilde{D}_0^x - \frac{v^{-a}}{x}\;e^{-px}\;{}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right), \\ \frac{d}{dx}\left[e^{px}\;{}_r\widetilde{D}_0^x\right] &= p\,e^{px}\,{}_r\widetilde{D}_0^x + v^{-a}\,x^{-u-1}\;{}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right). \end{aligned} \tag{25}$$

3.2. The Probability Density Function $f_1(x)$

The probability density function (p.d.f) of a random variable X_1 associated with equation (14) is defined by

$$f_1(x) = \frac{\beta\,v^{-a}\,\alpha^{\frac{m}{\beta}+1}\;x^{m+\beta-1}\,e^{-\delta x^\beta}\;{}_r\tilde{F}\left(a,b;c;-\frac{\alpha x^\beta}{v}\right)}{{}_r\widetilde{D}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}} \times 1\,[x>0]\,. \tag{26}$$

It is obvious that $\int_0^\infty f_1(t)\,dt = 1$. We observe that the behavior of $f_1(x)$ at zero depends on $m+\beta$, that is :

$$f_1(0) = \begin{cases} 0 & , \quad m+\beta > 1 \\ \beta\,\alpha^{\frac{1}{\beta}}\,v^{-a} \times \left[{}_r\widetilde{D}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{1}{\beta},v \end{pmatrix}\right]^{-1} & , \quad m+\beta = 1 \end{cases}$$

Moreover, we have $\lim_{x\to 0^+} f_1(x) = \infty$ for $m+\beta < 1$ and $\lim_{x\to\infty} f_1(x) = 0$ for $\beta > 1$. Using the differentiation formulas for ${}_r\tilde{F}(a,b;c;x)$ (11). It is easy to show that

$$\frac{d}{dx}\, f_1(x) = \left[\frac{m+\beta-1}{x} - \delta\beta\, x^{\beta-1} - \frac{a\, b\, \alpha\beta}{v\, c}\ x^{\beta-1} \times H(x)\right] \quad f_1(x),$$

where

$$H(x) \triangleq \frac{{}_r\tilde{F}\left(a+1, b+1; c+1; -\frac{\alpha x^\beta}{v}\right)}{{}_r\tilde{F}\left(a, b; c; -\frac{\alpha x^\beta}{v}\right)}.$$

3.3. Some Statistical Functions

The aim of this section is to obtain some basic functions associated with the p.d.f $f_1(x)$, such as the population moments, the cumulative distribution function (c.d.f.), the survivor function, the hazard rate function and the mean residue life function. We start with,

3.3.1. Population Moments

In this subsection we derive several types of moments such as the k -th moment and the moment generating function. We begin by evaluating the k -th moment, since it will be used to obtain the remaining basic moments, such as the mean, the variance and the moment generating function.

The k-th Moment: The k-th Moment about the origin of the random variable X_1 is defined by

$$E\left[X_1^k\right] \triangleq \int_0^\infty t^k\, f_1(t)\ dt.$$

Substituting with the value of the p.d.f $f_1(x)$ given by the equation (26) yields

$$E\left[X_1^k\right] = \frac{\beta\, v^{-a}\, \alpha^{\frac{m}{\beta}+1}}{{}_r\widetilde{D}\begin{pmatrix} a, b; c; \frac{\delta}{\alpha} \\ \frac{m}{\beta}+1, v \end{pmatrix}} \int_0^\infty x^{m+\beta+k-1}\, e^{-\delta x^\beta}\, {}_r\tilde{F}\left(a, b; c; -\frac{\alpha x^\beta}{v}\right) dx,$$

then by means of the transformation $t = \alpha x^\beta$ and by virtue of equation (26), we get

$$E\left[X_1^k\right] = \frac{\alpha^{-\frac{k}{\beta}} \times {}_r\widetilde{D}\begin{pmatrix} a, b; c; \frac{\delta}{\alpha} \\ \frac{m+k}{\beta}+1, v \end{pmatrix}}{{}_r\widetilde{D}\begin{pmatrix} a, b; c; \frac{\delta}{\alpha} \\ \frac{m}{\beta}+1, v \end{pmatrix}}. \tag{27}$$

Now since the mean, expected value of the random variable X_1, is a special case of this moment, namely, the mean is the 1-st Moment

$$E[X_1] \triangleq \int_0^\infty t\, f_1(t)\ dt = \frac{\alpha^{-\frac{1}{\beta}} \times {}_r\widetilde{D}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m+1}{\beta}+1,v \end{pmatrix}}{{}_r\widetilde{D}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}}. \tag{28}$$

