LE MATEMATICHE
Vol. LXV (2010) – Fasc. I, pp. 127–150
doi: 10.4418/2010.65.1.11

MONOTONE SOLUTIONS FOR NONCONVEX FUNCTIONAL DIFFERENTIAL INCLUSIONS OF SECOND ORDER WITH CARATHEODORY PERTURBATION

A. G. IBRAHIM - F. A. ALADSANI

We give sufficient conditions to assure the existence of a monotone solution for a functional differential inclusions of second order with Caratheodory perturbation. No convexity condition is involved on the values of the right hand side in the construction. This work generalizes some a recent papers, for example [1,13,15].

1. Introduction

Let K be a closed subset of $\mathbb{R}^n$, Ω an open subset of $\mathbb{R}^n$ and P a lower semicontinuous set-valued map from K to the family of all nonempty subsets of K, with closed graph and satisfies the following two conditions

$$\text{(i) } \forall x \in K, x \in P(x).$$
$$\text{(ii) } \forall x, y \in K, y \in P(x) \Rightarrow P(y) \subseteq P(x).$$

Under these conditions, a preorder (reflexive and transitive) relation on K is defined as

$$x \preceq y \Leftrightarrow y \in P(x).$$

Entrato in redazione: 2 luglio 2010

AMS 2000 Subject Classification: 34A60, 49K24.
Keywords: Functional differential inclusion, monotone solution, second order contingent cone.

Let $\sigma > 0$ and $C([-\sigma,0],\mathbb{R}^n)$ be the space of continuous functions from $[-\sigma,0]$ to $\mathbb{R}^n$ with the uniform norm

$$||x||_\sigma = \sup\{||x(t)|| : t \in [-\sigma,0]\}.$$

For each $t \in [0,T]$; $T > 0$, we define the operator $\tau(t)$ from $C([-\sigma,T],\mathbb{R}^n)$ to $C([-\sigma,0],\mathbb{R}^n)$ as

$$(\tau(t)x)(s) = x(t+s), \text{ for all } s \in [-\sigma,0].$$

Here $\tau(t)x$ represents the history of the state from the time $t-\sigma$ to the present time t.

Let $K_0 = \{\varphi \in C([-\sigma,T],\mathbb{R}^n) : \varphi(0) \in K\}$ and F be a set-valued map defined from $K_0 \times \Omega$ to the family of nonempty compact subsets (not necessarily convex) in $\mathbb{R}^n$ and f be a Caratheodory function from $\mathbb{R}\times K \times \Omega$ to $\mathbb{R}^n$. Let (φ_0, y_0) be a given element in $K_0 \times \Omega$, we consider the second order functional differential inclusion with perturbation

$$(\mathrm{Q}) \begin{cases} x''(t) \in F(\tau(t)x, x'(t)) + f(t, x(t), x'(t)) \text{ a.e. on } [0,T], \\ x(t) = \varphi_0(t), \forall t \in [-\sigma,0], \\ x'(0) = y_0, \\ x(t) \in P(x(t)) \subseteq K, \ \forall t \in [0,T], \\ x(t) \preceq x(s), \text{ whenever } 0 \le t \le s \le T \ . \end{cases}$$

In this paper, we prove under reasonable conditions that there is a function $x : [-\sigma,T] \to \mathbb{R}^n$ such that

(1) the function x is absolutely continuous on $[0,T]$ with absolutely continuous derivative.

(2) $\tau(t)x \in K_0$, for all $t \in [0,T]$.

(3) $x'(t) \in \Omega$, a.e on $[0,T]$.

(4) the functions x, x', x'' satisfy (Q).

In order to explain the mathematical motivation to this study we refer to, Ibrahim and AL-Adsani [13] proved the existence of monotone solution for (Q) without perturbation, (that is $f \equiv 0$).

Further, Amine et al [1] proved the existence of a local solutions, not necessarily monotone, for (Q) in the particular case $P(x) = K$, for all $x \in K$ and without delay. Thus, the result, we are going to prove, generalizes the results of [1] and [13].

We mention, among others works, [9,11,12,13,15,20] for the proof of existence of monotone solutions for differential inclusions or functional differential inclusions and the works [3,4,7,8,10,14,16,17,18,19,21] for solutions not necessarily monotone. Note that the case where the solutions are not necessarily monotone has been widely investigated compared with that of monotone solutions which has been rarely investigated.

The present paper is organized as follows: In section 2, some definitions and facts to be used later introduced. In section 3, the main result is proved.

2. Preliminaries and notations

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $||.||$ and the scalar product $< \cdot,\cdot >$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : ||y-x|| < r\}$ denote the open ball centered at x of radius r and $\overline{B(x,r)}$ be its closure.

For $\sigma > 0$ and $\varphi \in C([-\sigma,0],\mathbb{R}^n)$, let $B_\sigma(\varphi,r) = \{\psi \in C([-\sigma,0],\mathbb{R}^n) : ||\psi - \varphi||_\sigma < r\}$ and $\overline{B_\sigma(\varphi,r)} = \{\psi \in C([-\sigma,0],\mathbb{R}^n) : ||\psi-\varphi||_\sigma \leq r\}$. We also, denote by $d(x,A) = \inf\{||x-y|| : y \in A\}$ the distance from $x \in \mathbb{R}^n$ to a closed subset A of $\mathbb{R}^n$. We denote also by $||.||_2$to the norm $L^2(C[-\sigma,0],\mathbb{R}^n))$.

A function $V : \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ is said to be proper if its effective domain $D(V) = \{x \in \mathbb{R}^n : V(x) < \infty\}$ is nonempty. The subdifferential of a proper convex lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ is defined (in the sense of convex analysis) by

$$\partial V(x) = \{\xi \in \mathbb{R}^n : V(y) - V(x) \geq < \xi, y-x >, \ \forall y \in \mathbb{R}^n\}.$$

The second-order contingent cone of a nonempty closed subset $C \subset \mathbb{R}^n$ and $(x,y) \in C \times \mathbb{R}^n$ is defined by

$$T_C^2(x,y) = \{z \in \mathbb{R}^n : \lim_{t\to 0^+} \inf \frac{d(x+ty+\frac{t^2}{2}z,C)}{t^2} = 0\}.$$

For the properties of the second-order contingent see [3,7,15].

Let M be a metric space, a multifunction $F : M \to 2^{\mathbb{R}^n}$ is said to be upper semicontinuous at a point $y \in M$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(z) \subset F(y) + B(0,\varepsilon),$$

for all $z \in B(y,\delta)$. For more informations about the continuity properties for multifunction we refer to [2,6].

3. Main result

We start by the following lemma which plays an important rule in the proof of our main result.

Lemma 3.1. *Let K be a nonempty closed subset of $\mathbb{R}^n$, Ω a nonempty open subset of $\mathbb{R}^n$, P a set-valued map from K to the family of nonempty closed subsets of K and $K_0 = \{\varphi \in C([-\sigma,0],\mathbb{R}^n), \varphi(0) \in K\}$. Let F be an upper semicontinuous set-valued map from $K_0 \times \Omega$ to the family of nonempty compact subsets of $\mathbb{R}^n$. Let f be a function from $\mathbb{R} \times K \times \Omega$ into $\mathbb{R}^n$. Assume also the following conditions:*

(H1) For all $x \in K$, $x \in P(x)$.

(H2) There exists a proper convex lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that $F(\varphi, y) \subseteq \partial V(y)$, for every $(\varphi, y) \in K_0 \times \Omega$.

(H3) For all $(t, \varphi, y) \in I \times K_0 \times \Omega$, there exists $z \in F(\varphi, y)$ such that

$$\lim_{h\to 0^+} \inf \frac{1}{h^2} d(\varphi(0) + hy + \frac{h^2}{2} z + \\ + \int_t^{t+h} (t+h-s) f(s, \varphi(0), y) ds, P(\varphi(0))) = 0.$$

(H4) $f : \mathbb{R} \times K \times \Omega \to \mathbb{R}^n$ is a Caratheodory function, (i.e. for each $(x,y) \in \Omega$, $t \to f(t,x,y)$ is measurable and for all $t \in \mathbb{R}$, $(x,y) \to f(t,x,y)$ is continuous) and there exists $m \in L^2(\mathbb{R})$ such that $||f(t,x,y)|| \leq m(t)$ for all $(t,x,y) \in \mathbb{R} \times K \times \Omega$.

Let (φ_0, y_0) be a fixed element in $K_0 \times \Omega$. Then there are two positive real numbers r and T such that for each positive integer m there are:

(1) A positive integer ν_m.

(2) A set of points
$$P_m = \{t_0^m = 0, t_1^m, ..., t_{\nu_m}^m\},$$
with
$$t_0^m < t_1^m < ... < t_{\nu_{m-1}}^m \leq T < t_{\nu_m}^m.$$

(3) Three sets of elements in $\mathbb{R}^n$
$$\begin{aligned} X_m &= \{x_p^m : p = 0,1,...,\nu_{m-1}\}, \\ Y_m &= \{y_p^m : p = 0,1,...,\nu_{m-1}\}, \end{aligned}$$
and
$$Z_m = \{z_p^m : p = 0,1,...,\nu_{m-1}\},$$
with $x_0^m = \varphi_0(0)$ and $y_0^m = y_0$.

