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SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS

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Sufficient conditions for the existence of solutions of the periodic and anti-periodic boundary value problems for nonlinear functional difference equations are established, respectively.

1. Introduction

In this paper, we study the following boundary value problems for nonlinear functional difference equations

$$\begin{cases} \Delta^3 x(n) = f(n, x(n), x(n+1), x(n+2), x(n+3), \\ \quad x(n-\tau_1(n)), \dots, x(n-\tau_m(n)),\ n \in [0,T], \\ x(0) = x(T+1), \\ \Delta x(0) = \Delta x(T+1), \\ \Delta^2 x(0) = \Delta^2 x(T+1), \\ x(i) = \phi(i), \quad i \in [-\tau, -1], \\ x(i) = \psi(i), \ \ i \in [T+4, T+\delta], \end{cases} \tag{1}$$

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and

$$\begin{cases} \Delta^3 x(n) = f(n, x(n), x(n+1), x(n+2), x(n+3), \\ \quad x(n-\tau_1(n)), \ldots, x(n-\tau_m(n)),\ n \in [0,T], \\ x(0) = -x(T+1), \\ \Delta x(0) = -\Delta x(T+1), \\ \Delta^2 x(0) = -\Delta^2 x(T+1), \\ x(i) = \phi(i), \quad i \in [-\tau, -1], \\ x(i) = \psi(i), \quad i \in [T+4, T+\delta], \end{cases} \tag{2}$$

where $T \geq 1,\ \tau_i : [0,T] \to Z \setminus \{0,-1,-2,-3,\},\ i = 1,\ldots,m,\ [a,b] = \{a, a+1,\ldots,b\}$ for the integers a and b with $a \leq b$, $\Delta x(n) = x(n+1) - x(n)$, $\Delta^i x(n) = \Delta(\Delta x^{i-1} x(n))$, $f(n, y_1, y_2, y_3, x_1, \ldots, x_m)$ is continuous for each $n \in [0,T]$ with

$$\tau = -\min\left\{\min_{n\in[0,T]}\{n - \tau_i(n)\} : \ i = 1,\ldots,m\right\},$$

and

$$\delta = \max\left\{\max_{n\in[0,T]}\{n - \tau_i(n)\} : \ i = 1,\ldots,m\right\} - T.$$

Recently there has been a large number of authors paid attention to the existence solutions of boundary value problems for the differential equations that arise from various applied problems. Similarly there has been a parallel interest in results for the analogous discrete problems, see the papers [1-27] and the references therein.

Particular significance in these points lies in the fact that when a BVP is discretized, strange and interesting changes can occur in the solutions. For example, properties such as existence, uniqueness and multiplicity of solutions may not be shared between the continuous differential equation and its related discrete difference equation [28, p. 520]. Moreover, when investigating difference equations, as opposed to differential equations, basic ideas from calculus are not necessarily available to use, such as the intermediate value theorem, the mean value theorem and Rolle′s theorem. Thus, new challenges are faced and innovation is required [29].

In paper [1], the authors studied the anti-periodic boundary value problems for equations

$$\begin{cases} x'''(t) + f(x'(t))x''(t) + h(x(t)) = g(t, x(t), x'(t), x''(t)) + e(t),\ t \in [0,1], \\ x^{(i)}(0) = -x^{(i)}(1), \quad i = 0,1,2 \end{cases} \tag{3}$$

under the following assumptions:

i) f is a continuous even function;

ii) h is a continuous odd function;

iii) g is continuous on $[0,1]\times R^3$ satisfying Caratheodory′s conditions.

In [2], [3], the authors studied the existence of solutions of the following periodic boundary value problems or its special cases for third-order differential equations

$$\begin{cases} x'''(t)+f(x'(t))x''(t)+h(x(t))=g(t,x(t),x'(t),x''(t))+e(t),\ t\in[0,1], \\ x^{(i)}(0)=x^{(i)}(1),\ \ i=0,1,2. \end{cases} \tag{4}$$

We note that the discrete analogous of equation (3) is a special case of BVP(1). The discrete analogous of equation (4) is a special case of BVP(2).

In paper [4], the authors studied the existence of positive solutions of the boundary value problem of third order differential equation

$$y'''(t)+g(t,y(t))=0,\ t\in[a,b],\ \ a_{i,1,}y^{(i-1)}(a)=a_{i,2}y^{(i-1)}(b),\ i=1,2,3 \tag{5}$$

under the assumptions $\gamma_i=a_{i,1}-a_{i,2}>0$ for all $i=1,2,3$. We note that BVP(5) becomes a periodic boundary value problem when $\gamma_i=0(i=1,2,3)$, BVP(5) an anti-periodic boundary value problem when $a_{i,1}=-a_{i,2}(i=1,2,3)$. The methods used in [4] can not be applied to these cases. The discrete form of BVP(5) is as follows

$$\Delta x^3x(n)+g(n,x(n))=0,\ \ n\in[a,b],\ \ a_{i,1}\Delta x^{i-1}(a)=a_{i,2}\Delta x^{i-1}(b+1),$$

which is a spacial case of BVP(1) when $\gamma_i=0(i=1,2,,3)$, and BVP(2) when $a_{i,1}=-a_{i,2}(i=1,2,3)$.