Similarly, we can obtain the variance of the random variable X_1 , $\sigma^2_{X_1}$, using the equation (27) with $k=2$, besides the equation (28), since it is defined as

$$\sigma^2_{X_1} \triangleq E\left[X_1^2\right] - \left(E[X_1]\right)^2.$$

Moment generating function: The moment generating function of the random variable X_1 is is defined by

$$M_1(t) \triangleq E\left[e^{tX_1}\right] = \int_0^\infty e^{tx}\, f_1(x)\ dx.$$

To avoid the difficulty of this integration, we observe, using Taylor expansion, that

$$E\left[e^{tX_1}\right] = \sum_{k=0}^\infty \frac{t^k}{k!}\, E\left[X_1^k\right]. \tag{29}$$

Using this beside the equation (27), we obtain

$$M_1(t) = \sum_{k=0}^\infty \frac{\alpha^{-\frac{k}{\beta}} \times t^k}{k!} \times \frac{{}_r\widetilde{D}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m+k}{\beta}+1,v \end{pmatrix}}{{}_r\widetilde{D}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}}.$$

3.3.2. The Distribution Function

The cumulative distribution function $F_1(x)$ of the random variable X_1 is given by

$$F_1(x) \triangleq P(X_1 \le x) = \int_0^x f_1(t)\, dt = \frac{{}_r\widetilde{D}_0^{\alpha x^\beta}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}}{{}_r\widetilde{D}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}},$$

hence the survivor function $S_1(x)$can be expressed as

$$S_1(x) \triangleq P(X_1 \geq x) = 1 - F_1(x) = \int_x^\infty f_1(t)\,dt = \frac{{}_r\widetilde{D}^{\infty}_{\alpha x^\beta}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}}{{}_r\widetilde{D}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}}. \quad (30)$$

3.3.3. The Hazard Rate Function

For a p.d.f $f_1(x)$ the hazard rate function is defined by

$$h_1(x) \triangleq \frac{f_1(x)}{S_1(x)}.$$

Using the equations (27) and (30), it follows that

$$h_1(x) = \frac{\beta\ \alpha^{\frac{m}{\beta}+1}\ v^{-a}\ x^{m+\beta-1}\ e^{-\delta x^\beta}\ {}_r\tilde{F}\left(a,b;c;-\frac{\alpha x^\beta}{v}\right)}{{}_r\widetilde{D}^{\infty}_{\alpha x^\beta}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}}. \quad (31)$$

3.3.4. The Mean Residue Life Function.

For a random variable X_1 the mean residue life function is defined by

$$K_1(x) = E\left[X_1 - x\ /X_1 \geq x\right] =$$

$$= \frac{1}{S_1(x)}\int_x^\infty (t-x)\,f_1(t)\,dt = \frac{1}{S_1(x)}\int_x^\infty t\ f_1(t)\,dt - x. \quad (32)$$

Using equation (30), and

$$\int_x^\infty t\ f_1(t)\ dt = \alpha^{-\frac{1}{\beta}} \times \frac{{}_r\widetilde{D}^{\infty}_{\alpha x^\beta}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m+1}{\beta}+1,v \end{pmatrix}}{{}_r\widetilde{D}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}},$$

we get

$$K_1(x) = \alpha^{-\frac{1}{\beta}} \times \frac{{}_r\widetilde{D}^{\infty}_{\alpha x^\beta}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m+1}{\beta}+1,v \end{pmatrix}}{{}_r\widetilde{D}^{\infty}_{\alpha x^\beta}\begin{pmatrix} a,b;c;\frac{\delta}{\alpha} \\ \frac{m}{\beta}+1,v \end{pmatrix}} - x. \quad (33)$$

4. The $r-$Generalized Beta Function ${}_r\tilde{B}$

We begin this section by introduce our generalized $r-$Beta function in the following form

Definition 4.1. Continue with the preceding assumptions on the parameters a, b, c, r, u and v. Then for Re $(a+\mu)$ and Re $(b+\mu)>0$

$$ {}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = v^{-a}\int_0^\infty t^{u-1}\ (1+t)^{-\mu-u}\ \ {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\, dt, \tag{34} $$