(4) *A continuous function* $x_m : [-\sigma, T] \to \mathbb{R}^n$ *with* $x_m(t) = \varphi_0(t)$, *for all* $t \in [-\sigma, 0]$ *and such that for each* $p = 0, 1, ..., \nu_{m-1}$, *the following properties are satisfied:*

(i) $h^m_{p+1} = t^m_{p+1} - t^m_p < \frac{1}{m}$.

(ii) $z^m_p = u^m_p + w^m_p$ *where* $u^m_p \in F(\tau(t^m_p)x_m, y^m_p)$ *and* $w^m_p \in \frac{1}{m}B(0,1)$.

(iii) $x_m(t) = x^m_p + (t - t^m_p)y^m_p + \frac{1}{2}(t - t^m_p)^2 z^m_p + \int_{t^m_p}^t (t-s) f(s, x^m_p, y^m_p) ds$, $\forall t \in [t^m_p, t^m_{p+1}]$.

(iv) $x^m_{p+1} = x^m_p + h^m_{p+1} y^m_p + \frac{1}{2}(h^m_{p+1})^2 z^m_p +$

$$+ \int_{t^m_p}^{t^m_p + h^m_{p+1}} (t^m_p + h^m_{p+1} - s) f(s, x^m_p, y^m_p) ds.$$

(v) $x^m_{p+1} \in P(x^m_p) \cap B(\varphi_0(0), r) \subseteq K$ *and*

$$y^m_{p+1} = y^m_p + h^m_{p+1} z^m_p \in \overline{B(y_0, r)} \subseteq \Omega.$$

(vi) $\tau(t^m_{p+1})x_m \in B_\sigma(\varphi_0, r) \cap K_0$.

Proof. We follow the techniques developed in [13,18]. From [6, Prop. I.26], for each $y \in \mathbb{R}^n$, the subset $\partial V(y)$ is closed, convex and bounded. Moreover, by [2, Th. 0.7.2] the multifunction $y \to \partial V(y)$ is upper semicontinuous. So, by [2, Prop. 1.1.3] there are two positive real numbers r and M such that

$$\sup\{||z|| : z \in \partial V(y)\} \leq M\},$$

for all $y \in \overline{B(y_0, r)}$. Using condition (H2), we get

$$\sup\{||z|| : z \in F(\psi, y)\} \leq M, \tag{1}$$

for all $(\psi, y) \in (K_0 \cap B_\sigma(\varphi_0, r)) \times \overline{B(y_0, r)}$. Since Ω is open we can choose r such that $\overline{B(y_0, r)} \subseteq \Omega$. It is obvious that the closedness of K implies the closedness of K_0 in $C([-\sigma, 0], \mathbb{R}^n)$. From the continuity of φ_0 on $[-\sigma, 0]$ there is $\mu > 0$ such that for all $t, s \in [-\sigma, 0]$, we have

$$|t - s| < \mu \Longrightarrow ||\varphi_0(t) - \varphi_0(s)|| < \frac{r}{8}. \tag{2}$$

Let r and M be the positive real numbers defined above and choose $T_1 > 0$ such that

$$\int_0^{T_1} (m(s) + M + 1) ds < \frac{r}{4}.$$

Set

$$T_2 = \min\left\{\mu, \frac{r}{8(M+1)}, \frac{r}{8(||y_0|| + 1)}, \sqrt{\frac{r}{8(M+1)}}\right\}.$$

Choose T such that

$$T \in]0, \min\{T_1, T_2\}[. \tag{3}$$

Thus the numbers r and T are well defined. Now let m be a fixed positive integer. We put $t_0^m = 0$, $x_0^m = \varphi_0(0)$ and $y_0^m = y_0$. The sets P_m, X_m, Y_m and Z_m will be defined by induction. We first define x_1^m, t_1^m, y_1^m, z_0^m and x_m on $[0, t_1^m]$ such that the properties (i)-(vi) are satisfied for $p = 0$. Using condition (H3), there is $u_0^m \in F(\varphi_0, y_0)$ such that

$$\lim_{h\to 0^+} \inf \frac{1}{h^2} d(\varphi_0(0) + h y_0^m + \frac{h^2}{2} u_0^m +$$

$$+ \int_0^{0+h} (t_0^m + h - s) f(s, \varphi_0(0), y_0^m) ds, P(\varphi_0(0))) = 0.$$

So, there exists a positive number h_1^m such that $h_1^m \le \min\{\frac{1}{m}, T\}$ and

$$d(\varphi_0(0) + h_1^m y_0^m + \frac{(h_1^m)^2}{2} u_0^m +$$

$$+ \int_0^{h_1^m} (t_0^m + h - s) f(s, \varphi_0(0), y_0^m) ds, P(\varphi_0(0))) \le \frac{(h_1^m)^2}{4m}.$$

Since $P(\varphi_0(0))$ is closed, there is $x_1^m \in P(\varphi_0(0))$ with

$$||\varphi_0(0) + h_1^m y_0^m + \frac{(h_1^m)^2}{2} u_0^m + \int_0^{h_1^m} (t_0^m + h - s) f(s, \varphi_0(0), y_0^m) ds - x_1^m|| \le \frac{(h_1^m)^2}{4m}.$$

Consequently, there is $w_0^m \in \mathbb{R}^n$ such that $||w_0^m|| \le \frac{1}{2m}$ and

$$x_1^m = \varphi_0(0) + h_1^m y_0^m + \frac{(h_1^m)^2}{2} u_0^m + \frac{(h_1^m)^2}{2} w_0^m + \int_0^{h_1^m} (t_0^m + h - s) f(s, \varphi_0(0), y_0^m) ds.$$

Now we define $z_0^m = u_0^m + w_0^m$, then $z_0^m \in F(\varphi_0, y_0^m) + \frac{1}{2m} B(0,1)$ and

$$x_1^m = \varphi_0(0) + h_1^m y_0^m + \frac{(h_1^m)^2}{2} z_0^m + \int_0^{h_1^m} (t_0^m + h - s) f(s, \varphi_0(0), y_0^m) ds.$$

We put

$$y_1^m = y_0^m + h_1^m z_0^m + \int_0^{h_1^m} f(s, \varphi_0(0), y_0^m) ds.$$

and $t_1^m = t_0^m + h_1^m$ and for $t \in [t_0^m, t_1^m]$, we define

$$x_m(t) = \varphi_0(0) + (t - t_0^m) y_0^m + \frac{(t - t_0^m)^2}{2} z_0^m + \int_{t_0^m}^{t} (t - s) f(s, \varphi_0(0), y_0^m) ds.$$

Thus, the properties (i)-(iv) are clearly satisfied for $p=0$.

Since $\tau(t_0^m)x_m = \varphi_0$, using relation (1), we obtain

$$\sup\{||v|| : v \in F(\tau(t_0^m)x_m, y_0^m)\} \leq M.$$

Therefore, $||z_0^m|| \leq M + \frac{1}{2m} < M+1$. We get from the definition of y_1^m,

$$\begin{aligned} ||y_1^m - y_0^m|| &\leq h_1^m ||z_0^m|| + || \int_{t_0^m}^{t_1^m} f(s, x_0^m, y_0^m) ds || \\ &\leq T(M+1) + \frac{r}{4} < r. \end{aligned}$$

Thus $y_1^m \in \overline{B(y_0, r)}$. Since $x_1^m \in P(x_0^m) \subseteq K$, then to prove property (v) for $p=0$, it is sufficient to show that

$$||x_1^m - \varphi_0(0)|| < r.$$

We get using (1) and (3)

$$\begin{aligned} ||x_1^m - \varphi_0(0)|| &\leq h_1^m ||y_0^m|| + \frac{(h_1^m)^2}{2} ||z_0^m|| + \int_0^{h_1^m} |t_1^m - s| \; ||f(s, \varphi_0(0), y_0^m)|| ds \\ &\leq T||y_0^m|| + \frac{T^2}{2}(M+1) + \int_0^{h_1^m} m(s) ds \\ &\leq \frac{r}{8\left(||y_0^m||+1\right)} ||y_0^m|| + \frac{r}{8(M+1)} \frac{(M+1)}{2} + \frac{r}{4} \\ &< \frac{r}{8} + \frac{r}{16} + \frac{r}{4} < r, \end{aligned}$$

and hence (v) is satisfied for $p=0$.To prove (vi) for $p=0$, let $v \in [-\sigma, 0]$. By (2) and (3), we get

$$||\tau(t_1^m)x_m - \varphi_0||_\sigma = \sup_{-\sigma \leq v \leq 0} ||x_m(t_1^m + v) - \varphi_0(v)||$$

$$\leq \sup_{\substack{-\sigma \leq v \leq 0 \\ -\sigma \leq t_1^m + v \leq 0}} ||x_m(t_1^m + v) - \varphi_0(v)|| + \sup_{\substack{-\sigma \leq v \leq 0 \\ 0 \leq t_1^m + v}} ||x_m(t_1^m + v) - \varphi_0(v)||$$