Recent studies on the existence of positive solutions of boundary value problems of third-order difference equations have been made in [30-33] To the authors's knowledge, there has been few paper concerned with the solvability of BVP(1) and BVP(2). The purpose of this paper is to establish sufficient conditions for the existence of at least one solution of BVP(1) and BVP(2), respectively. Our methods, based upon the Mawhin′s coincidence degree theory, are different from those used in [5-26] and those in [1-3]. Our results are different from those ones obtained in [6,7,9-12,21,23,27].

This paper is organized as follows. In section 2, we give the main results, and in section 3, examples to illustrate the main results are presented.

2. Main Results

Let X and Y be Banach spaces, $L:\ \mathrm{Dom}\,L\subset X\to Y$ be a Fredholm operator of index zero, $P:\ X\to X$, $Q:\ Y\to Y$ be projectors such that

$$\mathrm{Im}\,P=\mathrm{Ker}\,L,\ \mathrm{Ker}\,Q=\mathrm{Im}\,L,\ \ X=\mathrm{Ker}\,L\oplus\mathrm{Ker}\,P,\ \ Y=\mathrm{Im}\,L\oplus\mathrm{Im}\,Q.$$

It follows that

$$L|_{\mathrm{Dom}\,L\cap \mathrm{Ker}\,P}:\ \mathrm{Dom}\,L\cap \mathrm{Ker}\,P\to \mathrm{Im}\,L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $\mathrm{Dom}\,L\cap\overline{\Omega}\neq\emptyset$, the map $N:\ X\to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\ \overline{\Omega}\to X$ is compact.

Lemma 2.1 (14)**.** . *Let L be a Fredholm operator of index zero and let N be $L-$compact on Ω. Assume that the following conditions are satisfied:*

i) $Lx\neq\lambda Nx$ for every $(x,\lambda)\in[(DomL\setminus KerL)\cap\partial\Omega]\times(0,1)$;

ii) $Nx\notin ImL$ for every $x\in KerL\cap\partial\Omega$;

iii) $deg(\wedge QN\big|_{KerL},\ \Omega\cap KerL,0)\neq 0$, where $\wedge:\ Y/ImL\to KerL$ is an isomorphism.

Then the equation $Lx=Nx$ has at least one solution in $DomL\cap\overline{\Omega}$.

Lemma 2.2 (14)**.** *Let X and Y be Banach spaces. Suppose $L: DomL\subset X\to Y$ is a Fredholm operator of index zero with $KerL=\{0\}$, $N: X\to Y$ is $L-$compact on any open bounded subset of X. If $0\in\Omega\subset X$ is a open bounded subset and $Lx\neq\lambda Nx$ for all $x\in DomL\cap\partial\Omega$ and $\lambda\in[0,1]$, then there is at least one $x\in\Omega$ so that $Lx=Nx$.*

Let $X=R^{T+\tau+\delta+1}$ be endowed with the norm $||x||_X=\max_{n\in[1,T+\tau+\delta+1]}|x(n)|$ for $x\in X$, $Y=R^{T+1}$ be endowed with the norm $||y||_Y=\max_{n\in[0,T]}|y(n)|$ for $y\in Y$. It is easy to see that X and Y are Banach spaces. Choose $\mathrm{Dom}\,L=$

$$=\left\{x\in X:\ \begin{array}{ll} x(i)=0,\ i\in[-\tau,\dots,-1], & x(0)=x(T+1)\\ x(i)\in R,\ \ i\in[0,T+3], & \Delta x(0)=\Delta x(T+1)\\ x(i)=0,\ \ i\in[T+4,\dots,T+\delta], & \Delta^2x(0)=\Delta^2x(T+1)\end{array}\right\}$$

Set

$$L:\mathrm{Dom}\,L\cap X\to X,\quad Lx(n)=\Delta^3x(n),\ \ n\in[0,T],$$

and $N: X\to Y$ by

$$Nx(n)=f(n,x(n)+x_0(n),x(n+1)+x_0(n+1),x(n+2)+x_0(n+2)+$$
$$+x_0(n+3)+x(n-\tau_1(n))+x_0(n-\tau_1(n)),\dots,x(n-\tau_m(n))+x_0(n-\tau_m(n)))$$

$n\in[0,T]$, for all $x\in X$, where

$$x_0(n)=\begin{cases}\phi(n), & n\in[-\tau,-1],\\ 0, & n\in[0,T+3],\\ \psi(n), & n\in[T+4,T+\delta].\end{cases}$$

It is easy to show that $\Delta^3 x_0(n)=0$ for $n\in[0,T]$ and that $x\in \mathrm{Dom}L$ is a solution of $Lx=Nx$ implies that $x+x_0$ is a solution of BVP(1).

It is easy to check the following results.

i) $\mathrm{Ker}L=\left\{x\in R^{T+\delta+\tau+1}:\ x(n)=\left\{\begin{array}{l}0,\ n\in[-\tau,\ldots,-1],\\ c,\ \ n\in[0,T+3],\, c\in R\\ 0,\ \ n\in[T+4,\ldots,T+\delta],\end{array}\right.\right\}.$

ii) $\mathrm{Im}L=\left\{\, y\in R^{T+1}:\quad \sum_{n=0}^{T} y(n)=0\right\}.$

iii) L is a Fredholm operator of index zero.

iv) There exist projectors $P:X\to X$ and $Q:Y\to Y$ such that $\mathrm{Ker}L=\mathrm{Im}P$, $\mathrm{Ker}Q=\mathrm{Im}L$. Furthermore, let $\Omega\subset X$ be an open bounded subset with $\overline{\Omega}\cap \mathrm{Dom}L\neq\emptyset$, then N is $L-$compact on $\overline{\Omega}$.