For $r=0$, since the Gauss hypergeometric function is the classical Gauss hypergeometric function therefore equation (34) becomes

$$ B\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = v^{-a}\int_0^\infty t^{u-1}\ (1+t)^{-\mu-u}\ {}_2F_1\left(a,b;c;-\frac{t}{v}\right)\, dt\ , \tag{35} $$

and by letting $b=c$, we have

$$ B\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = \int_0^\infty \frac{t^{u-1}\ (1+t)^{-\mu-u}}{(v+t)^a}\, dt. $$

Moreover, if we take $a=0$ then equation (34) reduces to the ordinary Beta function, i.e.

$$ B(u,\mu) = \int_0^\infty t^{u-1}\ (1+t)^{-\mu-u}\ dt $$

(34) can be rewritten as

$$ {}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = v^{u-a}\int_0^\infty t^{u-1}\ (1+vt)^{-\mu-u}\ {}_r\tilde{F}(a,b;c;-t)\ \ dt. \tag{36} $$

The following recurrence relations for ${}_r\tilde{B}$ can be easily derived from its definition besides the recurrence relations of ${}_r\tilde{F}$ given in (12) and (13).

Theorem 4.2. *The following relations hold:*

$$ (c-b-1)\ {}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = $$

$$ = (c-1)\ {}_r\tilde{B}\left(\begin{array}{c} a,b;c-1;v \\ u,\mu \end{array}\right) - \ b\ {}_r\tilde{B}\left(\begin{array}{c} a,b+1;c;v \\ u,\mu \end{array}\right) \tag{37} $$

$$c\ {}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) =$$

$$= (c-b)\ {}_r\tilde{B}\left(\begin{array}{c} a,b;c+1;v \\ u,\mu \end{array}\right) + b\ {}_r\tilde{B}\left(\begin{array}{c} a,b+1;c+1;v \\ u,\mu \end{array}\right) \qquad (38)$$

Moreover, using (11) and integration by parts to the integral representation of ${}_r\tilde{B}$ we obtain the following result.

Lemma 4.3. *With the preceding assumptions, we have*

$${}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u+1,\mu \end{array}\right) =$$

$$= \frac{u}{u+\mu}\ {}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) - \frac{ab}{(u+\mu)\ c}\ {}_r\tilde{B}\left(\begin{array}{c} a+1,b+1;c+1;v \\ u+1,\mu-1 \end{array}\right). \qquad (39)$$

We end this section by giving the partial derivatives of the generalized Beta function

Lemma 4.4. *The partial derivatives of* ${}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right)$ *are*

$$\frac{\partial^n}{\partial u^n}\ {}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) =$$

$$= v^{-a} \int_0^\infty (-1)^{n+1}\ t^{-1}\ [\ln(1+t)]^n\ (1+t)^{-\mu-u}\ {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\ dt, \qquad (40)$$

$$\frac{\partial^n}{\partial v^n}\ {}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = (-1)^n\ (a)_n\ \ {}_r\tilde{B}\left(\begin{array}{c} a+n,b;c;v \\ u,\mu \end{array}\right). \qquad (41)$$

Proof. The first formula is obtained by observing that

$$\frac{\partial^n}{\partial u^n}\ \left[t^{u-1}\ (1+t)^{-\mu-u}\right] = (-1)^{n+1}\ t^{-1}\ [\ln(1+t)]^n\ (1+t)^{-\mu-u}.$$

The second formula is obtained by using the integral representation of ${}_r\tilde{F}$, that is

$${}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right) =$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 \frac{s^{b-1}\ (1-s)^{c-b-1}}{\left(1+\frac{st}{v}\right)^a} {}_1\overset{*}{\Psi}_1 \left[(\lambda,\tau);(\gamma,\eta);\frac{r}{s\ (s\ -1)}\right]\ ds,$$

to write

$${}_r\tilde{B} = \frac{v^{-a}}{L} \int_0^\infty t^{u-1}\ (1+t)^{-\mu-u}\ \cdot$$

$$\cdot \left(\int_0^1 \frac{s^{b-1}\ (1-s)^{c-b-1}}{\left(1+\frac{st}{v}\right)^a} {}_1\overset{*}{\Psi}_1 \left[(\lambda,\tau);(\gamma,\eta);\frac{r}{s\ (s\ -1)}\right]\ ds \right) dt =$$

