LE MATEMATICHE
Vol. LXV (2010) – Fasc. I, pp. 73–82
doi: 10.4418/2010.65.1.5

POLYNOMIALS EXPANSIONS FOR SOLUTION OF WAVE EQUATION IN QUANTUM CALCULUS

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In this paper, using the q^2-Laplace transform early introduced by Abdi [1], we study q-Wave polynomials related with the q-difference operator $\Delta_{q,x}$. We show in particular that they are linked to the q-little Jacobi polynomials $p_n(x;\alpha,\beta \mid q^2)$.

1. Introduction and preliminaries

In a recent paper [6], the authors have shown that the solutions of certain q-elliptic problem can be expressed in terms of solutions of a parabolic problem by means of the inverse q^2-Laplace transform.

In this paper, our interest is to obtain series representations of solutions of a q-Wave problem. The initial data in these cases is taken to be analytic, and the representations sets of polynomials involve the q-Laguerre polynomials and q-little Jacobi polynomials. These polynomials are obtainable from the q-Heat polynomials studied by A. Fitouhi and F. Bouzeffour [3] by the use of the inverse q^2-Laplace transform. We also study the series representations of solutions of the q-Wave problem concerning the q-difference operator $\Delta_{q,x}$.

Throughout this paper, we fix $q \in]0,1[$ and suppose that $\log(1-q^2)/\log q^2 \in \mathbb{N}$. We recall some usual notions and notations used in the q-theory.

Entrato in redazione: 9 novembre 2009

AMS 2000 Subject Classification: 33D60, 26D15, 33D05, 33D15, 33D90.
Keywords: Quantum calculus, Wave polynomial, q-analysis, q-Integral Transform.

The q-shifted factorials are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{1}$$

and more generally:

$$(a_1,\cdots,a_r;q)_n = \prod_{k=1}^{r}(a_k;q)_n. \tag{2}$$

A basic hypergeometric series is

$${}_r\varphi_s(a_1,\cdots,a_r;b_1,\cdots,b_s;q,x) = \sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(b_1,\cdots,b_s;q)_n(q;q)_n}[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r}x^n.$$

A function f is said to be q-regular at zero [2] if $\lim_{n\to\infty} f(xq^n) = f(0)$ exists and does not depend of x. The q-derivative $D_q f$ [9] of a function f is defined by:

$$D_{q,x}f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad x \neq 0. \tag{3}$$

The q-derivative at zero [2] is defined by

$$D_{q,x}f(0) = \lim_{n\to+\infty}\frac{f(xq^n)-f(0)}{xq^n}, \tag{4}$$

where the limit exists and independent of x.

For $n \in \mathbb{N}$,

$$D^n_{q,x}f(x) = \frac{(-1)^n}{x^n(1-q)^n}\sum_{k=0}^{n}(-1)^k\frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}q^{-(n-k)(n-k-1)/2}f(q^{n-k}x) \tag{5}$$

The q-analogue of $(a+b)^n$ is a non commutative term $(a+b)^n_q$ given by

$$(a+b)^n_q = \begin{cases} a^n(-\frac{b}{a};q)_n, & a \neq 0 \\ q^{n(n-1)/2}b^n, & a = 0. \end{cases} \tag{6}$$

It is clear that $(a+b)^n_q$ and $(b+a)^n_q$ are not always the same.

Some q-functional spaces will be used to establish our result. We begin by putting

$$\mathbb{R}_q = \{\pm q^k, k\in\mathbb{Z}\}\cup\{0\}, \quad \mathbb{R}_{q,+} = \{+q^k, k\in\mathbb{Z}\} \tag{7}$$

and we define $\mathscr{E}_{q,*}(\mathbb{R}_q)$ the space of even functions infinitely q-differentiable at zero.