The projectors $P:X\to X$ and $Q:Y\to Y$, the isomorphism $\wedge:\mathrm{Ker}L\to Y/\mathrm{Im}L$ and the generalized inverse $K_p:\mathrm{Im}L\to \mathrm{Dom}L\cap \mathrm{Im}P$ are as follows:

$$
\begin{aligned}
(Px)(n) &= \begin{cases}0, & n\in[-\tau,-1],\\ x(0), & n\in[0,T+3],\\ 0, & n\in[T+4,T+\delta],\end{cases}\qquad \text{for } x\in X,\\
(Qy)(n) &= \frac{1}{T+1}\sum_{n=0}^{T} y(n),\quad n\in[0,T],\ \text{for } y\in Y,\\
\wedge(x_c) &= (\overbrace{c,\ldots,c}^{T+1}),\ \ \text{for } x_c=(\overbrace{0,\ldots,0}^{\tau},\overbrace{c,\ldots,c}^{T+4},\overbrace{0,\ldots,0}^{\delta-3})\in \mathrm{Ker}L,\\
(K_p y)(n) &= \begin{cases}0, & n\in[-\tau,-1],\\ \sum_{s=0}^{n-1}\sum_{j=0}^{s-1}\sum_{k=0}^{j-1}y(k) & \\ \quad -\frac{n-1}{T+1}\Big(\sum_{s=0}^{T}\sum_{j=0}^{s-1}\sum_{k=0}^{j-1}y(k) & \\ \quad\quad -\frac{T+1}{2}\sum_{s=0}^{T}\sum_{j=0}^{s-1}y(j)\Big) & n\in[0,T+3],\qquad y\in Y.\\ \quad\quad\quad -\frac{T}{2}\sum_{s=0}^{T}\sum_{j=0}^{s-1}y(j), & \\ 0, & n\in[T+4,T+\delta],\end{cases}
\end{aligned}
$$

Suppose the followings:

(B) let

$$
x_{\tau_i,c,0}(n)=\begin{cases}\phi(n-\tau_i(n)), & n-\tau_i(n)\in[-\tau,-1],\\ \psi(n-\tau_i(n)), & n-\tau_i(n)\in[T+4,T+\delta],\\ c, & n-\tau_i(n)\in[0,T+3].\end{cases}
$$

There exists a constant $M > 0$ such that

$$c\left[\sum_{n=0}^{T} f(n,c,c,c,c,x_{\tau_1,c,0}(n),\ldots,x_{\tau_m,c,0}(n))\right] > 0$$

for all $|c| > M$ or

$$c\left[\sum_{n=0}^{T} f(n,c,c,c,c,x_{\tau_1,c,0}(n),\ldots,x_{\tau_m,c,0}(n))\right] < 0$$

for all $|c| > M$.
(C_1) There exist numbers $\beta > 0$, $\theta > 1$, nonnegative sequences $p_i(n)$ ($i = 1,2,3,4$), $q_i(n)$ ($i = 1,\ldots,m$), $r(n)$, functions $g(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m)$, and $h(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m)$ such that

$$f(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m) =$$

$$= g(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m) + h(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m)$$

and

$$g(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m)y_2 \le -\beta|y_2|^{\theta+1},$$

and

$$|h(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m)| \le \sum_{i=1}^{4} p_i(n)|y_i|^{\theta} + \sum_{s=1}^{m} q_i(n)|x_i|^{\theta} + r(n),$$

for all $n \in \{0,\ldots,T\}$, $(y_1,y_2,y_3,y_4,x_1,\ldots,x_m) \in R^{m+4}$.
(C_2) There exist numbers $\beta > 0$, $\theta > 1$, nonnegative sequences $p_i(n)$ ($i = 1,2,3,4$), $q_i(n)$ ($i = 1,\ldots,m$), $r(n)$, functions $g(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m)$ and $h(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m)$ such that (6) holds and

$$g(n,y_1,y_2,y_3,y_4,x_1,x_1,\ldots,x_m)y_3 \ge \beta|y_3|^{\theta+1},$$

and

$$|h(n,y_1,y_2,y_3,y_4,x_1,\ldots,x_m)| \le \sum_{I=1}^{4} p_i(n)|y_i|^{\theta} + \sum_{s=1}^{m} q_i(n)|x_i|^{\theta} + r(n),$$

for all $n \in \{0,\ldots,T\}$, $(y_1,y_2,y_3,y_4,x_1,x_1,\ldots,x_m) \in R^{m+4}$.

Theorem 2.3. *Suppose that (B) and (C_1) hold. Then BVP(1) has at least one solution if*

$$\sum_{i=1}^{4} ||p_i|| + (T+1)^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||q_i|| < \beta. \tag{6}$$

Proof. To apply Lemma 2.1, we will construct an open bounded subset Ω of X such that (i), (ii) and (iii) in Lemma 2.1 hold. So the proof is divided into four steps.

