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# A NOTE ON CERTAIN CLASSES OF MULTIVARIABLE $q$-SERIES TRANSFORMATIONS

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The present note envisage to derive some new classes of multiple $q$-series transformations and reduction formulae. Special cases in terms of the known and new results are also derived.

## 1. Introduction

In many areas of pure as well as applied mathematics, various types of special functions (including the $q$-extensions) have become essential tools for the scientists and engineers. Transformation, generating and reduction (or summation) formulae involving these functions of one and more variables have a wide range of applications to various fields of mathematical, physical and engineering sciences. This has led various workers in the field of $q$-theory for exploring the possible $q$-extensions to all the important results involving various ordinary special functions. With this objective in mind, we derive some new classes of transformation formulae in terms of certain general multiple $q$-series. These results have found applications to a substantially more general classes of $q$-special functions and orthogonal $q$-polynomials, which is illustrated in the concluding section.

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The $q$-shifted factorial is defined for $q \in \mathbb{R}$, $\alpha \in \mathbb{C}$ as a product of $n$ factors, by

$$(\alpha;q)_n = \begin{cases} 1 & ; \quad n=0 \\ (1-\alpha)(1-\alpha q)\ldots(1-\alpha q^{n-1}) & ; \quad n \in \mathbb{N} \end{cases} \tag{1.1}$$

in terms of the basic analogue of the gamma function, we have

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}(1-q)^n, \quad 0<q<1,\, n \in \mathbb{N}\ , \tag{1.2}$$

where the $q$-gamma function is defined by [2, p. 16, eqn. (1.10.1)].
If $|q|<1$, the definition (1.1) remains meaningful for $n=\infty$ as a convergent infinite product:

$$(\alpha;q)_\infty = \prod_{j=0}^{\infty}(1-\alpha q^j)\ . \tag{1.3}$$

The generalized basic hypergeometric series (cf. Gasper and Rahman [2]) is given by

$${}_r\Phi_s\left[\begin{array}{cc} a_1,\ldots,a_r; & \\ & q,x \\ b_1,\ldots,b_s; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}x^n\left\{(-1)^n q^{n(n-1)/2}\right\}^{(1+s-r)} \tag{1.4}$$

where $r$ and $s$ are positive integers, $q \neq 0$ when $r>s+1$, the numerator parameters $a_1,\ldots,a_r$, and the denominator parameters $b_1,\ldots,b_s$ being complex quantities provided that

$$b_j \neq q^{-m}, m=0,1,\ldots;\ j=1,2,\ldots,s.$$

If $0<|q|<1$, the series (1.4) converges absolutely for all $x$ if $r \leq s$ and for $|x|<1$ if $r=s+1$. This series also converges absolutely if $|q|>1$ and $|x|< |b_1 b_2 \cdots b_s|/|a_1 a_2 \cdots a_r|$.
The abnormal type of generalized basic hypergeometric series ${}_r\Phi_s(.)$ (cf. Agarwal and Verma [1]), is defined by

$${}_r\Phi_s\left[\begin{array}{ll} a_1,\ldots,a_r & ;q,x \\ b_1,\ldots,b_s & ;q^{\lambda} \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}q^{\lambda n(n+1)/2}\,x^n \tag{1.5}$$

where $\lambda>0$ and $0<|q|<1$.
As the $q$-analogue of Kampé de Fériet function, Jain [3] defined the generalized bivariate basic hypergeometric function as:

$$\Phi^{A:B';B''}_{C:D';D''}\left[\begin{array}{ll} (a):(b');(b'') & ;q,x,y \\ (c):(d');(d'') & ;i,j,k \end{array}\right] =$$

$$\sum_{m,n=0}^{\infty} \frac{\prod_{t=1}^{A}(a_t;q)_{m+n}\prod_{t=1}^{B'}(b'_t;q)_m\prod_{t=1}^{B''}(b''_t;q)_n}{\prod_{t=1}^{C}(c_t;q)_{m+n}\prod_{t=1}^{D'}(d'_t;q)_m\prod_{t=1}^{D''}(d''_t;q)_n} \times \frac{x^m y^n q^{im(m-1)/2+jn(n-1)/2+kmn}}{(q;q)_m(q;q)_n}, \tag{1.6}$$

where $|x| < 1$, $|y| < 1$, $0 < |q| < 1$, $i, j, k$ are integers and $(a)$ denotes a sequence of $A$ parameters $a_1, a_2, \ldots, a_A$.