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}. \tag{8}$$

The q-shift operators are

$$(\Lambda_{q,x} f)(x) = f(qx), \qquad (\Lambda_{q,x}^{-1} f)(x) = \Lambda_{q^{-1},x} f(x). \tag{9}$$

We consider the q-difference operator

$$\Delta_{q,x} = \Lambda_{q,x}^{-1} D_{q,x}^2. \tag{10}$$

Koornwinder and Swarttouw introduced q-trigonometric function denoted in [10] by $\cos(x;q^2)$ and $\sin(x;q^2)$, we have in particular:

$$\cos(x;q^2) = {}_1\varphi_1(0,q,q^2;(1-q)^2x^2) = \sum_{n=0}^{\infty}(-1)^n b_n(x;q^2) \tag{11}$$

where we have put

$$b_n(x;q^2) = b_n(1;q^2)x^{2n} = q^{n(n-1)}\frac{(1-q)^{2n}}{(q;q)_{2n}}x^{2n}. \tag{12}$$

More generally the normalized q-Bessel function [4] is given by

$$j_\alpha(x;q^2) = \Gamma_{q^2}(\alpha+1)q^{n(n-1)}\frac{q^\alpha(1+q)^\alpha}{x^\alpha}J_\alpha((1-q)x;q^2) \tag{13}$$

$$= \sum_{n=0}^{\infty}(-1)^n b_{n,\alpha}(x,q^2) \tag{14}$$

where $J_\alpha(x;q^2)$ is the Hahn Exton q-Bessel function [12] and

$$b_{n,\alpha}(x,q^2) = b_{n,\alpha}(1,q^2)x^{2n} = \frac{\Gamma_{q^2}(\alpha+1)q^{n(n-1)}}{(1+q)^{2n}\Gamma_{q^2}(n+1)\Gamma_{q^2}(\alpha+n+1)}x^{2n}. \tag{15}$$

The q-j_α Bessel function $j_\alpha(x;q^2)$ is entire function and tends to the normalized j_α Bessel function as $q \longrightarrow 1^-$.
One can see, after simple computation, that

$$j_{-\frac{1}{2}}(x;q^2) = \cos(x;q^2), \tag{16}$$

$$j_{\frac{1}{2}}(x;q^2) = \frac{\sin(x;q^2)}{x}. \tag{17}$$

The q^2-Jackson integral from 0 to a and from 0 to ∞ are respectively defined by

$$\int_0^a f(x)d_{q^2}x = (1-q^2)a\sum_{n=0}^{\infty} f(aq^{2n})q^{2n}, \ \int_0^{\infty} f(x)d_{q^2}x = (1-q^2)\sum_{-\infty}^{+\infty} f(q^{2n})q^{2n}.$$

Note that for $n \in \mathbb{Z}$ and $a \in \mathbb{R}_q$, we have

$$\int_0^{\infty} f(q^{2n}x)d_{q^2}x = \frac{1}{q^{2n}}\int_0^{\infty} f(x)d_{q^2}x, \quad \int_0^{a} f(q^{2n}x)d_{q^2}x = \frac{1}{q^{2n}}\int_0^{aq^{2n}} f(x)d_{q^2}x. \tag{18}$$

The q^2-integration by parts is given for suitable function f and g regular at zero by:

$$\int_a^b f(x)D_{q^2,x}g(x)d_{q^2}x = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(q^2x)D_{q^2,x}g(x)d_{q^2}x. \tag{19}$$

The improper integral is defined in the following way (see [10]; [11])

$$\int_0^{\infty/A} f(x)d_{q^2}x = (1-q^2)\sum_{-\infty}^{+\infty} f\left(\frac{q^{2n}}{A}\right)\frac{q^{2n}}{A}, \quad A \neq 0. \tag{20}$$

We remark that, for $n \in \mathbb{Z}$, we have

$$\int_0^{\infty/q^{2n}} f(x)d_{q^2}x = \int_0^{\infty} f(x)d_{q^2}x. \tag{21}$$

The q^2-analogue of the exponential function [7] are given by

$$\begin{aligned} E_{q^2}(x) &= {}_0\varphi_0(-,-;q^2,-(1-q^2)x) = \sum_{n=0}^{\infty} q^{n(n-1)}\frac{(1-q^2)^n}{(q^2;q^2)_n}x^n \\ &= (-(1-q^2)x;q^2)_\infty, \quad \text{for} \quad x \in \mathbb{C} \end{aligned} \tag{22}$$

and

$$\begin{aligned} e_{q^2}(x) &= {}_1\varphi_0(0,-;q^2,(1-q^2)x) = \sum_{n=0}^{\infty} \frac{(1-q^2)^n}{(q^2;q^2)_n}x^n \\ &= \frac{1}{((1-q^2)x;q^2)_\infty}, \quad \text{for} \quad |x| < \frac{1}{1-q^2}. \end{aligned} \tag{23}$$