Step 1. Let

$$\Omega_1 = \{x : Lx = \lambda Nx,\ (x,\lambda) \in [(\mathrm{Dom}L \setminus \mathrm{Ker}L)] \times (0,1)\},$$

we prove that Ω_1 is bounded. For $x \in \Omega_1$, we have $Lx = \lambda Nx$, $\lambda \in (0,1)$, so

$$\Delta^3 y(n) = \lambda f(n, y(n), y(n+1), y(n+2), y(n+3), y(n-\tau_1(n)), \ldots, y(n-\tau_m(n))), \tag{7}$$

where $y(n) = x(n) + x_0(n)$. We get that $[\Delta^3 y(n)]y(n+1) =$

$$= \lambda f(n, y(n), y(n+1), y(n+2), y(n+3), y(n-\tau_1(n)), \ldots, y(n-\tau_m(n))y(n+1).$$

Since $x(0) = x(T+1), \Delta x(0) = \Delta x(T+1), \Delta^2 x(0) = \Delta^2 x(T+1)$, we get $y(0) = y(T+1), \Delta y(0) = \Delta y(T+1), \Delta^2 y(0) = \Delta^2 y(T+1)$. Then

$$\begin{aligned}
&\sum_{n=0}^{T} [\Delta^3 y(n)]y(n+1) = \sum_{n=0}^{T} [\Delta^2 y(n+1) - \Delta^2 y(n)][y(n+2) - \Delta y(n+1)] \\
= &\ -\frac{1}{2}\left[-\sum_{n=0}^{T} (\Delta y(n+2) - \Delta y(n+1))^2 - [\Delta y(1)]^2 + [\Delta y(T+2)]^2\right] \\
= &\ \frac{1}{2}\sum_{n=0}^{T} (\Delta y(n+2) - \Delta y(n+1))^2 \geq 0.
\end{aligned}$$

So

$$\sum_{n=0}^{T} f(n, y(n), y(n+1), y(n+2), y(n+3), y(n-\tau_1(n)), \ldots$$

$$\ldots, y(n-\tau_m(n)))y(n+1) \geq 0.$$

It follows from (C_1) that

$$\begin{aligned}
&\beta\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\leq\\
\leq\ & -\sum_{n=0}^{T}g(n,y(n),y(n+1),y(n+2),y(n+3),y(n-\tau_1(n)),\dots\\
&\dots,y(n-\tau_m(n))y(n+1)\leq\\
\leq\ & \sum_{n=0}^{T}h(n,y(n),y(n+1),y(n+2),y(n+3),y(n-\tau_1(n)),\dots\\
&\dots,y(n-\tau_m(n))y(n+1)\leq\\
\leq\ & \sum_{n=0}^{T}|h(n,y(n),y(n+1),y(n+2),y(n+3),y(n-\tau_1(n)),\dots\\
&\dots,y(n-\tau_m(n))|\,|y(n+1)|\leq\\
\leq\ & \sum_{n=0}^{T}\sum_{i=1}^{4}p_i(n)|y(n+1)||y(n+i-1)|^{\theta}+\\
&+\sum_{i=1}^{m}\sum_{n=0}^{T}q_i(n)|y(n-\tau_i(n))|^{\theta}|y(n+1)|+\sum_{n=0}^{T}r(n)|y(n+1)|\leq\\
\leq\ & \sum_{i=1}^{4}||p_i||\sum_{n=0}^{T}|y(n+1)||y(n+i-1)|^{\theta}+\\
&+\sum_{i=1}^{m}||q_i||\sum_{n=0}^{T}|y(n-\tau_i(n))|^{\theta}|y(n+1)|+||r||\sum_{n=0}^{T}|y(n+1)|.
\end{aligned}$$

Hence by Holder's inequality, we get

$$\begin{aligned}
&\beta\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\leq\left[\sum_{i=1}^{4}||p_i||\right]\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\\
&+\left[\sum_{i=1}^{m}||q_i||\left(\tau\sum_{u+1=-\tau}^{-1}|\phi(u+1)|^{\theta+1}+|\delta-3|\sum_{u+1=T+4}^{T+\delta}|\psi(u+1)|^{\theta+1}\right.\right.\\
&\left.\left.+(T+1)\sum_{u=0}^{T}|y(u+1)|^{\theta+1}\right)^{\frac{\theta}{\theta+1}}\right]\left[\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}\\
&+||r||(T+1)^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}.
\end{aligned}$$

Since $\lim_{x\to 0^+}\frac{(1+x)^y-1}{(1+y)x}=\frac{y}{1+y}<1$ for $y>0$, then there is $\sigma>0$ such that $(1+x)^y\leq 1+(1+y)x$ for $0\leq x\leq\sigma$. We consider two cases.

Case 1.

$$\sum_{n=0}^{T}|y(n+1)|^{\theta+1} \leq$$

$$\frac{\tau\sum_{u+1=-\tau}^{-1}|\phi(u+1)|^{\theta+1}+|\delta-3|\sum_{u+1=T+4}^{T+\delta}|\psi(u+1)|^{\theta+1}}{(T+1)\sigma}=:Q.$$

In this case, we get

$$\beta\sum_{n=0}^{T}|y(n+1)|^{\theta+1} \leq \left[\sum_{i=1}^{4}||p_i||\right]\sum_{n=0}^{T}|y(n+1)|^{\theta+1}$$
$$+\left[\sum_{i=1}^{m}||q_i||\left((T+1)\sigma Q+(T+1)Q\right)^{\frac{\theta}{\theta+1}}\right]\left[\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}$$
$$+||r||(T+1)^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}.$$

It follows that

$$\left[\beta-\sum_{i=1}^{4}||p_i||\right]\sum_{n=0}^{T}|y(n+1)|^{\theta+1}$$
$$\leq \left[\sum_{i=1}^{m}||q_i||(T+1)^{\frac{\theta}{\theta+1}}\left[(\sigma+1)Q\right]^{\frac{\theta}{\theta+1}}\right]\left[\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}$$
$$+||r||(T+1)^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}.$$

From (6), there is $M_1>0$ such that $\sum_{u=0}^{T}|y(u+1)|^{\theta+1}\leq M_1$.