## 2. Main Results

For non-negative integers $m_j$ and $n_j$ we define

$$A_i(m_i,n_i) = \sum_{k_i=0}^{[m_i/n_i]} (q^{-m_i};q)_{n_ik_i}\, B_i(m_i,k_i,q) \quad (i \in \{1,\ldots,r\}), \tag{2.1}$$

where $[x]$ denotes the greatest integer in $x$, and $B_i(m_i,k_i,q)$ are bounded sequences of real (or complex) parameters, $m_i,\ k_i \geq 0$, $i \in \{1,\ldots,r\}$ .

**Theorem 2.1.** *In correspondence to the bounded sequence $B_i(m_i,k_i,q)$, let $A_i(m_i,n_i)$ be defined by (2.1), $\forall\, i \in \{1,\ldots,r\}$ where $m_i$; $n_i$ are non-negative integers. Then*

$$\sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r) \prod_{i=1}^{r}\left\{A_i(m_i,n_i)\ \frac{x_i^{m_i}}{(q;q)_{m_i}}\right\}$$

$$= \sum_{m_1,k_1,\ldots,m_r,k_r=0}^{\infty} \Delta(m_1+n_1k_1,\ldots,m_r+n_rk_r)$$

$$\prod_{i=1}^{r}\left\{(-1)^{n_ik_i}\ q^{-n_ik_i(n_ik_i+1)/2-m_in_ik_i} B_i(m_i+n_ik_i,k_i,q)\ \frac{x_i^{m_i+n_ik_i}}{(q;q)_{m_i}}\right\}, \tag{2.2}$$

*where $\Delta(m_1,\ldots,m_r)$ is a single-valued, bounded multiple sequence of real (or complex) parameters such that each of the series involved is absolutely convergent.*

*Proof.* To prove the theorem, we denote the multiple-series in the left-hand side by $I$ and thus

$$I = \sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r) \prod_{i=1}^{r}\left\{A_i(m_i,n_i)\ \frac{x_i^{m_i}}{(q;q)_{m_i}}\right\}. \tag{2.3}$$

On using (2.1) and the elementary $q$-identity [2, p. 233, I. 12], namely

$$(q^{-m};q)_k = \frac{(q;q)_m}{(q;q)_{m-k}}(-1)^k\ q^{k(k-1)/2-mk}, \tag{2.4}$$

the equation (2.3) yields to

$$I = \sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r)$$

$$\prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{[m_i/n_i]} (-1)^{n_i k_i} \, q^{n_i k_i (n_i k_i - 1)/2 - m_i n_i k_i} \, B_i(m_i,k_i,q) \, \frac{x_i^{m_i}}{(q;q)_{m_i - n_i k_i}} \right\}. \quad (2.5)$$

On interchanging the order of summations, and making an appeal to the series arrangement property [6, p. 101, Lemma 3, eqn. (6)], namely

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} B(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k,n+km), \quad (2.6)$$

we are lead to the right-hand side of the result (2.2) after some simplifications.
□

Now, we consider another variation of the above theorem. Let us put the sequence

$$\Delta(m_1,\ldots,m_r) = \Omega(m_1,\ldots,m_r) \prod_{i=1}^{r} (q;q)_{m_i}, \quad (2.7)$$

and

$$B_i(m_i,k_i,q) = \frac{C_i(k_i;q)}{(q;q)_{m_i}}, \quad (2.8)$$

so that from equation (2.1), we write

$$A_i^*(m_i,n_i) = \sum_{k_i=0}^{[m_i/n_i]} (q^{-m_i};q)_{n_i k_i} \frac{C_i(k_i;q)}{(q;q)_{m_i}} \quad (i \in \{1,\ldots,r\}), \quad (2.9)$$

then the Theorem yields the following result:

**Corollary 2.2.** *For non-negative integers $m_i$; $n_i$ $(i = 1,\ldots,r)$ there exists the following multiple-series identity:*

$$\sum_{m_1,\ldots,m_r=0}^{\infty} \Omega(m_1,\ldots,m_r) \prod_{i=1}^{r} \{A_i^*(m_i,n_i) \, x_i^{m_i}\}$$

$$= \sum_{m_1,k_1,\ldots,m_r,k_r=0}^{\infty} \Omega(m_1+n_1k_1,\ldots,m_r+n_rk_r)$$

$$\prod_{i=1}^{r} \left\{ (-1)^{n_i k_i} \, q^{-n_i k_i (n_i k_i + 1)/2 - m_i n_i k_i} C_i(k_i,q) \, \frac{x_i^{m_i+n_i k_i}}{(q;q)_{m_i}} \right\}, \quad (2.10)$$

*where $A_i^*(m_i,n_i)$ is defined by (2.9) and $\Omega(m_1,\ldots,m_r)$ is a single-valued, bounded multiple sequence of real (or complex) parameters such that each of the series is absolutely convergent.*

## 3. Applications of the main results

It is interesting to observe that in view of the following limiting cases:

$$\lim_{q\to 1^-} \Gamma_q(a) \;=\; \Gamma(a) \quad \textit{and} \quad \lim_{q\to 1^-} \frac{(q^a;q)_n}{(1-q)^n} \;=\; (a)_n \;, \tag{3.1}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1), \tag{3.2}$$

the main theorem and corollary provide, respectively, the $q$-extensions of the known results due to Raina and Ladha [4, pp.519-520, Theorem 1 and 2].