$$\leq \sup_{\substack{-\sigma \leq v \leq 0 \\ -\sigma \leq t_1^m + v \leq 0}} ||\varphi_0(t_1^m + v) - \varphi_0(v)|| + \sup_{\substack{-\sigma \leq v \leq 0 \\ 0 \leq t_1^m + v}} ||x_m(t_1^m + v) - \varphi_0(v)||$$

$$\leq \frac{r}{8} + ||x_m(t_1^m + v) - \varphi_0(0)|| + ||\varphi_0(0) - \varphi_0(v)||$$

$$\leq \frac{r}{8} + (h_1^m + v)||y_0^m|| + \frac{(h_1^m + v)^2}{2} ||z_0^m|| + \int_0^T m(s) ds + \frac{r}{8}$$

$$\leq \frac{r}{8} + T||y_0^m|| + \frac{T^2}{2}||z_0^m|| + \frac{r}{4} + \frac{r}{8} \leq$$

$$\leq \frac{r}{8} + \frac{r}{8\left(||y_0^m||+1\right)}||y_0^m|| + \frac{r}{16(M+1)}(M+1) + \frac{r}{4} + \frac{r}{8} \leq$$

$$\leq \frac{r}{8} + \frac{r}{8} + \frac{r}{16} + \frac{r}{4} + \frac{r}{8} < r,$$

which shows that $\tau(t_1^m)x_m \in B_\sigma(\varphi_0, r)$ and hence (vi) is proved.
Now we suppose that $t_{p+1}^m, x_{p+1}^m, y_{p+1}^m, z_p^m$ are well defined for $p = 0, 1, ..., (q-1)$ and x_m is defined on the interval $[-\sigma, t_q^m]$ such that all the properties (i)-(vi) are satisfied for $p = 0, 1, ..., (q-1)$. We define t_{q+1}^m, x_{q+1}^m, y_{q+1}^m, z_q^m and x_m on $[t_q^m, t_{q+1}^m]$ such that the properties (i)-(vi) are satisfied for $p = q$. We denote by H_q^m the set of all $h \in \left]0, \frac{1}{m}\right[$ for which the following conditions are satisfied:

(a) $0 < h < T - t_q^m$.

(b) there exists $u_q^m \in F(\tau(t_q^m)x_m, y_q^m)$ such that

$$d(x_q^m + h\, y_q^m + \frac{h^2}{2}u_q^m + \int_{t_q^m}^{t_q^m+h}(t_q^m + h - s)f(s, x_q^m, y_q^m)ds, p(x_q^m)) \leq \frac{h^2}{4m}.$$

From the fact that (v) and (vi) are true for $p = q-1$, we get $y_q^m \in \Omega$ and $\tau(t_q^m)x_m \in K_0$. Moreover, since (iv) is true for $p = q-1$, then $\tau(t_q^m)x_m(0) = x_m(t_q^m) = x_q^m$. Therefore there is $u_q^m \in F(\tau(t_q^m)x_m, y_q^m)$ such that

$$\liminf_{h \to 0^+} \frac{1}{h^2} d(x_q^m + h y_q^m + \frac{h^2}{2}u_q^m + \int_{t_q^m}^{t_q^m+h}(t_q^m + h - s)f(s, x_q^m, y_q^m)ds, P(x_q^m)) = 0,$$

which assures the existence of a positive number h such that $h < \min\{\frac{1}{m}, T - t_q^m\}$ and

$$d(x_q^m + h\, y_q^m + \frac{h^2}{2}u_q^m + \int_{t_q^m}^{t_q^m+h}(t_q^m + h - s)f(s, x_q^m, y_q^m)ds, P(x_q^m)) \leq \frac{h^2}{4m},$$

hence $h \in H_q^m$. Since H_q^m is bounded by the number T, there is a number d_q^m such that $d_q^m = \sup\{\alpha : \alpha \in H_q^m\}$. Since $H_q^m \cap [\frac{d_q^m}{2}, d_q^m] \neq \phi$, an element $h_{q+1}^m \in H_q^m \cap [\frac{d_q^m}{2}, d_q^m]$ is found such that

$$d(x_q^m + h_{q+1}^m\, y_q^m + \frac{(h_{q+1}^m)^2}{2}u_q^m +$$

$$+ \int_{t_q^m}^{t_q^m+h_{q+1}^m}(t_q^m + h_{q+1}^m - s)f(s, x_q^m, y_q^m)ds, P(x_q^m)) \leq \frac{(h_{q+1}^m)^2}{4m}.$$

From the closedness of $P(x_q^m)$, there is $x_{q+1}^m \in P(x_q^m) \subseteq K$ with

$$||x_q^m + h_{q+1}^m y_q^m + \frac{(h_{q+1}^m)^2}{2} u_q^m +$$

$$+ \int_{t_q^m}^{t_q^m + h_{q+1}^m} (t_q^m + h_{q+1}^m - s) f(s, x_q^m, y_q^m) ds - x_{q+1}^m|| \le \frac{(h_{q+1}^m)^2}{4m}.$$

Consequently, there is $w_q^m \in \mathbb{R}^n$ with $||w_q^m|| \le \frac{1}{2m} < \frac{1}{m}$ such that

$$x_{q+1}^m = x_q^m + h_{q+1}^m y_q^m + \frac{(h_{q+1}^m)^2}{2} u_q^m + \frac{\left(h_{q+1}^m\right)^2}{2} w_q^m +$$

$$+ \int_{t_q^m}^{t_q^m + h_{q+1}^m} (t_q^m + h_{q+1}^m - s) f(s, x_q^m, y_q^m) ds =$$

$$= x_q^m + h_{q+1}^m y_q^m + \frac{(h_{q+1}^m)^2}{2} (u_q^m + w_q^m) + \int_{t_q^m}^{t_q^m + h_{q+1}^m} (t_q^m + h_{q+1}^m - s) f(s, x_q^m, y_q^m) ds.$$

We define $z_q^m = u_q^m + w_q^m$. Then

$$z_q^m \in F(\tau(t_q^m) x_m, y_q^m) + \frac{1}{m} B(0,1)$$

and

$$x_{q+1}^m = x_q^m + h_{q+1}^m y_q^m + \frac{(h_{q+1}^m)^2}{2} z_q^m + \int_{t_q^m}^{t_q^m + h_{q+1}^m} (t_q^m + h_{q+1}^m - s) f(s, x_q^m, y_q^m) ds.$$

We put

$$y_{q+1}^m = y_q^m + h_{q+1}^m z_q^m + \int_{t_q^m}^{t_q^m + h_{q+1}^m} f(s, x_q^m, y_q^m) ds$$

and $t_{q+1}^m = t_q^m + h_{q+1}^m$.

For $t \in [t_q^m, t_{q+1}^m]$, we define

$$x_m(t) = x_q^m + (t - t_q^m) y_q^m + \frac{(t - t_q^m)^2}{2} z_q^m + \int_{t_q^m}^{t} (t - s) f(s, x_q^m, y_q^m) ds.$$

Obviously the relations (i)-(iv) are satisfied for $p = q$.

Now we prove that (v) is true for $p = q$. Since (v) and (vi) are true for $p = q - 1$, then $\tau(t_q^m) x_m \in B_\sigma(\varphi_0, r)$ and $y_q^m \in \overline{B(y_0, r)}$, and hence by (1) we get $||z_q^m|| \le M + 1$. Let us prove that $||y_{q+1}^m - y_0|| < r$. We note that

$$y_{q+1}^m = y_q^m + h_{q+1}^m z_q^m + \int_{t_q^m}^{t_q^m + h_{q+1}^m} f(s, x_q^m, y_q^m) ds =$$

$$y_{q-1}^m + h_q^m z_{q-1}^m + \int_{t_{q-1}^m}^{t_{q-1}^m + h_q^m} f(s, x_{q-1}^m, y_{q-1}^m) ds + h_{q+1}^m z_q^m + \int_{t_q^m}^{t_q^m + h_{q+1}^m} f(s, x_q^m, y_q^m) ds.$$

By iterating we get

$$y_{q+1}^m = y_0^m + \sum_{j=0}^{q} h_{j+1}^m z_j^m + \sum_{j=0}^{q} \int_{t_j^m}^{t_{j+1}^m} f(s, x_j^m, y_j^m) ds.$$

Thus

$$\begin{aligned} ||y_{q+1}^m - y_0^m|| &\leq \sum_{j=0}^{q} h_{j+1}^m ||z_j^m|| + \sum_{j=0}^{q} \int_{t_j^m}^{t_{j+1}^m} ||f(s, x_j^m, y_j^m)|| ds \\ &\leq (M+1) \sum_{j=0}^{q} h_{j+1}^m + \frac{r}{4} \\ &\leq (M+1)T + \frac{r}{4} < \frac{r}{8} + \frac{r}{4} < r. \end{aligned}$$