Case 2.

$$\sum_{n=0}^{T}|y(n+1)|^{\theta+1} >$$

$$> \frac{\tau\sum_{u+1=-\tau}^{-1}|\phi(u+1)|^{\theta+1}+|\delta-3|\sum_{u+1=T+4}^{T+\delta}|\psi(u+1)|^{\theta+1}}{(T+1)\sigma}=:Q.$$

In this case, we get

$$0<\frac{\tau\sum_{u+1=-\tau}^{-1}|\phi(u+1)|^{\theta+1}+|\delta-3|\sum_{u+1=T+4}^{T+\delta}|\psi(u+1)|^{\theta+1}}{(2T+2)\sum_{u=0}^{T}|y(u+1)|^{\theta+1}}<\sigma.$$

Thus

$$\beta\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\le$$

$$\le\left[\sum_{i=1}^{4}||p_i||\right]\sum_{n=0}^{T}|y(n+1)|^{\theta+1}+\left[(T+1)^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||q_i||\times\left(1+\left(1+\frac{\theta}{\theta+1}\right)\times\right.\right.$$
$$\left.\left.\frac{\tau\sum_{u+1=-\tau}^{-1}|\phi(u+1)|^{\theta+1}+|\delta-3|\sum_{u+1=T+4}^{T+\delta}|\psi(u+1)|^{\theta+1}}{(T+1)\sum_{u=0}^{T}|y(u+1)|^{\theta+1}}\right)\right]\times$$
$$\times\sum_{n=0}^{T}|y(n+1)|^{\theta+1}+||r||(T+1)^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}=$$
$$=\left[\sum_{i=1}^{4}||p_i||\right]\sum_{n=0}^{T}|y(n+1)|^{\theta+1}+(T+1)^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||q_i||\sum_{n=0}^{T}|y(n+1)|^{\theta+1}+$$
$$+(T+1)^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||q_i||\left(1+\frac{\theta}{\theta+1}\right)\sigma Q+||r||(T+1)^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}.$$

We get

$$\left[\beta-\sum_{i=1}^{4}||p_i||-(T+1)^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||q_i||\right]\sum_{u=0}^{T}|y(u+1)|^{\theta+1}$$
$$\le(T+1)^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||q_i||\left(1+\frac{\theta}{\theta+1}\right)\sigma Q+$$
$$+||r||(T+1)^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}.$$

It follows from (6) that there is $M_1>0$ such that $\sum_{u=0}^{T}|y(u+1)|^{\theta+1}\le M_1$.

Hence $|y(n+1)|\le M_1^{1/(\theta+1)}$ for all $n\in\{0,\dots,T\}$ in each cases. Thus we get

$$|x(n+1)|\le|y(n+1)|+|x_0(n+1)|\le M_1^{1/(\theta+1)}+||x_0||_X,\ \ n\in[0,\dots,T].$$

Hence $||x||\le M_1^{1/(\theta+1)}+||x_0||$. So Ω_1 is bounded. This completes the Step 1.

Step 2. Prove that the set $\Omega_2=\{x\in\mathrm{Ker}L:\ Nx\in\mathrm{Im}L\}$ is bounded.

For $x\in\mathrm{Ker}L$, we have $x(n)=(\overbrace{0,\dots,0}^{\tau},\overbrace{c,\dots,c}^{T+4},\overbrace{0,\dots,0}^{\delta-3})$. Thus, for $n=$

$0,\dots,T$, we have

$$\begin{aligned} Nx(n) &= f(n,x(n)+x_0(n),x(n+1)+x_0(n+1),x(n+2)+x_0(n+2),\\ &\quad \dots,x(n-\tau_1(n))+x_0(n-\tau_1(n)),\dots,x(n-\tau_m(n))+x_0(n-\tau_m(n)))\\ &= f(n,c,c,c,c,x_{\tau_1,c,0},\dots,x_{\tau_m,c,0}), \end{aligned}$$

where

$$x_{\tau_i,c,0} = \begin{cases} \phi(n-\tau_i(n)), & n-\tau_i(n)\in[-\tau,-1],\\ \psi(n-\tau_i(n)), & n-\tau_i(n)\in[T+4,T+\delta],\\ c, & n-\tau_i(n)\in[0,T+3]. \end{cases}$$

$Nx \in \mathrm{Im}L$ implies that

$$\sum_{n=0}^{T-1} f(n,c,c,c,c,x_{\tau_1,c,0},\dots,x_{\tau_m,c,0}) = 0.$$

It follows from condition (B) that $|c|\le M$. Thus Ω_2 is bounded.

Step 3. Prove the set $\Omega_3 = \{x\in \mathrm{Ker}L:\ \pm\lambda\wedge x+(1-\lambda)QNx=0,\ \lambda\in[0,1]\}$ is bounded.