They satisfy

$$e_{q^2}(x).E_{q^2}(-x) = 1, \quad D_{q^2,x}e_{q^2}(x) = e_{q^2}(x), \quad D_{q^2,x}E_{q^2}(x) = E_{q^2}(q^2x). \tag{24}$$

The little q-Jacobi polynomials [7] is defined by

$$\begin{aligned} p_n(x;\alpha,\beta \mid q) &= \frac{q^{(n+\alpha)n}(q^{-n-\alpha};q)_n}{(q^{n+\alpha+\beta+1};q)_n}\,{}_2\varphi_1(q^{-n},q^{n+\alpha+\beta+1};q^{\alpha+1};q,qx) \quad (25)\\ &= \frac{q^{(n+\alpha)n}(q^{-n-\alpha};q)_n}{(q^{n+\alpha+\beta+1};q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q^{n+\alpha+\beta+1};q)_k}{(q^{\alpha+1};q)_k}\frac{q^k x^k}{(q;q)_k}. \end{aligned}$$

Jackson [8] defined a q^2-analogue of the Gamma function by

$$\Gamma_{q^2}(x) = \frac{(q^2;q^2)_\infty}{(q^{2x};q^2)_\infty}(1-q^2)^{1-x}. \tag{26}$$

Abdi in [1], introduced the q^2-Laplace transform by

$$\begin{aligned} \varphi(s) &= {}_{q^2}\mathscr{L}_s\{f(t)\}_{t\to s} &(27)\\ &= \int_0^{1/(1-q^2)s} E_{q^2}(-q^2st)f(t)d_{q^2}t &(28)\\ &= \int_0^{\infty/(1-q^2)s} E_{q^2}(-q^2st)f(t)d_{q^2}t. &(29) \end{aligned}$$

Moreover, since $\log(1-q^2)/\log(q^2)\in\mathbb{N}$ and $s\in\mathbb{R}_{q,+}$, we obtain from (21) that the following q^2-integral representations hold:

$$\varphi(s) = {}_{q^2}\mathscr{L}_s\{f(t)\}_{t\to s} = \int_0^{\infty/s} E_{q^2}(-q^2st)f(t)d_{q^2}t = \int_0^{\infty} E_{q^2}(-q^2st)f(t)d_{q^2}t \tag{30}$$

and (see [5] Theorem 1)

$$\Gamma_{q^2}(s) = \int_0^{+\infty} t^{s-1}E_{q^2}(-q^2t)d_{q^2}t. \tag{31}$$

In [6], we have defined the q-Wave polynomials associated with the q-difference operator $\Delta_{q,x}$ by defining $w_{1,2n}$ and $w_{2,2n}$ given, for x,t in $\mathbb{R}$, by:

$$\begin{cases} w_{1,2n}(x,t;q^2) = [2n]_q!\displaystyle\sum_{k=0}^{n} q^{k^2} b_{n-k}(x;q^2)\frac{t^{2k}}{[2k]_q!}\\ w_{2,2n}(x,t;q^2) = [2n]_q!\displaystyle\sum_{k=0}^{n} q^{k^2} b_{n-k}(x;q^2)\frac{t^{2k+1}}{[2k+1]_q!}. \end{cases} \tag{32}$$

wich can be expressed in term of q-little Jacobi polynomial $p_n(x;\alpha,\beta \mid q^2)$ [6] as follows

Proposition 1.1. *For x, t in $\mathbb{R}_q$ we have:*

$$w_{1,2n}(x,t;q^2) = (-1)^n q^{-n(n-1)}(q^2;q^2)_n t^{2n} p_n(q^{2n}x^2/q^2t^2;\text{-}1/2,\text{-}2n \mid q^2)$$

$$w_{2,2n}(x,t;q^2) = (-1)^n q^{-n(n+1)}(q^2;q^2)_n t^{2n+1} p_n(q^{2n}x^2/t^2;\text{-}1/2,\text{-}2n\text{-}1 \mid q^2).$$

Proposition 1.2. *For x, t in $\mathbb{R}$ we have:*

$$\begin{aligned} \mid w_{1,2n}(x,t;q^2) \mid &\leq q^{-n^2}\frac{[2n]_q!}{2}[(x+q^nt)_q^{2n}+(x-q^nt)_q^{2n}] \\ \mid w_{2,2n}(x,t;q^2) \mid &\leq q^{-n^2}\frac{[2n]_q!}{2}t[(x+q^nt)_q^{2n}+(x-q^nt)_q^{2n}] \end{aligned}$$

where $(a+b)_q^n$ is given by (6).