To prove that $x_{q+1}^m \in B_\sigma(\varphi_0(0), r)$ we first use the induction technique to prove the relation

$$\begin{aligned} x_{p+1}^m = \varphi_0(0) + (\sum_{j=0}^{p} h_{j+1}^m) y_0^m + \tfrac{1}{2} \sum_{j=0}^{p} (h_{j+1}^m)^2 z_j^m + \\ + \sum_{i=0}^{p-1} \sum_{j=i+1}^{p} h_{i+1}^m h_{j+1}^m z_i^m + \sum_{i=0}^{p} \int_{t_i^m}^{t_{i+1}^m} (t_{i+1}^m - s) f(s, x_i^m, y_i^m) ds + \\ + h_{p+1}^m \sum_{j=0}^{p-1} \int_{t_j^m}^{t_{j+1}^m} f(s, x_j^m, y_j^m) ds \end{aligned} \tag{4}$$

for $p = 1, ..., q$. For $p = 1$ we note that

$$x_2^m = x_1^m + h_2^m y_1^m + \frac{1}{2} (h_2^m)^2 z_1^m + \int_{t_1^m}^{t_2^m} (t_2^m - s) f(s, x_1^m, y_1^m) ds =$$

$$= x_1^m + h_2^m (y_0^m + h_1^m z_0^m + \int_{t_0^m}^{t_1^m} f(s, x_0^m, y_0^m) ds) + \frac{1}{2} (h_2^m)^2 z_1^m +$$

$$+ \int_{t_1^m}^{t_2^m} (t_2^m - s) f(s, x_1^m, y_1^m) ds =$$

$$= (x_0^m + h_1^m y_0^m + \frac{1}{2} (h_1^m)^2 z_0^m + \int_{t_0^m}^{t_1^m} (t_1^m - s) f(s, x_0^m, y_0^m) ds) +$$

$$+ h_2^m (y_0^m + h_1^m z_0^m + \int_{t_0^m}^{t_1^m} f(s, x_0^m, y_0^m) ds) + \frac{1}{2} (h_2^m)^2 z_1^m + \int_{t_1^m}^{t_2^m} (t_2^m - s) f(s, x_1^m, y_1^m) ds$$

$$= x_0^m + (h_1^m + h_2^m) y_0^m + \frac{1}{2} ((h_1^m)^2 z_0^m + (h_2^m)^2 z_1^m) + h_1^m h_2^m z_0^m + h_2^m \int_{t_0^m}^{t_1^m} f(s, x_0^m, y_0^m) ds$$

$$+\int_{t_0^m}^{t_1^m}(t_1^m-s)f(s,x_0^m,y_0^m)ds+\int_{t_1^m}^{t_2^m}(t_2^m-s)f(s,x_1^m,y_1^m)ds=$$

$$=\varphi_0(0)+(\sum_{j=0}^{1}h_{j+1}^m)y_0^m+\frac{1}{2}\sum_{j=0}^{1}(h_{j+1}^m)^2z_j^m+(h_1^m h_2^m z_0^m)+$$

$$+\sum_{j=0}^{1}\int_{t_i^m}^{t_{i+1}^m}(t_{i+1}^m-s)f(s,x_i^m,y_i^m)ds+h_2^m\int_{t_0^m}^{t_1^m}f(s,x_0^m,y_0^m)ds.$$

Then the relation (4) is true for $p=1$. Suppose that (4) is true for $p=q-1$. This gives us

$$\begin{aligned} x_q^m &= \varphi_0(0)+(\sum_{j=0}^{q-1}h_{j+1}^m)y_0^m+\frac{1}{2}\sum_{j=0}^{q-1}(h_{j+1}^m)^2z_j^m+\sum_{i=0}^{q-2}\sum_{j=i+1}^{q-1}(h_{i+1}^m h_{j+1}^m z_i^m) \\ &\quad+\sum_{j=0}^{q-1}\int_{t_j^m}^{t_{j+1}^m}(t_{j+1}^m-s)f(s,x_j^m,y_j^m)ds+h_q^m(\sum_{j=0}^{q-1}\int_{t_i^m}^{t_{i+1}^m}f(s,x_j^m,y_j^m)ds). \end{aligned}$$

So, according to the definition of x_{q+1}^m we have

$$x_{q+1}^m=x_q^m+h_{q+1}^m y_q^m+\frac{1}{2}(h_{q+1}^m)^2z_q^m+\int_{t_q^m}^{t_{q+1}^m}(t_{q+1}^m-s)f(s,x_q^m,y_q^m)ds=$$

$$=\varphi_0(0)+(\sum_{j=0}^{q-1}h_{j+1}^m)y_0^m+\frac{1}{2}\sum_{j=0}^{q-1}(h_{j+1}^m)^2z_j^m+\sum_{i=0}^{q-2}\sum_{j=i+1}^{q-1}h_{i+1}^m h_{j+1}^m z_i^m+$$

$$+\sum_{i=0}^{q-1}\int_{t_i^m}^{t_{i+1}^m}(t_{i+1}^m-s)f(s,x_i^m,y_i^m)ds+$$

$$+h_{q+1}^m\left(y_0^m+\sum_{j=0}^{q-1}h_{j+1}^m z_j^m+\sum_{j=0}q-1\int_{t_i^m}^{t_{i+1}^m}f(s,x_i^m,y_i^m)ds\right)+$$

$$+\frac{1}{2}(h_{q+1}^m)^2z_q^m+\int_{t_q^m}^{t_{q+1}^m}(t_{q+1}^m-s)f(s,x_q^m,y_q^m)ds=$$

$$=\varphi_0(0)+(\sum_{j=0}^{q}h_{j+1}^m)y_0^m+\frac{1}{2}\sum_{j=0}^{q}(h_{j+1}^m)^2z_j^m+\sum_{i=0}^{q-1}\sum_{j=i+1}^{q}h_{i+1}^m h_{j+1}^m z_i^m+$$

$$+\sum_{i=0}^{q}\int_{t_i^m}^{t_{i+1}^m}(t_{i+1}^m-s)f(s,x_i^m,y_i^m)ds+h_{q+1}^m\sum_{j=0}^{q-1}\sum_{j=0}^{q-1}\int_{t_i^m}^{t_{i+1}^m}f(s,x_i^m,y_i^m)ds.$$

This implies that the relation (4) is true for $p = q$. Now, from the fact that $||z_p^m|| \leq M+1$, for all $p = 0, 1, ..., q$ we get

$$||x_{q+1}^m - \varphi_0(0)|| \leq ||y_0^m||(\sum_{j=0}^{q} h_{j+1}^m) + \frac{1}{2}\sum_{j=0}^{q}(h_{j+1}^m)^2(M+1)+$$

$$+(M+1)\sum_{i=0}^{q-1}\sum_{j=i+1}^{q} h_{i+1}^m h_{j+1}^m + \sum_{i=0}^{q}\int_{t_i^m}^{t_{i+1}^m}(t_{i+1}^m - s)f(s, x_i^m, y_i^m)ds + \frac{r}{4} \leq$$

$$\leq ||y_0^m||T + \frac{1}{2}(M+1)T^2 + (M+1)T^2 + \frac{r}{4} + \frac{r}{4} <$$

$$< \frac{r}{8} + \frac{r}{16} + \frac{r}{8} + \frac{r}{4} + \frac{r}{4} < r.$$

Thus (v) is true for $p = q$. We prove (vi) for $p = q$.

$$||\tau(t_{q+1}^m)x_m - \varphi_0||_\sigma = \sup_{-\sigma \leq v \leq 0} ||\tau(t_{q+1}^m)x_m(v) - \varphi_0(v)|| =$$

$$= \sup_{-\sigma \leq v \leq 0} ||x_m(t_{q+1}^m + v) - \varphi_0(v)|| \leq$$

$$\leq \sup_{\substack{-\sigma \leq v \leq 0 \\ v + t_{q+1}^m \leq 0}} ||x_m(t_{q+1}^m + v) - \varphi_0(v)|| + \sup_{\substack{-\sigma \leq v \leq 0 \\ 0 \leq v + t_{q+1}^m}} ||x_m(t_{q+1}^m + v) - \varphi_0(v)|| \leq$$

$$\leq \sup_{\substack{-\sigma \leq v \leq 0 \\ v + t_{q+1}^m \leq 0}} ||\varphi_0(t_{q+1}^m + v) - \varphi_0(v)|| + \sup_{\substack{-\sigma \leq v \leq 0 \\ 0 \leq v + t_{q+1}^m}} ||x_m(t_{q+1}^m + v) - \varphi_0(0)||+$$

$$+ \sup_{\substack{-\sigma \leq v \leq 0 \\ 0 \leq v + t_{q+1}^m}} ||\varphi_0(0) - \varphi_0(v)|| < \frac{r}{8} + \frac{r}{8} + \frac{r}{16} + \frac{r}{8} + \frac{r}{4} + \frac{r}{8} < r.$$