If the first inequality in (B) holds, let

$$\Omega_3 = \{x\in \mathrm{Ker}L:\ \lambda\wedge x+(1-\lambda)QNx=0,\ \lambda\in[0,1]\}.$$

We will prove that Ω_3 is bounded. For $x(n) = (\overbrace{0,\dots,0}^{\tau},\overbrace{c,\dots,c}^{T+4},\overbrace{0,\dots,0}^{\delta-3}) \in \Omega_3$, and $\lambda\in[0,1]$, we have

$$-(1-\lambda)\sum_{n=0}^{T} f(n,c,c,c,c,x_{\tau_1,c,0},\dots,x_{\tau_m,c,0}) = \lambda cT.$$

If $\lambda = 1$, then $c = 0$. If $\lambda\ne 1$ and $|c|>M$, then

$$0\ge -(1-\lambda)c\sum_{n=0}^{T} f(n,c,c,c,c,x_{\tau_1,c,0},\dots,x_{\tau_m,c,0}) = \lambda c^2T>0,$$

from (B), a contradiction. So $|c|\le M$.

If the second inequality in (B) holds, let

$$\Omega_3 = \{x\in \mathrm{Ker}L:\ -\lambda\wedge x+(1-\lambda)QNx=0,\ \lambda\in[0,1]\},$$

Similarly, we can get a contradiction. So $|c|\le M$. Hence Ω_3 is bounded.

Step 4. Obtain open bounded set Ω such that (i), (ii) and (iii) of Lemma 2.1 hold.

Set Ω be a open bounded subset of X such that $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i}$. We know that L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. By the definition of Ω, we have $\Omega \supset \overline{\Omega_1}$ and $\Omega \supset \overline{\Omega_2}$, thus $Lx \neq \lambda Nx$ for $x \in (\mathrm{Dom}L/\mathrm{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $Nx \notin \mathrm{Im}L$ for $x \in \mathrm{Ker}L \cap \partial\Omega$.

In fact, let $H(x,\lambda) = \pm\lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $\Omega \supset \overline{\Omega_3}$, thus $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \mathrm{Ker}L$, thus by homotopy property of degree,

$$\begin{aligned} & \deg(QN|\mathrm{Ker}L, \Omega \cap \mathrm{Ker}L, 0) = \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0) \\ = \ & \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0) = \deg(\pm\wedge, \Omega \cap \mathrm{Ker}L, 0) \neq 0. \end{aligned}$$

Thus by Lemma 2.1, $Ly = Ny$ has at least one solution in $\mathrm{Dom}L \cap \overline{\Omega}$, which is a solution of BVP(1). The proof is completed. □

Theorem 2.4. *Suppose that (B) and (C_2) hold. Then BVP(1) has at least one solution if (7) holds.*

Proof. The proof of this theorem is divided into four steps.
Step 1. Let

$$\Omega_1 = \{x : Lx = \lambda Nx,\ (x,\lambda) \in [(\mathrm{Dom}L \setminus \mathrm{Ker}L)] \times (0,1)\}.$$

For $x \in \Omega_1$, we have $Lx = \lambda Nx$, $\lambda \in (0,1)$, so

$$\Delta^3 y(n) = \lambda f(n, y(n), y(n+1), y(n+2), y(n+3), y(n-\tau_1(n)), \dots, y(n-\tau_m(n))), \tag{8}$$

where $y(n) = x(n) + x_0(n)$. Then

$$[\Delta^2 y(n)]y(n+2) =$$

$$= \lambda f(n, y(n), y(n+1), y(n+2), y(n+3), y(n-\tau_1(n)), \dots, y(n-\tau_m(n))y(n+2).$$

Since $y(0)=-y(T+1), \Delta y(0)=-\Delta y(T+1), \Delta^2 y(0)=-\Delta^2 y(T+1)$, we get

$$\begin{aligned}
&\sum_{n=0}^{T}[\Delta^3 y(n)]y(n+2)=\sum_{n=0}^{T}[\Delta^2 y(n+1)-\Delta^2 y(n)][y(n+3)-\Delta y(n+2)]\\
=\ & \sum_{n=0}^{T}[(\Delta^2 y(n+1))y(n+3)-(\Delta^2 y(n))y(n+2)-\Delta^2 y(n+1)\Delta y(n+2)]\\
=\ & (\Delta^2 y(T+1))y(T+3)-(\Delta^2 y(0))y(2)-\sum_{n=0}^{T}\Delta^2 y(n+1)\Delta y(n+2)\\
=\ & -\sum_{n=0}^{T}\Delta^2 y(n+1)\Delta y(n+2)\\
=\ & -\sum_{n=0}^{T}\big((\Delta y(n+2))^2-(\Delta y(n+1))(\Delta y(n+2))\big)\\
=\ & -\frac{1}{2}\left[\sum_{n=0}^{T}(\Delta y(n+2)-\Delta y(n+1))^2-[\Delta y(1)]^2+[\Delta y(T+2)]^2\right]\\
=\ & -\frac{1}{2}\sum_{n=0}^{T}(\Delta y(n+2)-\Delta y(n+1))^2\\
\leq\ & 0.
\end{aligned}$$

So, we get $0\geq$

$$\sum_{n=0}^{T-1} f(n,y(n),y(n+1),y(n+2),y(n+3),y(n-\tau_1(n)),\dots,y(n-\tau_m(n))y(n+2).$$

The proof of the remainder steps is just similar to those of the proof of Theorem 2.3 and is omitted. $\square$

For BVP(2), choose $\mathrm{Dom}L=$

$$\left\{x\in X:\ \begin{array}{ll} x(i)=0,\ i\in[-\tau,\dots,-1], & x(0)=-x(T+1)\\ x(i)\in R,\quad i\in[0,T+3], & \Delta x(0)=-\Delta x(T+1)\\ x(i)=0,\ \ i\in[T+4,\dots,T+\delta], & \Delta^2 x(0)=-\Delta^2 x(T+1)\end{array}\right\}.$$