$$= \frac{1}{L} \int_0^1 s^{b-1}\ (1-s)^{c-b-1}\ {}_1\overset{*}{\Psi}_1 \left[(\lambda,\tau);(\gamma,\eta);\frac{r}{s\ (s\ -1)}\right] \cdot$$

$$\cdot \left(\int_0^\infty t^{u-1}\ (1+t)^{-\mu-u}\quad (v+st)^{-a}\ dt \right) ds$$

where

$${}_r\tilde{B} = {}_r\tilde{B} \left(\begin{array}{c} a,b;c;p \\ u,v \end{array} \right) \text{ and } L = B(b,c-b)$$

besides noting that

$$\frac{\partial^n}{\partial v^n}\ (v+st)^{-a}\ = (-1)^n\ (a)_n\ (v+st)^{-a-n}.$$

□

4.1. The Generalized Incomplete Beta Functions ${}_r\tilde{B}$

Generalizing ${}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array} \right)$, we define, for x, Re u , Re $(a+\mu)$, Re $(b+\mu) > 0$, and $|\arg v| < \pi$, the related functions

$${}_r\tilde{B}_0^x \left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array} \right) = v^{-a} \int_0^x t^{u-1}\ (1+t)^{-\mu-u}\quad {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\ dt, \qquad (42)$$

and call it the *generalized incomplete Beta function*, and its companion function

$${}_r\tilde{B}_x^\infty \left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array} \right) = v^{-a} \int_x^\infty t^{u-1}\ (1+t)^{-\mu-u}\quad {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\ dt, \qquad (43)$$

and call it the *generalized complementary incomplete Beta function*. In other words we have

$$ {}_r\tilde{B}\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) = {}_r\tilde{B}_0^x\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) + {}_r\tilde{B}_x^\infty\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) \tag{44} $$

The next theorem lists some differential properties and recurrence relations of these incomplete functions. For simplicity, we let

$$ {}_r\tilde{B}_0^x \triangleq {}_r\tilde{B}_0^x\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) \quad \text{and} \quad {}_r\tilde{B}_x^\infty \triangleq {}_r\tilde{B}_x^\infty\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) $$

Theorem 4.5. *Continuing with the preceding notations, we have*

$$ \begin{aligned} {}_r\tilde{B}_0^x\left(\begin{array}{c} a,b;c;v \\ u+1,\mu \end{array}\right) = \frac{u}{u+\mu}\, {}_r\tilde{B}_0^x\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) - \\ - \frac{x^{u-1}\,(1+x)^{-\mu-u}}{(u+\mu)\,\times v^a}\, {}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right) - \\ - \frac{ab}{(u+\mu)\,c}\, {}_r\tilde{B}_0^x\left(\begin{array}{c} a+1,b+1;c+1;v \\ u+1,\mu-1 \end{array}\right) \end{aligned} \tag{45} $$

$$ \begin{aligned} {}_r\tilde{B}_x^\infty\left(\begin{array}{c} a,b;c;v \\ u+1,\mu \end{array}\right) = \frac{u}{u+\mu}\, {}_r\tilde{B}_x^\infty\left(\begin{array}{c} a,b;c;v \\ u,\mu \end{array}\right) + \\ + \frac{x^{u-1}\,(1+x)^{-\mu-u}}{(u+\mu)\,\times v^a}\, {}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right) - \\ - \frac{ab}{(u+\mu)\,c}\, {}_r\tilde{B}_x^\infty\left(\begin{array}{c} a+1,b+1;c+1;v \\ u+1,\mu-1 \end{array}\right) \end{aligned} \tag{46} $$

$$ \begin{aligned} \frac{d}{dx}\left[x^{1-u}\,{}_r\tilde{B}_0^x\right] &= (1-u)\,x^{-u}\,{}_r\tilde{B}_0^x + v^{-a}\,(1+x)^{-\mu-u}\,{}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right), \\ \frac{d}{dx}\left[(1+x)^{\mu+u}\,{}_r\tilde{B}_0^x\right] &= (u+\mu)\,(1+x)^{\mu+u-1}\,{}_r\tilde{B}_0^x + v^{-a}x^{u-1}\,{}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right), \\ \frac{d}{dx}\left[x^{1-u}\,{}_r\tilde{B}_x^\infty\right] &= (1-u)\,x^{-u}\,{}_r\tilde{B}_x^\infty - v^{-a}\,(1+x)^{-\mu-u}\,{}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right), \\ \frac{d}{dx}\left[(1+x)^{\mu+u}\,{}_r\tilde{B}_x^\infty\right] &= (u+\mu)\,(1+x)^{\mu+u-1}\,{}_r\tilde{B}_x^\infty - v^{-a}x^{u-1}\,{}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right). \end{aligned} $$