It remains to show that there is a positive number ν_m such that $t_{\nu_{m-1}}^m \leq T < t_{\nu_m}^m$. Therefore, we have to prove that the iterative process is finite. For this purpose suppose that the iterative process is not finite. So, for each non negative integer number p, there are $t_p^m \in [0, T[$, x_p^m, y_p^m, z_p^m such that the relations (i)-(vi) are satisfied. Since the sequence $\{t_p\}_{p \geq 1}$ is bounded and increasing, there is $t_\alpha^m \in]0, T]$ such that $\lim_{p \to \infty} t_p^m = t_\alpha^m$. Let us show that the sequence $\{x_p\}_{p \geq 1}$ and $\{y_p\}_{p \geq 1}$ are Cauchy sequences. Let p and q be two positive integers such that $p > q$. From the relation (4) we have

$$||x_p^m - x_q^m|| = ||(\sum_{j=q}^{p-1} h_{j+1}^m)y_0^m + \frac{1}{2}\sum_{j=q}^{p-1}(h_{j+1}^m)^2 z_j^m + \sum_{j=q}^{p-1} h_1^m h_{j+1}^m z_0^m + \sum_{j=q}^{p-1} h_2^m h_{j+1}^m z_1^m +$$

$$\cdots + \sum_{j=p-2}^{p-1} h_{p-2}^m h_{j+1}^m z_{p-3}^m + h_{p-1}^m h_p^m z_{p-2}^m + \sum_{i=q}^{p-1}\int_{t_i^m}^{t_{i+1}^m}(t_{i+1}^m - s)f(s, x_i^m, y_i^m)ds+$$

$$+h_p^m(\sum_{j=0}^{p-2}\int_{t_i^m}^{t_{i+1}^m} f(s,x_j^m,y_j^m)ds) - h_q^m(\sum_{j=0}^{q-2}\int_{t_i^m}^{t_{i+1}^m} f(s,x_j^m,y_j^m)ds)|| \leq$$

$$\leq ||y_0^m||(t_p^m - t_q^m) + \frac{1}{2}(M+1)(t_p^m - t_q^m)^2 + (M+1)(\sum_{j=q}^{p-1} h_{j+1}^m)(h_1^m + \ldots + h_{p-1}^m) +$$

$$+\sum_{i=q}^{p-1}\int_{t_i^m}^{t_{i+1}^m} ||f(s,x_i^m,y_i^m)||ds \leq$$

$$\leq ||y_0^m||(t_p^m - t_q^m) + \frac{1}{2}(M+1)(t_p^m - t_q^m)^2 + (M+1)(t_p^m - t_q^m)(t_{p-1}^m - t_0^m) +$$
$$+(t_p^m - t_q^m)\sup_s ||f(s,x_i^m,y_i^m)|| + (t_p^m - t_{p-1}^m)\sup_s ||f(s,x_i^m,y_i^m)|| +$$
$$+(t_q^m - t_{q-1}^m)\sup_s ||f(s,x_i^m,y_i^m)|| \leq$$

$$\leq ||y_0^m||(t_p^m - t_q^m) + \frac{1}{2}(M+1)(t_p^m - t_q^m)^2 + (M+1)T(t_p^m - t_q^m) +$$
$$+m(s)(t_p^m - t_q^m) + (t_p^m - t_{p-1}^m)m(s) + (t_q^m - t_{q-1}^m)m(s).$$

Since the sequence $\{t_p\}_{p\geq 1}$ is convergent, the sequence $\{x_p^m\}_{p\geq 1}$ is Cauchy. Then there is $x_\alpha^m \in \mathbb{R}^n$ such that $\lim_{p\to\infty} x_p^m = x_\alpha^m$. Also,

$$\begin{aligned} ||y_p^m - y_q^m|| &= ||\sum_{j=q}^{p-1} h_{j+1}^m z_j^m + \sum_{i=q}^{p-1}\int_{t_i^m}^{t_{i+1}^m} f(s,x_i^m,y_i^m)ds|| \\ &\leq (M+1)(t_p^m - t_q^m) + m(s)(t_p^m - t_q^m). \end{aligned}$$

Thus the sequence $\{y_p\}_{p\geq 1}$ is a Cauchy sequence in $\mathbb{R}^n$. Hence there is $y_\alpha^m \in \mathbb{R}^n$ such that $y_\alpha^m = \lim_{p\to\infty} y_p^m$. From property (v) we note that

$$x_p^m \in P(x_p^m) \cap \overline{B(\varphi_0(0),r)} \subseteq K, \tag{5}$$

and

$$y_p^m \in \overline{B(y_0,r)} \subset \Omega.$$

Thus $x_\alpha^m \in K$ and $y_\alpha^m \in \overline{B(y_0,r)} \subset \Omega$.

Now we put $x_m(t_\alpha^m) = x_\alpha^m$. To show that x_m is continuous at t_α^m let $\{s_p^m : p \geq 1\}$ be a sequence in $[0,t_\alpha^m[$ such that $\lim_{p\to\infty} s_p^m = t_\alpha^m$ and $t_p^m \leq s_p^m \leq t_{p+1}^m$ for every $p \geq 1$. We have

$$||x_m(s_p^m) - x_m(t_\alpha^m)|| \leq ||x_m(s_p^m) - x_m(t_p^m)|| + ||x_m(t_p^m) - x_\alpha^m|| \leq$$
$$\leq (s_p^m - t_p^m)||y_0^m|| + \frac{1}{2}(s_p^m - t_p^m)^2(M+1) + (M+1)T(s_p^m - t_p^m) +$$
$$+(s_p^m - t_p^m)m(s) + ||x_m(t_p^m) - x_\alpha^m||.$$

By taking the limit as $p \to \infty$, we obtain

$$\lim_{p\to\infty} ||x_m(s_p^m) - x_m(t_\alpha^m)|| = 0,$$

which prove that x_m is continuous at t_α^m. Hence x_m is continuous on $[-\sigma, t_\alpha^m]$. Consequently,

$$\lim_{p\to\infty} \tau(t_p^m)x_m = \tau(t_\alpha^m)x_m.$$

Note that from (vi), $\tau(t_p^m)x_m \in K_0 \cap \overline{B_\sigma(\varphi_0, r)}$. Since the subset $K_0 \cap \overline{B_\sigma(\varphi_0, r)}$ is closed, we obtain

$$\tau(t_\alpha^m)x_m \in K_0 \cap \overline{B_\sigma(\varphi_0, r)}.$$

Furthermore, by (ii) and the relation (1), the sequences $\{z_p\}_{p\geq 1}$ and $\{u_p\}_{p\geq 1}$ are bounded in $\mathbb{R}^n$. So, there are two convergent subsequences, denoted again by $\{z_p\}_{p\geq 1}$ and $\{u_p\}_{p\geq 1}$. Thus there are two elements z_α^m and u_α^m of $\mathbb{R}^n$ such that $\lim\limits_{p\to\infty} z_p^m = z_\alpha^m$ and $\lim\limits_{p\to\infty} u_p^m = u_\alpha^m$.

Now since F is upper semicontinuous on $K_0 \times \Omega$ with compact values and since $u_p^m \in F(\tau(t_p^m)x_m, y_p^m)$, for all $p \geq 1$, it follows that $u_\alpha^m \in F(\tau(t_\alpha^m)x_m, y_\alpha^m)$. Applying condition (H3),

$$\lim_{h\to 0^+} \inf d(x_m(t_\alpha^m) + hy_\alpha^m + \frac{h^2}{2}u_\alpha^m + \int_{t_\alpha^m}^{t_\alpha^m+h} (t_\alpha^m + h - s)f(s, x_\alpha^m, y_\alpha^m)ds, P(x_m(t_\alpha^m)))$$

vanishes. Hence, there is $h \in]0, T - t_\alpha^m[$ such that

$$d(x_\alpha^m + hy_\alpha^m + \frac{h^2}{2}u_\alpha^m + \int_{t_\alpha^m}^{t_\alpha^m+h}(t_\alpha^m + h - s)f(s, x_\alpha^m, y_\alpha^m)ds, P(x_\alpha^m)) \leq \frac{h^2}{16m}. \quad (6)$$

We prove that h belongs to H_p^m for every p sufficient large. Since $\{t_p^m\}_p$ is an increasing sequence to t_α^m, $\lim\limits_{p\to\infty} x_p^m = x_\alpha^m$, $\lim\limits_{p\to\infty} y_p^m = y_\alpha^m$ and $\lim\limits_{p\to\infty} u_p^m = u_\alpha^m$, then we can find a natural number p_1 such that for every $p > p_1$ we have $t_p^m < t_\alpha^m < t_p^m + h < t_\alpha^m + h$,

$$||x_p^m - x_\alpha^m|| \leq \frac{h^2}{24m}, \quad (7)$$

$$||y_p^m - y_\alpha^m|| \leq \frac{h}{24m} \quad (8)$$

and

$$||u_p^m - u_\alpha^m|| \leq \frac{1}{12m}. \quad (9)$$

From the lower semicontinuity of P at x_p^m, there is a natural number p_2 such that $P(x_\alpha^m) \subseteq P(x_p^m) + \frac{h^2}{16m}B(0,1)$, for all $p > p_2$. This gives that if $z \in \mathbb{R}^n$, then

$$d(z, P(x_p^m)) \leq d(z, P(x_\alpha^m)) + \frac{h^2}{16m}, \forall p > p_2. \quad (10)$$