By suitably specializing the arbitrary sequences in accordance with the involvement of various summation theorems relating to the basic hypergeometric functions, our main results can be applied to derive certain interesting transformation and reduction (or summation) formulae for the generalized basic hpergeometric functions of one and more variables. To illustrate this, we deduce certain transformation formulae from our main result (2.2).

(i) We put $n_i = 1$ and

$$B_i(m_i,k_i,q) = \frac{z^{k_i}}{(q;q)_{k_i}}, \;\; |z|<1, \tag{3.3}$$

so that from summation formula [2, p. 236, II. 4], namely

$$_1\Phi_0\left(q^{-n};-;q,z\right) = (zq^{-n};q)_n, \tag{3.4}$$

we have

$$A_i(m_i,1) = (z\,q^{-m_i};q)_{m_i}. \tag{3.5}$$

Using the above relations (3.3) and (3.5), result (2.2) leads to

$$\sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r) \prod_{i=1}^{r} \left\{ \frac{(z\,q^{-m_i};q)_{m_i}\, x_i^{m_i}}{(q;q)_{m_i}} \right\} =$$

$$\sum_{m_1,k_1,\ldots,m_r,k_r=0}^{\infty} \Delta(m_1+k_1,\ldots,m_r+k_r) \prod_{i=1}^{r} \left\{ \frac{(-1)^{k_i}\, q^{-k_i(k_i+1)/2-m_ik_i}\, z^{k_i}\, x_i^{m_i+k_i}}{(q;q)_{k_i}\,(q;q)_{m_i}} \right\}. \tag{3.6}$$

On setting $r=1$ and

$$\Delta(m_1) = \frac{(\alpha_1;q)_{m_1}\ldots(\alpha_p;q)_{m_1}}{(\beta_1;q)_{m_1}\ldots(\beta_s;q)_{m_1}}, \tag{3.7}$$

the relation (3.6), after replacing $x_1$ by $x$ and in view of the definitions (1.5) and (1.6) leads to the following transformation formula:

$$ {}_{p+1}\Phi_s \left[ \begin{array}{ll} (\alpha_p),\ q/z & ;q,-zx \\ (\beta_s) & ;q^{-1} \end{array} \right] = \Phi^{p:0;0}_{s:0;0} \left[ \begin{array}{llll} (\alpha_p) & : - ; - ; & q,x,-zx/q \\ (\beta_s) & : - ; - ; & 0,-1,-1 \end{array} \right], \tag{3.8} $$

where $(\alpha_p)$ denotes a sequence of $p$ parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$.

(ii) Again, if we put $n_i = 1$ and

$$ B_i(m_i,k_i,q) = \frac{(a_i;q)_{k_i}\ (b_i;q)_{k_i}\ q^{k_i}}{(c_i;q)_{k_i}\ (q^{1-m_i}a_ib_i/c_i;q)_{k_i}\ (q;q)_{k_i}}, \tag{3.9} $$

then on making use of $q$-Pfaff-Saalschütz summation theorem [2, p. 237, II. 12], namely

$$ {}_3\Phi_2 \left[ \begin{array}{ll} a,b,q^{-n} & ; \\ & q,q \\ c,abc^{-1}q^{1-n} & ; \end{array} \right] = \frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}, \tag{3.10} $$

we obtain

$$ A_i(m_i,1) = \frac{(c_i/a_i;q)_{m_i}\ (c_i/b_i;q)_{m_i}}{(c_i;q)_{m_i}\ (c_i/a_ib_i;q)_{m_i}}. \tag{3.11} $$