4.2. The Probability Density Function $f_2(x)$

The probability density function (p.d.f) of a random variable X_2 associated with equation (34) is defined by

$$f_2(x) = \frac{v^{-a}\, x^{u-1}\, (1+x)^{-\mu-u}\ {}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right)}{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}} \times 1\,[x>0] \tag{47}$$

It is obvious that $\int_0^\infty f_2(t)\, dt = 1$. We observe that the behavior of $f_2(x)$ at zero depends on u, that is:

$$f_2(0) = \begin{cases} 0 & , \quad u>1 \\ \left[v^a\ {}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ 1,\mu \end{pmatrix}\right]^{-1} & , \quad u=1 \end{cases}$$

Moreover, we have $\lim\limits_{x\to 0^+} f_2(x) = \infty \quad , u<1$ and $\lim\limits_{x\to\infty} f_2(x) = 0$. It can be easily shown that

$$\frac{d}{dx} f_2(x) = \left[\frac{u-1}{x} - \frac{\mu+u}{1+x} - \frac{ab}{vc}\ \frac{{}_r\tilde{F}\left(a+1,b+1;c+1;-\frac{x}{v}\right)}{{}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right)}\right] f_2(x)\,. \tag{48}$$

4.3. Some Statistical Functions

The aim of this section is to obtain some basic functions associated with the p.d.f $f_2(x)$, such as the population moments, the cumulative distribution function (c.d.f.), the survivor function, the hazard rate function and the mean residue life function. We start with

4.3.1. Population Moments

Now we derive several types of moments such as the k-th moment and the moment generating function. We begin by evaluating the k-th moment, since it will be used to obtain the remaining basic moments, such as the mean, the variance and the moment generating function.

The k-th Moment:

The k-th Moment about the origin of the random variable X_2 whose p.d.f

$f_2(x)$ given by the equation (47) is

$$E\left[X_2^k\right] = \frac{v^{-a}\int_0^\infty t^{k+u-1}\,(1+t)^{-\mu-u}\ {}_r\tilde{F}\left(a,b;c;-\frac{t}{v}\right)\,dt}{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}} = \frac{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u+k,\mu-k \end{pmatrix}}{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}} \tag{49}$$

In particular, the expected value of the random variable X_2, is

$$E\left[X_2\right] = \frac{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u+1,\mu-1 \end{pmatrix}}{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}}. \tag{50}$$

Similarly, we can obtain the variance of the random variable X_2 , $\sigma_{X_2}^2$, using the equation (49) with $k=2$, besides the equation (50).

Moment generating function:

The moment generating function of the random variable X_2 is

$$M_2(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\,E\left[X_2^k\right] = \sum_{k=0}^{\infty}\frac{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u+k,\mu-k \end{pmatrix}}{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}}\times\frac{t^k}{k!}.$$

4.3.2. The Distribution Function

The cumulative distribution function $F_2(x)$ of the random variable X_2 is given by

$$F_2(x) \triangleq P\left(X_2 \le x\right) = \frac{{}_r\tilde{B}_0^x\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}}{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}},$$

hence the survivor function $S_2(x)$ can be expressed as

$$S_2(x) = 1 - F_2(x) = \frac{{}_r\tilde{B}_x^\infty\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}}{{}_r\tilde{B}\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}}. \tag{51}$$

4.3.3. The Hazard Rate Function

Using the equations (49) and (51) , the hazard rate function for a p.d.f $f_2(x)$ is given by

$$h_2(x) = \frac{v^{-a}\, x^{u-1}\, (1+x)^{-\mu-u}\ \ {}_r\tilde{F}\left(a,b;c;-\frac{x}{v}\right)}{{}_r\tilde{B}_x^{\infty}\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}}. \tag{52}$$

4.3.4. The Mean Residue Life Function

Using (32), the mean residue life function for a random variable X_2 is given by

$$K_2(x) = \frac{{}_r\tilde{B}_x^{\infty}\begin{pmatrix} a,b;c;v \\ u+1,\mu-1 \end{pmatrix}}{{}_r\tilde{B}_x^{\infty}\begin{pmatrix} a,b;c;v \\ u,\mu \end{pmatrix}} - x.$$

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