Set

$$L:\mathrm{Dom}L\cap X\to X,\quad Lx(n)=\Delta^3 x(n),\ \ n\in[0,T],$$

and $N:X\to Y$ by

$$\begin{aligned}
Nx(n)=f(n,x(n)+x_0(n),x(n+1)+x_0(n+1),x(n+2)+x_0(n+2),&\\
x(n+3)+x_0(n+3),x(n-\tau_1(n))+x_0(n-\tau_1(n)),\dots&\\
\dots,x(n-\tau_m(n))+x_0(n-\tau_m(n)))&
\end{aligned}$$

$n \in [0,T]$, for all $x \in X$, where

$$x_0(n) = \begin{cases} \phi(n), & n \in [-\tau,-1], \\ 0, & n \in [0,T+3], \\ \psi(n), & n \in [T+4,T+\delta]. \end{cases}$$

It is easy to show that $\Delta^3 x_0(n) = 0$ for $n \in [0,T]$ and that $x \in \mathrm{Dom}L$ is a solution of $Lx = Nx$ implies that $x + x_0$ is a solution of BVP(2) and

i) $\mathrm{Ker}L = \{(0,\ldots,0) \in X\}$.

ii) L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$ with Ω being an open bounded nonempty subset of X.

Theorem 2.5. *Suppose that (C_1) holds. Then BVP(2) has at least one solution if (6) holds.*

Proof. Let $\Omega_1 = \{x : Lx = \lambda Nx,\ (x,\lambda) \in (\mathrm{Dom}L) \times (0,1)\}$. For $x \in \Omega_1$, we have $Lx = \lambda Nx$, $\lambda \in (0,1)$, so

$$\Delta^3 y(n) = \lambda f(n,y(n),y(n+1),y(n+2),y(n+3),y(n-\tau_1(n)),\ldots,y(n-\tau_m(n))), \tag{9}$$

where $y(n) = x(n) + x_0(n)$. Then $[\Delta^3 y(n)]y(n+1) =$

$$= \lambda f(n,y(n),y(n+1),y(n+2),y(n+3),y(n-\tau_1(n)),\ldots,y(n-\tau_m(n))y(n+1).$$

As in the proof of Theorem 2.3, we get $0 \le$

$$\sum_{n=0}^{T-1} f(n,y(n),y(n+1),y(n+2),y(n+3),y(n-\tau_1(n)),\ldots,y(n-\tau_m(n))y(n+1).$$

Similar to that of proof of Step 1 in the proof of Theorem 2.3, we can prove that Ω_1 is bounded.

Let $\Omega \supset \overline{\Omega_1}$ be an open bounded subset of X, it is easy to see that $Lx \neq \lambda Nx$ for all $x \in \mathrm{Dom}L \cap \partial\Omega$ and $\lambda \in [0,1]$.

Thus by Lemma 2.2, $Lx = Nx$ has at least one solution in $\mathrm{Dom}L \cap \overline{\Omega}$, so $x + x_0$ is a solution of BVP(2). The proof is completed. □

Theorem 2.6. *Suppose that (C_2) holds. Then BVP(2) has at least one solution if (6) holds.*

Proof. Let $\Omega_1 = \{x : Lx = \lambda Nx,\ (x,\lambda) \in (\mathrm{Dom}L) \times (0,1)\}$. For $x \in \Omega_1$, we have $Lx = \lambda Nx$, $\lambda \in (0,1)$, so

$$\Delta^3 y(n) = \lambda f(n,y(n),y(n+1),y(n-\tau_1(n)),\ldots,y(n-\tau_m(n))), \tag{10}$$

where $y(n) = x(n) + x_0(n)$. Then

$$[\Delta^3 y(n)]y(n+2) = \lambda f(n, y(n), y(n+1), y(n+2), y(n+3), y(n-\tau_1(n)), \ldots$$
$$\ldots, y(n-\tau_m(n))y(n+2).$$

As in the proof of Theorem 2.4, we get $0 \geq$

$$\sum_{n=0}^{T-1} f(n, y(n), y(n+1), y(n+2), y(n+3), y(n-\tau_1(n)), \ldots, y(n-\tau_m(n))y(n+2).$$

The remainder of the proof is just similar to that of the step 1 of the proof of Theorem 2.3, we can get that Ω_1 is bounded.

Let $\Omega \supset \overline{\Omega_1}$ be an open bounded subset of X, it is easy to see that $Lx \neq \lambda Nx$ for all $x \in \mathrm{Dom}L \cap \partial\Omega$ and $\lambda \in [0,1]$.

Thus by Lemma 2.2, $Lx = Nx$ has at least one solution in $\mathrm{Dom}L \cap \overline{\Omega}$, so $x + x_0$ is a solution of BVP(2). The proof is completed. $\square$

3. Examples

In this section, we present some examples to illustrate the main results in section 2.