Proof. Owing to the relation

$$(q^{-2n};q^2)_k = (-1)^k q^{k(k-1)-2nk}\frac{(q^2;q^2)_n}{(q^2;q^2)_{n-k}},$$

we deduce, from Proposition 1.1, that

$$\begin{aligned} & w_{1,2n}(x,t;q^2) = \\ &= (-1)^n q^{n^2}(q;q)_{2n} t \sum_{k=0}^{n}(-1)^k \frac{q^{k(k-1)}q^k(q^{-2n};q^2)_k}{(q;q^2)_k(q^2;q^2)_k}\frac{(q^2;q^2)_{n-k}}{(q;q)_{2n-2k}} x^{2k}t^{2n-2k-1} \\ &= (-1)^n q^{n^2}(q;q)_{2n}t^{2n}\sum_{k=0}^{n}\frac{q^{k(k-1)}q^{k^2-2nk}(q^2;q^2)_n}{(q;q^2)_k(q^2;q^2)_k(q^2;q^2)_{n-k}(q;q^2)_{n-k}}\frac{x^{2k}}{t^{2k}} \\ &= (-1)^n q^{n^2}(q;q)_{2n}t^{2n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^2}\frac{q^{k(k-1)}q^{k^2-2nk}}{(q;q^2)_k(q;q^2)_{n-k}}\frac{x^{2k}}{t^{2k}}\ . \end{aligned}$$

Hence

$$\begin{aligned} \mid w_{1,2n}(x,t;q^2) \mid &\leq q^{n^2}\frac{(q;q)_{2n}}{(1-q)^n}t^{2n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^{k(k-1)}\left[\frac{x^2}{q^{2n}t^2}\right]^k \\ &\leq q^{n^2}\frac{(q;q)_{2n}}{(1-q)^n}t^{2n}(1+\frac{x^2}{q^{2n}t^2})_q^n \\ &\leq q^{-n^2}[2n]_q!(x^2+q^{2n}t^2)_q^n. \end{aligned}$$

The result is then deduced by the fact that:

$$(x^2+q^{2n}t^2)_q^n = \frac{1}{2}\left[(x+q^nt)_q^{2n}-(x-q^nt)_q^{2n}\right]. \tag{33}$$

□

Proposition 1.3. *For x, t in $\mathbb{R}$ and $n \geq 0$, we have:*

$$|D_{q,t}w_{1,2n}(x,t;q^2)| \leq q^{4n}C(q,n)t\left[(x+q^nt)_q^{2n-2}+(x-q^nt)_q^{2n-2}\right]$$

$$|\Delta_{q,t}w_{1,2n}(x,t;q^2)| \leq C(q,n)\left(x^2+t^2+q(1+q)[n-1]_{q^2}t^2\right)\left[(x+q^nt)_q^{2n-4}+(x-q^nt)_q^{2n-4}\right].$$

Furthermore

$$|\Delta_{q,x}w_{1,2n}(x,t;q^2)| \leq C(q,n)(x^2+q^2(n-1)t^2+q(1+q) \times [n-1]_{q^2}x^2)[(x+q^{n-1}t)_q^{2n-4}+(x-q^{n-1}t)_q^{2n-4}]$$

where the constant $C(q,n)$ is given by

$$C(q,n) = q^{-n^2-2n}\,\frac{1+q}{2}\,[2n]_q!\,[n]_{q^2}.$$

2. Convergence of series $\sum_{n=0}^{+\infty}\frac{\alpha_n}{[2n]_q!}w_{1,2n}(x,t;q^2)$ and $\sum_{n=0}^{+\infty}\frac{\alpha_n}{[2n]_q!}w_{2,2n}(x,t;q^2)$

In this section, we prove that the series $\sum_{n=0}^{+\infty}\frac{\alpha_n}{[2n]_q!}w_{1,2n}(x,t;q^2)$ and $\sum_{n=0}^{+\infty}\frac{\alpha_n}{[2n]_q!}w_{2,2n}(x,t;q^2)$ converge in the strip $\left\{(x,t)/\mid x\mid+\mid t\mid< R\right\}$, $R>0$.