Now let $p > \max(p_1, p_2)$. By (7)-(10), we have

$$\begin{aligned}
& d(x_p^m + hy_p^m + \frac{h^2}{2}u_p^m + \int_{t_p^m}^{t_p^m+h}(t_p^m+h-s)f(s,x_p^m,y_p^m)ds, P(x_p^m)) \\
\leq\ & d(x_p^m + hy_p^m + \frac{h^2}{2}u_p^m + \int_{t_p^m}^{t_p^m+h}(t_p^m+h-s)f(s,x_p^m,y_p^m)ds, x_\alpha^m + hy_\alpha^m + \frac{h^2}{2}u_\alpha^m \\
& + \int_{t_\alpha^m}^{t_\alpha^m+h}(t_\alpha^m+h-s)f(s,x_\alpha^m,y_\alpha^m)ds) + d(x_\alpha^m + hy_\alpha^m + \frac{h^2}{2}u_\alpha^m \\
& + \int_{t_\alpha^m}^{t_\alpha^m+h}(t_\alpha^m+h-s)f(s,x_\alpha^m,y_\alpha^m)ds, P(x_\alpha^m)) + \frac{h^2}{16m} \\
\leq\ & ||x_p^m - x_\alpha^m|| + h||y_p^m - y_\alpha^m|| + \frac{h^2}{2}||u_p^m - u_\alpha^m|| \\
& + ||\int_{t_p^m}^{t_p^m+h}(t_\alpha^m+h-s)f(s,x_p^m,y_p^m)ds - \int_{t_\alpha^m}^{t_\alpha^m+h}(t_\alpha^m+h-s)f(s,x_\alpha^m,y_\alpha^m)ds|| \\
& + \frac{h^2}{16m} + \frac{h^2}{16m} \\
\leq\ & \frac{h^2}{24m} + \frac{h^2}{24m} + \frac{h^2}{24m} + \frac{h^2}{16m} + \frac{h^2}{16m} = \frac{h^2}{8m} + \frac{h^2}{8m} = \frac{h^2}{4m}.
\end{aligned}$$

Thus $h \in H_p^m$, for all $p \geq \max(p_1, p_2)$. From the choice of h_p^m we have

$$\frac{1}{2}\sup H_p^m \leq h_p^m \leq \sup H_p^m.$$

Hence, $h_p^m \geq \frac{h}{2} > \frac{h}{4}$, for all $p \geq \max(p_1, p_2)$, this means that $\lim_{p\to\infty} h_p^m = \lim_{p\to\infty}(t_{p+1}^m - t_p^m)$ can not equal to zero, which contradicts with the fact that the sequence $\{t_p^m\}_{p\geq 1}$ is convergent. So, the process must be finite. □

Theorem 3.2. *In addition to the assumptions of Lemma 3.1 we suppose that the graph of P is closed and the following condition is satisfied.*

(H5) for all $x \in K$ and all $y \in P(x)$ we have $P(y) \subseteq P(x)$.

Then for all $(\varphi_0, y_0) \in K_0 \times \Omega$ there exist $T > 0$ and an absolutely continuous function $x : [0,T] \to K$ with absolutely continuous derivative such that

$$(Q)\ \begin{cases} x''(t) \in F(\tau(t)x, x'(t)) + f(t, x(t), x\prime(t)) \text{ a.e. on } [0,T], \\ x(t) = \varphi_0(t), \forall t \in [-\sigma, 0], \\ x'(0) = y_0, \\ x(t) \in P(x(t)) \subseteq K, \forall t \in [0,T], \\ x(t) \preceq x(s), \text{ whenever } 0 \leq t \leq s \leq T\ . \end{cases}$$

Proof. According to the definition of x_m, for all $m \geq 1$ and all $p = 0, 1, \dots, \nu_{m-1}$ we have $\forall t \in \left[t_p^m, t_{p+1}^m\right]$

$$x_m'(t) = y_p^m + (t - t_p^m) z_p^m + \frac{d}{dt}\left[t \int_{t_p^m}^t f(s, x_p^m, y_p^m) ds - \int_{t_p^m}^t s f(s, x_p^m, y_p^m) ds\right] =$$

$$= y_p^m + (t - t_p^m) z_p^m + t \frac{d}{dt} \int_{t_p^m}^t f(s, x_p^m, y_p^m) ds +$$

$$+ \int_{t_p^m}^t f(s, x_p^m, y_p^m) ds - \frac{d}{dt} \int_{t_p^m}^t s f(s, x_p^m, y_p^m) ds =$$

$$= y_p^m + (t - t_p^m) z_p^m + t f(t, x_p^m, y_p^m) ds + \int_{t_p^m}^t f(s, x_p^m, y_p^m) ds - t f(t, x_p^m, y_p^m) ds,$$

and

$$x_m''(t) = z_p^m + f(t, x_p^m, y_p^m), \ \forall t \in \left]t_p^m, t_{p+1}^m\right[.$$

Then from (ii) and (v) of Lemma 3.1 we get

$$\begin{aligned} ||x_m'(t)|| &\leq ||y_p^m|| + (t - t_p^m)||z_p^m|| + ||\int_{t_p^m}^t f(s, x_p^m, y_p^m) ds|| \qquad (11) \\ &= ||y_0^m|| + \frac{r}{4} + T(M+1) + \frac{r}{4} \\ &= ||y_0^m|| + \frac{r}{4} + \frac{r}{4} + \frac{r}{4} \\ &= ||y_0^m|| + \frac{3r}{4}, \forall t \in [0, T], \end{aligned}$$

and

$$||x_m''(t)|| \leq ||z_p^m|| + ||f(t, x_p^m, y_p^m)|| \leq M + 1 + m(t), \text{ a.e on } t \in [0, T]. \qquad (12)$$

Then the sequences (x_m) and (x_m') are equicontinuous in $C([0,T], \mathbb{R}^n)$. Applying Ascoli-Arzela theorem, there is a subsequence of (x_m) denoted again by (x_m), an absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ with absolutely continuous derivative x' such that (x_m) converges uniformly to x on $[0, T]$, (x_m') converges uniformly to x' on $[0, T]$ and (x_m'') converges weakly to x'' in $L^2([0,T], \mathbb{R}^n)$. Furthermore, since all the functions x_m equal φ_0 on $[-\sigma, 0]$, we can say that x_m converges uniformly to x on $[-\sigma, T]$ where $x = \varphi_0$ on $[-\sigma, 0]$.

Now, for each $t \in [0, T]$ and each $m \geq 1$, let $\delta_m(t) = t_p^m$, $\theta_m(t) = t_{p+1}^m$. If $t \in \left]t_p^m, t_{p+1}^m\right[$ and $\delta_m(0) = \theta_m(0) = 0$, for $t \in \left]t_p^m, t_{p+1}^m\right[$ we get

$$\begin{aligned} x_m''(t) &= z_p^m + f(t, x_p^m, y_p^m) \\ &\in F(\tau(t_p^m) x_m, y_p^m) + \frac{1}{m} B(0,1) + f(t, x_p^m, y_p^m) \\ &= F(\tau(\delta_m(t)) x_m, x_m'(t_p^m)) + \frac{1}{m} B(0,1) + f(t, x_p^m, y_p^m). \end{aligned}$$

Thus for all $m \geq 1$ and a.e. on $[0,T]$,

$$x''_m(t) \in F(\tau(\delta_m(t))x_m, x'_m(\delta_m(t))) + \frac{1}{m}B(0,1) + f(t, x_p^m, y_p^m). \tag{13}$$

Also, for all $m \geq 1$ and all $t \in [0,T]$,

$$\tau(\theta_m(t))x_m \in B_\sigma(\varphi_0, r) \cap K_0, \tag{14}$$

$$x_m(t) \in B(\varphi_0(0), r), \tag{15}$$

$$x_m(\theta_m(t)) \in P(x_m(\delta_m(t))) \subseteq K. \tag{16}$$

Claim. For each $t \in [0,T]$, $\lim_{m\to\infty} \tau(\theta_m(t))x_m = \tau(t)x$ in $C([-\sigma,0],\mathbb{R}^n)$.
Let $t \in [0,T]$, then