Using the above relations (3.9) and (3.11), (2.2) leads to

$$ \sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r) \prod_{i=1}^{r} \left\{ \frac{(c_i/a_i;q)_{m_i}\ (c_i/b_i;q)_{m_i}\ x_i^{m_i}}{(c_i;q)_{m_i}\ (c_i/a_ib_i;q)_{m_i}\ (q;q)_{m_i}} \right\} $$

$$ = \sum_{m_1,k_1,\ldots,m_r,k_r=0}^{\infty} \Delta(m_1+k_1,\ldots,m_r+k_r) $$

$$ \prod_{i=1}^{r} \left\{ \frac{(-1)^{k_i}\ q^{-k_i(k_i+1)/2-m_ik_i}\ q^{k_i}\ (a_i;q)_{k_i}\ (b_i;q)_{k_i}\ x_i^{m_i+k_i}}{(c_i;q)_{k_i}\ (q^{1-m_i-k_i}a_ib_i/c_i;q)_{k_i}\ (q;q)_{k_i}\ (q;q)_{m_i}} \right\}. \tag{3.12} $$

On setting $r = 1$ and

$$ \Delta(m_1) = \frac{(\alpha_1;q)_{m_1} \cdots (\alpha_p;q)_{m_1}}{(\beta_1;q)_{m_1} \cdots (\beta_s;q)_{m_1}} \left\{ (-1)^{m_1} q^{m_1(m_1-1)/2} \right\}^{(1+s-p)}, \tag{3.13} $$

the relation (3.12), after replacing $x_1$ by $x$, and in light of the definitions (1.4) and (1.6), leads to the following transformation:

$$ {}_{p+2}\Phi_{s+2} \left[ \begin{array}{ll} \alpha_1,\ldots,\alpha_p,c_1/a_1,c_1/b_1 & ; \\ & q,x \\ \beta_1,\ldots,\beta_s,c_1,c_1/a_1b_1 & ; \end{array} \right] = $$

$$\Phi^{p\ :1;2}_{s+1:0;1}\left[\begin{array}{lllll}(\alpha_p) & : \frac{c_1}{a_1 b_1}; a_1, b_1; & q, (-1)^{1+s-p}\, x, (-1)^{1+s-p}\, \frac{c_1 x}{a_1 b_1} \\ (\beta_s), c_1/a_1 b_1 & : \ -- \ ; \ c_1 \ ; & 1+s-p, 1+s-p, 1+s-p\end{array}\right]. \tag{3.14}$$

(iii) Finally, if we put $n_i = 1$ and

$$B_i(m_i, k_i, q) = \frac{(q^{\alpha_i+1};q)_{m_i}\ (t_i q^{m_i})^{k_i}\ q^{k_i(k_i-1)/2}}{(q;q)_{m_i}\ (q^{\alpha_i+1};q)_{k_i}\ (q;q)_{k_i}}, \tag{3.15}$$

then from (2.1), we get

$$A_i(m_i, 1) = L^{(\alpha)}_{m_i}(t_i;q), \tag{3.16}$$

where $L^{(\alpha)}_n(x;q)$ denotes the $q$-Laguerre polynomials cf. [5, p. 102, eqn. (3.1)] given by

$$L^{(\alpha)}_n(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\ {}_1\Phi_1\left[\begin{array}{lll} q^{-n} & ; & \\ & & q, -x\, q^n \\ q^{\alpha+1} & ; & \end{array}\right]. \tag{3.17}$$

On using the above relations (3.15) and (3.16), in the main theorem, we obtain

$$\sum_{m_1,\ldots,m_r=0}^{\infty} \Delta(m_1,\ldots,m_r) \prod_{i=1}^{r}\left\{L^{(\alpha)}_{m_i}(t_i;q)\frac{x_i^{m_i}}{(q;q)_{m_i}}\right\}$$

$$= \sum_{m_1,k_1,\ldots,m_r,k_r=0}^{\infty} \Delta(m_1+k_1,\ldots,m_r+k_r)$$

$$\prod_{i=1}^{r}\left\{\frac{(q^{\alpha_i+1};q)_{m_i+k_i}(-1)^{k_i}\ (t_i\, q^{k_i-1})^{k_i}\ x_i^{m_i+k_i}}{(q;q)_{m_i+k_i}(q^{\alpha_i+1};q)_{k_i}(q;q)_{k_i}\ (q;q)_{m_i}}\right\}, \tag{3.18}$$

which represents a transformation formula involving the product of the $q$-Laguerre polynomials.

We conclude with the remark that, the results deduced above are significant and can lead to yield numerous other transformation, generating and reduction (or summation) formulae involving various $q$-special functions and orthogonal $q$-polynomials by the suitable specializations of arbitrary sequences in the theorems. More importantly, as shown in the derivation of (3.18), on suitable specialization of the arbitrary sequences in the main theorem, one can deduce transformation formulae involving various orthogonal $q$-polynomials like the $q$-Hermite polynomials, the little $q$-Jacobi polynomials, the Wall polynomials, the $q$-Konhauser polynomials and several others.

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