Given R_0 such that the previous series converge for $\mid x\mid< R_0$. We consider the q-difference problems (I) and (II) given by:

$$(I)\begin{cases}\Delta_{q,t}w(x,t;q^2)=\Delta_{q,x}w(x,t;q^2)\\ w(x,0;q^2)=\phi(x)\\ \\ D_{q,t}w(x,t,q^2)_{|t=0}=0\end{cases} \qquad (II)\begin{cases}\Delta_{q,t}w(x,t;q^2)=\Delta_{q,x}w(x,t;q^2)\\ w(x,0;q^2)=0\\ \\ D_{q,t}w(x,t,q^2)_{|t=0}=\phi(x)\end{cases}$$

where $\Delta_{q,.}$ is given by (10) and ϕ being an entire even function defined on $\mathbb{C}$, infinitely q-differentiable at zero, having the following expansion:

$$\phi(x)=\sum_{n=0}^{+\infty}\alpha_n b_n(x;q^2) \tag{34}$$

the convergence holds for $\mid x\mid< R_0$.

In this section, We prove the solutions of (I) and (II) have respectively an expansion of the form $\sum_{n=0}^{+\infty}\frac{\alpha_n}{[2n]_q!}w_{1,2n}(x,t;q^2)$ and $\sum_{n=0}^{+\infty}\frac{\alpha_n}{[2n]_q!}w_{2,2n}(x,t;q^2)$, which converge respectively in the strip $\left\{(x,t)/\mid x\mid+\mid t\mid< R\right\}$, $R>0$.

Theorem 2.1. *Let $(\alpha_n)_n$ be a sequence of real or complex numbers such that the series $\sum \alpha_n b_n(x;q^2)$ converge for any sequence of real or complex numbers, for all $|x| < R_0$. Then:*

i) the series

$$\sum_{n=0}^{+\infty} \frac{\alpha_n}{[2n]_q!} w_{1,2n}(x,t;q^2) \tag{35}$$

is solution of the q-problem (I) in the strip $\{(x,t)/\ |x|+|t|<R_0\}$ and converges uniformly in any compact subset of this strip.

ii) the series

$$\sum_{n=0}^{+\infty} \frac{\alpha_n}{[2n]_q!} w_{2,2n}(x,t;q^2) \tag{36}$$

is the solution of the q-problem (II) in the strip $\{(x,t)/\ |x|+|t|<R_0\}$ and converges uniformly in any compact subset of this strip.

Proof. Given $R_1 < R_0$ and $K = \{(x,t)/\ |x|+|t|<R_1\}$, then for all (x,t) in K, $|x+q^n t| < R_1$ and $|x-q^n t| < R_1$. Furthermore, the fact that $\sum_{n=0}^{+\infty} \alpha_n b_n(x;q^2)$ converges for $|x| < R_0$ implies that there exist $M > 0$ such that

$$|\alpha_n| \leq \frac{M}{q^{-n^2}[2n]_q! R_1^{2n}}. \tag{37}$$

The Proposition 1.2, give

$$\left|\sum_{n=0}^{+\infty} \frac{\alpha_n}{[2n]_q!} w_{1,2n}(x,t;q^2)\right| \leq \frac{M}{2} \sum_{n=0}^{+\infty} \frac{1}{R_1^{2n}} [(x+q^n t)_q^{2n} + (x-q^n t)_q^{2n}].$$

Hence, the convergence of the last series holds for all (x,t) in K then $\sum_{n=0}^{+\infty} \frac{\alpha_n}{[2n]_q!} w_{1,2n}(x,t;q^2)$ converges uniformly in any compact subset of K to $w(x,t;q^2)$ and we have

$$\lim_{t\to 0} w(x,t) = \phi(x). \tag{38}$$

Now using Proposition 1.3, we can deduce easily that $\sum_{n=0}^{+\infty} \frac{\alpha_n}{[2n]_q!} D_{q,t} w_{1,2n}(x,t;q^2)$ converges uniformly in any compact subset of K with $D_{q,t} w(x,t;q^2)$ as sum and

$$\lim_{t\to 0} D_{q,t} w(x,t;q^2) = 0. \tag{39}$$

So i) is then proved.

To prove ii), we proceed with the same way. □

Acknowledgment. The authors would like to thank the Board of Editors for their helpful comments. They are thankful to the anonymous reviewer for the helpful remarks, valuable comments and suggestions.

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