$$\begin{aligned}
||\tau(\theta_m(t))x_m - \tau(t)x||_\sigma &\leq ||\tau(\theta_m(t))x_m - \tau(t)x_m||_\sigma + ||\tau(t)x_m - \tau(t)x||_\sigma \leq \\
&\leq \sup_{-\sigma\leq s\leq 0} ||x_m(\theta_m(t)+s) - x_m(t+s)|| + ||\tau(t)x_m - \tau(t)x||_\sigma \leq \\
&\leq \sup_{\substack{-\sigma\leq s_1\leq s_2\leq T\\ |s_2-s_1|\leq \frac{1}{m}}} ||x_m(s_2) - x_m(s_1)|| + ||\tau(t)x_m - \tau(t)x||_\sigma \leq \\
&\leq \sup_{\substack{-\sigma\leq s_1\leq s_2\leq T\\ |s_2-s_1|\leq \frac{1}{m}}} ||x_m(s_2) - x_m(s_1)|| + \sup_{\substack{-\sigma\leq s_1\leq 0\leq s_2\leq T\\ |s_2-s_1|\leq \frac{1}{m}}} ||x_m(s_2) - x_m(s_1)|| + \\
&\quad + \sup_{\substack{-\sigma\leq s_1\leq s_2\leq T\\ |s_2-s_1|\leq \frac{1}{m}}} ||x_m(s_2) - x_m(s_1)|| + ||\tau(t)x_m - \tau(t)x||_\sigma \leq \\
&\leq \sup_{\substack{-\sigma\leq s_1\leq s_2\leq 0\\ |s_2-s_1|\leq \frac{1}{m}}} ||\varphi_0(s_2) - \varphi_0(s_1)|| + \sup_{\substack{-\sigma\leq s_1\leq 0\\ |s_1|\leq \frac{1}{m}}} ||x_m(0) - x_m(s_1)|| + \\
&\quad + \sup_{\substack{s_2\leq T\\ |s_2|\leq \frac{1}{m}}} ||x_m(s_2) - x_m(0)|| + + \sup_{\substack{0\leq s_1\leq s_2\leq T\\ |s_2-s_1|\leq \frac{1}{m}}} ||x_m(s_2) - x_m(s_1)|| + ||\tau(t)x_m - \tau(t)x||_\sigma \leq \\
&\leq 2 \sup_{\substack{-\sigma\leq s_1\leq s_2\leq 0\\ |s_2-s_1|\leq \frac{1}{m}}} ||\varphi_0(s_2) - \varphi_0(s_1)|| + 2 \sup_{\substack{0\leq s_1\leq s_2\leq T\\ |s_2-s_1|\leq \frac{1}{m}}} ||x_m(s_2) - x_m(s_1)|| \\
&\quad + ||\tau(t)x_m - \tau(t)x||_\sigma
\end{aligned}$$

Using the continuity of φ_0, the fact that (x'_m) is uniformly bounded, the uniform convergence of (x_m) towards x and the preceding estimate, we get

$$\lim_{m\to\infty} ||\tau(\theta_m(t))x_m - \tau(t)x||_\sigma = 0.$$

Similarly, for each $t \in [0,T]$,

$$\lim_{m\to\infty} ||\tau(\delta_m(t))x_m - \tau(t)x||_\sigma = 0. \tag{17}$$

Also, since $\lim_{m\to\infty} \delta_m(t) = t$ and (x''_m) is uniformly bounded, then

$$\lim_{m\to\infty} x'_m(\delta_m(t)) = x'(t) \,, \forall t \in [0,T]. \tag{18}$$

Thus by the upper semicontinuity of F and by (14) we obtain

$$x''(t) \in \overline{Co}F(\tau(t)x, x'(t)) + f(t, x(t), x\prime(t)) \text{ a.e. on } [0,T]. \tag{19}$$

which implies to

$$x''(t) - f(t, x(t), x\prime(t)) \in \partial V(x'(t)) \text{ a.e. on } [0,T]. \tag{20}$$

By (20) and [5, Lemma 3.3], we obtain

$$\begin{aligned} \frac{d}{dt}(V(x'(t))) &= < x''(t) - f(t, x(t), x'(t)) > \text{ a.e. on } [0,T] \\ &= < x''(t), x''(t) > - < x''(t), f(t, x(t), x'(t)) > \\ &= ||x''(t)||^2 - < x''(t), f(t, x(t), x'(t)) > . \end{aligned}$$

therefore,

$$V(x'(T)) - V(x'(t_0)) = \int_0^T ||x''(t)||^2 \, dt - \int_0^T < x''(t), f(t, x(t), x'(t)) > dt. \tag{21}$$

On the other hand, for $p = 0, 1, ..., \nu_{m-2}$ and $t \in \left] t_p^m, t_{p+1}^m \right[$,

$$(x''_m(t) - f(t, x_m(t_p^m), x'_m(t_p^m))) \in F(\tau(t_p^m)x_m, x'_m(t_p^m)) + \frac{1}{m}B(0,1).$$

Then

$$(x''_m(t) - f(t, x_m(t_p^m), x'_m(t_p^m))) \in \partial V(x'_m(t_p^m)) + \frac{1}{m}B(0,1),$$

hence, there exists $b_p \in B(0,1)$ such that

$$(x''_m(t) - f(t, x_m(t_p^m), x'_m(t_p^m)) + \frac{1}{m}b_p) \in \partial V(x'_m(t_p^m)). \tag{22}$$

Definition and properties of the subdifferential of a convex function imply that for ξ in $\partial V(x'_m(t_p^m))$, we have

$$V(x'_m(t_{p+1}^m)) - V(x'_m(t_p^m)) \geq < x'_m(t_{p+1}^m) - x'_m(t_p^m), \xi > . \tag{23}$$

Then by (23)

$$
\begin{aligned}
& V(x'_m(t^m_{p+1})) - V(x'_m(t^m_p)) \\
\geq \quad & < x'_m(t^m_{p+1}) - x'_m(t^m_p), x''_m(t) - f(t, x_m(t^m_p), x'_m(t^m_p)) + \frac{1}{m} b_p > ,
\end{aligned}
$$

thus and since x''_m is constant in $\left] t^m_p, t^m_{p+1} \right[$, it follows that

$$
V(x'_m(t^m_{p+1})) - V(x'_m(t^m_p)) \geq
$$

$$
\geq \quad < \int_{t^m_p}^{t^m_{p+1}} x''_m(t)\, dt, x''_m(t) - f(t, x_m(t^m_p), x'_m(t^m_p)) + \frac{1}{m} b_p >
$$

$$
\geq \int_{t^m_p}^{t^m_{p+1}} < x''_m(t)\, , x''_m(t) > dt - \int_{t^m_p}^{t_{p+1}} < x''_m(t)\, , f(t, x_m(t^m_p), x'_m(t^m_p)) > dt +
$$

$$
+ \int_{t^m_p}^{t^m_{p+1}} < x''_m(t)\, , \frac{1}{m} b_p > dt,
$$

hence we have

$$
V(x'_m(T)) - V(y^m_0)) \geq \int_0^T ||x''_m(t)||^2\, dt -
$$

$$
- \sum_{p=0}^{\nu_{m-2}} \int_{t^m_p}^{t^m_{p+1}} < x''_m(t)\ , f(t, x_m(t^m_p), x'_m(t^m_p)) > dt + \sum_{p=0}^{\nu_{m-2}} \int_{t^m_p}^{t^m_{p+1}} < x''_m(t)\ , b_p > dt.
$$

Since $[0,T] = \cup_{p=0}^{\nu_{m-2}} [t^m_p, t^m_{p+1}]$, we have

$$
|| \sum_{p=0}^{\nu_{m-2}} \int_{t^m_p}^{t^m_{p+1}} < x''_m(t)\ , f(t, x_m(t^m_p), x'_m(t^m_p)) > dt - \int_0^T < x''(t), f(t, x(t), x\prime(t)) > dt||
$$

$$
= || \sum_{p=0}^{\nu_{m-2}} \int_{t^m_p}^{t^m_{p+1}} \left(< x''_m(t)\ , f(t, x_m(t^m_p), x'_m(t^m_p)) > - < x''(t), f(t, x(t), x\prime(t)) > \right) dt||
$$

$$
\leq \sum_{p=0}^{\nu_{m-2}} \int_{t^m_p}^{t^m_{p+1}} || < x''_m(t)\ , f(t, x_m(t^m_p), x'_m(t^m_p)) > - < x''(t), f(t, x(t), x\prime(t)) > ||dt \leq
$$

$$\leq \sum_{p=0}^{\nu_{m-2}} \int_{t_p^m}^{t_{p+1}^m} || < x_m''(t)\ , f(t, x_m(t_p^m), x_m'(t_p^m)) > - < x_m''(t), f(t, x_m(t), x_m'(t)) > || dt +$$

$$+ \sum_{p=0}^{\nu_{m-2}} \int_{t_p^m}^{t_{p+1}^m} || < x_m''(t)\ , f(t, x_m(t), x_m'(t)) > - < x_m''(t), f(t, x(t), x\prime(t)) > || dt +$$

$$+ \sum_{p=0}^{\nu_{m-2}} \int_{t_p^m}^{t_{p+1}^m} || < x_m''(t)\ , f(t, x(t), x'(t)) > - < x''(t), f(t, x(t), x\prime(t)) > || dt =$$

$$= \sum_{p=0}^{\nu_{m-2}} \int_{t_p^m}^{t_{p+1}^m} || < x_m''(t)\ , f(t, x_m(t_p^m), x_m'(t_p^m)) > - < x_m''(t), f(t, x_m(t), x_m'(t)) > || dt +$$

$$+ \int_0^T || < x_m''(t), f(t, x_m(t), x_m'(t)) > - < x_m''(t), f(t, x(t), x'(t)) > || dt +$$

$$+ \int_0^T || < x_m''(t), f(t, x(t), x'(t)) - \ < x''(t), f(t, x_m(t), x_m'(t)) > || dt.$$