Example 3.1. Consider the following problem

$$\begin{cases} \Delta^3 x(n) = p_1(n)[x(n)]^{2k+1} + p_2(n)[x(n+1)]^{2k+1} + \beta[x(n+2)]^{2k+1} \\ \qquad +p_4(n)[x(n+3)]^{2k+1} + \sum_{i=1}^m q_i(n+1)[x(n-T-3)]^{2k+1} \\ \qquad +\sum_{i=1}^m r_i(n+1)[x(n+T+4)]^{2k+1} + r(n), \\ x(0) = x(T+1), \\ \Delta x(0) = \Delta x(T+1), \\ \Delta^2 x(0) = \Delta^2 x(T+1), \\ x(i) = \phi(i), \quad i \in [-(T+3), -1], \\ x(i) = \psi(i), \quad i \in [T+4, 2T+4], \end{cases} \tag{11}$$

where $k \geq 0$ an integer, $\beta > 0$, $p_1(n), p_2(n), p_4(n), q_i(n)(i = 1, \ldots, m), r(n), \tau_i(n)$ are sequences. Corresponding to BVP(1), we find

$$f(n, y_1, y_2, y_3, x_1, \ldots, x_m) = p_1(n)y_1^{2k+1} + p_2(n)y_2^{2k+1} + \beta y_3^{2k+1} +$$
$$+p_4(n)y_4^{2k+1} + \sum_{i=1}^m q_i(n)x_i^{2k+1} + \sum_{i=m+1}^{2m} r_{i-m}(n)x_i^{2k+1} + r(n),$$
$$g(n, y_1, y_2, y_3, x_1, \ldots, x_m) = \beta y_3^{2k+1},$$
$$h(n, y_1, y_2, y_3, x_1, \ldots, x_m) = p_1(n)y_1^{2k+1} + p_2(n)y_2^{2k+1} + p_4(n)y_4^{2k+1} +$$
$$+\sum_{i=1}^m p_i(n)x_i^{2k+1} + \sum_{i=m+1}^{2m} r_{i-m}(n)x_i^{2k+1} + r(n).$$

It follows, for $n \in [0,T]$, that

$$c\left[\sum_{n=0}^{T} f(n,c,c,c,c,x_{\tau_1,c,0}(n),\dots,x_{\tau_m,c,0}(n))\right] =$$

$$= c\left[c^{2k+1}\sum_{n=0}^{T}(p_1(n)+p_2(n)+\beta+p_4(n)) + \sum_{n=0}^{T}\left(\sum_{i=1}^{m} p_i(n)\phi(n-T-3)\right.\right.$$

$$\left.\left.\sum_{n=0}^{T}\sum_{i=m+1}^{2m} r_{i-m}(n)\psi(n+T+4)+r(n)\right)\right] =$$

$$= c^{2k+2}\left[\sum_{n=0}^{T}(p_1(n)+p_2(n)+p_4(n)) + (T+1)\beta\right] +$$

$$+c\left[\sum_{n=0}^{T}\left(\sum_{i=1}^{m} p_i(n)\phi(n-T-3)\sum_{n=0}^{T}\sum_{i=m+1}^{2m} r_{i-m}(n)\psi(n+T+4)+r(n)\right)\right].$$

It is easy to see from Theorem L2 that BVP(11) has at least one solution for every $r(n)$ if

$$||p_1||_Y + ||p_2||_Y + ||p_4||_Y + (2T+2)^{\frac{2k+1}{2k+2}}\sum_{i=1}^{m}||q_i||_Y + (2T+2)^{\frac{2k+1}{2k+2}}\sum_{i=1}^{m}||r_i||_Y < \beta$$

and either

$$\sum_{n=0}^{T}(p_1(n)+p_2(n)+p_4(n)) > -(T+1)\beta$$

or

$$\sum_{n=0}^{T}(p_1(n)+p_2(n)+p_4(n)) < -(T+1)\beta.$$

Example 3.2. Consider the following problem

$$\begin{cases} \Delta x(n) = p_1(n)[x(n)]^{2k+1} + \beta[x(n+1)]^{2k+1} + p_3(n)[x(n+2)]^{2k+1} \\ +p_4(n)[x(n+3)]^{2k+1} + \sum_{i=1}^{m} p_i(n+1)[x(n-\tau_i(n))]^{2k+1} + r(n), \\ x(0) = -x(T+1), \\ \Delta x(0) = -\Delta x(T+1), \\ \Delta^2 x(0) = -\Delta^2 x(T+1), \\ x(i) = \phi(i), \quad i \in [-\tau,-1], \\ x(i) = \psi(i), \quad i \in [T+2,\delta], \end{cases} \tag{12}$$

where $k \geq 0$ an integer, $\beta < 0$, $p_1(n), p_3(n), p_4(n), q_i(n)(i=1,\dots,m), r(n), \tau_i(n)$

are sequences. Corresponding to BVP(2), we find

$$\begin{aligned} f(n,y_1,y_2,y_3,x_1,\ldots,x_m) &= p_1(n)y_1^{2k+1} + \beta y_2^{2k+1} + p_3(n)y_3^{2k+1} + \\ &\quad + p_4(n)y_4^{2k+1} + \sum_{i=1}^{m} p_i(n)x_i^{2m+1} + r(n), \\ g(n,y_1,y_2,y_3,x_1,\ldots,x_m) &= \beta y_3^{2k+1}, \\ h(n,y_1,y_2,y_3,x_1,\ldots,x_m) &= p_1(n)y_1^{2k+1} + p_3(n)y_3^{2k+1} + p_4(n)y_4^{2k+1} + \\ &\quad + \sum_{i=1}^{m} p_i(n)x_i^{2m+1} + r(n). \end{aligned}$$

It is easy to see from Theorem L3 that BVP(12) has at least one solution for every $r(n)$ if

$$||p_1||_Y + ||p_3||_Y + ||p_4||_Y + (2T+2)^{\frac{2k+1}{2k+2}} \sum_{i=1}^{m} ||q_i||_Y < -\beta.$$

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