Since f is a Caratheodory function, x_m and x_m' are uniformly continuous,

$$\left\| x_m''(s) \right\| \leq M + 1 + m(s),\ m \in L^2([0,T], \mathbb{R}^n),\ x_m \to x,\ x_m' \to x'$$

uniformly and $x_m'' \to x''$ weakly in $L^2([0,T], \mathbb{R}^n)$ then the last term converges to 0. Hence

$$\lim_{m\to\infty} \sum_{p=0}^{\nu_{m-2}} \int_{t_p^m}^{t_{p+1}^m} < x_m''(t)\ , f(t, x_m(t_p^m), x_m'(t_p^m)) > dt =$$

$$\int_0^T < x''(t), f(t, x(t), x\prime(t)) > dt.$$

Since

$$\lim_{m\to\infty} \sum_{p=0}^{\nu_{m-2}} \frac{1}{m} \int_{t_p^m}^{t_{p+1}^m} < x_m''(t), b_p > dt = 0,$$

by passing to the limit as $m \to \infty$ in (24) and using the contiuity of the function V on the ball $B(y_0^m, r)$, we obtain the estimate

$$V(x'(T)) - V(x'(t_0)) \geq \lim_{m\to\infty} \sup \int_0^T ||x''(t)||^2\, dt - \int_0^T < x''(t), f(t, x(t), x'(t)) > dt.$$

Moreover, by (21) we have

$$\left\|x''\right\|_2^2 \geq \lim_{m\to\infty} \sup ||x_m''||_2^2,$$

and by the weak lower semicontinuity of the norm, it follows that

$$\left\|x''\right\|_2^2 \leq \lim_{m\to\infty} \sup ||x_m''||_2^2.$$

Hence,

$$\lim_{m\to\infty} ||x_m''||_2^2 = \left\|x''\right\|_2^2. \tag{24}$$

Let us show that

$$x''(t) \in F(\tau(t)x, x'(t)) + f(t, x(t), x\prime(t)) \text{ a.e. on } [0, T]. \tag{25}$$

Let $t \in [0, T[$. By the above construction, there exists $p \in \{0, 1, ..., q\}$ such that $t \in \left[t_p^m, t_{p+1}^m\right[$ and $\lim\limits_{m\to\infty} t_p^m = t$. From (13), for each positive integer m,

$$x_m''(t) - f(t, x_m\left(t_p^m\right), x_m'\left(t_p^m\right)) \in F(\tau(t_p^m)x_m, x_m'(t_p^m)) + \frac{1}{m}B(0, 1).$$

Using the relations (17), (18), (25) and the fact that f is is a Caratheodory and the set-valued function F is u.s.c., we obtain the relation (26).

Furthermore, since K is closed and the graph of P is closed we get $x(t) \in P(x(t)) \subseteq K$.

It remains to prove the following two properties

(1) $(x(t), x'(t)) \in K \times \Omega$, $\forall t \in [0, T]$.

(2) $x(s) \in P(x(t))$, $\forall t, s \in [0, T]$ and $t \leq s$.

To prove the first property we note that the property (iii) of Lemma 3.1 implies that $x_m(\delta_m(t)) \in \overline{B(\varphi_0(0), r)} \cap K$ and $x_m'(\delta_m(t)) \in \overline{B(y_0, r)} \cap \Omega$. Since $\lim\limits_{m\to\infty} x_m(\delta_m(t)) = x(t)$ and $\lim\limits_{m\to\infty} x_m'(\delta_m(t)) = x'(t)$ then $x(t) \in \overline{B(\varphi_0(0), r)} \cap K$ and $x'(t) \in \overline{B(y_0, r)} \cap \Omega$.

To prove the second property, let $t, s \in [0, T]$ be such that $t \leq s$. Then for m large enough, we can find $p, q \in \{0, 1, 2, ..., \nu_{m-2}\}$ such that $p > q, t \in [t_q^m, t_{q+1}^m]$

and $s \in [t_p^m, t_{p+1}^m]$. Assume that $j = p - q$. Using property (v) of Lemma 3.1 and condition (H5) we get

$$P(x_m(t_p^m)) \subseteq P(x_m(t_{p-1}^m)) \subseteq P(x_m(t_{p-2}^m)) \subseteq \ldots \subseteq P(x_m(t_q^m)).$$

This implies $P(x_m(\delta_m(s))) \subseteq P(x_m(\delta_m(t)))$. Since $x_m(\delta_m(s)) \in P(x_m(\delta_m(s)))$, it follows that $x_m(\delta_m(s)) \in P(x_m(\delta_m(t)))$. By taking the limit as m tends to ∞ we obtain that $x(s) \in P(x(t))$ and hence the second property is proved. $\square$

Remark 3.3. If $f \equiv 0$, then the condition (H3) takes the following form: for all $(t, \varphi, y) \in I \times K_0 \times \Omega$, there exists $z \in F(\varphi, y)$,

$$\lim_{h \longrightarrow 0^+} \inf \frac{1}{h^2} d(\varphi(0) + hy + \frac{h^2}{2} z, P(\varphi(0))) = 0,$$

this means that for all $(t, \varphi, y) \in I \times K_0 \times \Omega$, there exists $z \in F(\varphi, y)$ such that

$$F(\varphi, y) \subseteq T^2_{P(\varphi(0))}(\varphi(0), y).$$

So, our work in the present paper generalizes the work authors in [13].

Acknowledgements

The authors would like to thank the anonymous referees for their valuable suggestions and pertinent comments which allow us to improve the presentation of the paper.

REFERENCES

[1] S. Amine - R. Morchadi - S. Sajid, *Caratheodory perturbation of a second-order differential inclusion with constrants*, Electronic J. Diff. Equs. 114 (2005), 1–11.

[2] J. P. Aubin - A. Cellina, *Differential inclusions*, Springer-Verlag, New York, Berlin, 1984.

[3] A. Auslender - J. Mechler, *Second order viability problems for differential inclusions*, J. Math. Anal. Appl. 181 (1994), 205–218.

[4] A. Bressen - A. Cellina - G. Colombo, *Upper semicontinuous differential inclusions without convexity*, Proc. Amer. Math. Soc. 106 (1989), 771–775.

[5] H. Brezis, *Operateurs Maximaux Monotones et Semigroups de Contractions dans les Espaces de Hilbert*, North Holland, Amesterdam, 1973.

[6] C. Castaing - M. Valadier, *Convex Analysis and Measurable Multifunctions*, Springer-Verlag, New York, Berlin, 1977.

[7] A. Cernea, *On the existence of viable solutions for a class of second order differential inclusions*, Discuss. Math. Differ. Incl. 22 (1) (2002), 67–78.

[8] A. Cernea - V. Lupulescu, *Viable solutions for a class of nonconvex functional differential inclusions*, Math. Reports 7 (57) N. 2 (2005), 91–103.

[9] F. H. Clarck - J. P. Aubin, *Monotone invariant solutions to differential inclusions*, J. London Math. Soc. 16 (2) (1977), 357–366.

[10] B. Cornet - G. Haddad, *Theorem de viabilité pour inclusions differentiells du second order*, Israel J. Math. 57 (1987),225–238.

[11] G. Haddad, *Monotone trajectories for functional differential inclusions with memory*, Israel J. Math. 39 (1981), 83–100.

[12] G. Haddad, *Monotone trajectories for functional differential inclusions*, J. Diff. Eqs. 42 (1981), 1–24.

[13] A. G. Ibrahim - F. A. AL-Adsani, *Monotone solutions for a nonconvex functional differential inclusions of second order*, Electronic J. Diff. Equs. 144 (2008), 1–16.

[14] A. G. Ibrahim - F. A. AL-Adsani, *Second Order Evolution Inclusions Governed By Sweeping Process In Banach Spaces*, Le Matematiche 64 (2) (2009), 17–39.

[15] A. G. Ibrahim - K. S. Alkulaibi, *On existence of monotone solutions for second-order non-convex differential inclusions in infinite dimensional spaces*, Portugalies Mathematica, 61 (2) (2004), 231–143.

[16] V. Lupulescu, *A viability result for nonconvex second order differential inclusions*, Electronic J.Diff. Equs. 76 (2002), 1–12.

[17] V. Lupulescu, *Existence of solutions for nonconvex second order differential inclusions*, Applied Mathematics E-Note, 3 (2003), 115–123.

[18] V. Lupulescu, *Existence of solutions for nonconvex functional differential inclusions*, Electronic J. Diff. Equs. 141 (2004), 1–6.

[19] V. Lupulescu, *Viable solutions for second order nonconvex functional differential inclusions*, Electronic J. Diff. Equs. 110 (2005),1–11.

[20] L. Malaguti, *Monotone trajectories differential inclusions in Banach spaces*, J. Convex Analysis 3 (2) (1996), 269–281.

[21] P. Rossi, *Viability for upper semicontinuous differential inclusions*, Diff. Integral Equations 5 (1992), 455–459.

A. G. IBRAHIM
Mathematics Department
Factuly of Science
King Faisal University
e-mail: agamal2000@yahoo.com

F. A. ALADSANI
Mathematics Department
Factuly of Science
King Faisal University
e-mail: faladsani@hotmail.com