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LINEAR FRACTIONAL TRANSPORTATION PROBLEM WITH VARYING DEMAND AND SUPPLY

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In this paper, we investigate the transportation problem with fractional objective function when the demand and supply quantities are varying. A set of mathematical programs is obtained to determine the objective value. Due to varying economic policies in the global world, it is hard to specify the supply and demand quantities for transportation problem. A transportation problem with fractional objective function is based on a network structure consisting of a finite number of nodes and arcs attached to them. After the derivation, we obtained the result in range, where the total transportation cost would appear. In addition to allowing for simultaneous changes in supply and demand values, the total cost bounds are calculated directly. This methodology would be very beneficial in the decision making. A numerical example is given in this paper for the support of theory.

1. Introduction

The fractional transportation problem was originally proposed by Swarup [3] in 1966 and it has an important role in logistics and supply chain management for reducing cost and improving service. Transportation problem could be defined as a linear fractional programming problem originates from network models consisting of a finite number of nodes and arcs. It deals with the situation in

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which a commodity is shipped from sources to destinations. The main objective of this paper is to determine the amounts shipped from each source to each destination that minimizes the total cost while satisfying both the supply limits and the demand requirements. The study about more-for less paradox [1], [5] in fractional transportation model can be help to increase demand and supply at some nodes. It would be helpful for us to save money and make better decisions to improving the routs.

In real world applications, however, the supply and demand quantities in the transportation problem are sometimes hardly specified precisely because of changing weather, social, or economic conditions. With the development of the market-oriented economy, market takes a more and more important role for adjusting the relationship of supply and demand. Under this background, the character and the rule of supply and demand will take on a flexible characteristic. Obviously, it is very feasible for the regulation of the supply and demand by using the concept of elasticity. Transportation models provide a powerful framework to meet this challenge.

Several efficient algorithms have been developed over the past decades for solving the transportation problem when the cost coefficients and the supply and demand values are known exactly. However, there are cases when these parameters may not be presented in a precise manner. For example, the customer demand and supplier capabilities might change over time and the supply-chain relationships might also evolve over time. Obviously, when the supply and demand quantities are varying in specific ranges, the total transportation cost will vary within an interval as well. In this paper, we develop a procedure that can directly calculate the total cost bounds of the transportation problem, where at least one of the supply or demand is varying. A pair of mathematical programs is constructed to calculate the lower and upper bounds of the total transportation cost. In this paper we extend the work done by S. T. Liu [4]. We studied the total cost bounds with varying demand and supply for the transportation problem with fractional objective function.

The work done in this article is as follows. Firstly, we define the fractional transportation problem with varying supply and demand quantities briefly. Then a pair of mathematical programs is formulated for calculating the bounds of the total transportation cost. A numerical example is given in this paper for the support of this theory. Finally, some conclusions are drawn in the favour of this work.

2. Mathematical Model

In the conventional transportation problem, a homogeneous product is to be transported from several sources to several destinations in such a way that the total transportation cost is a minimum. Suppose that there are m supply nodes and n demand nodes. The i^{th} supply node can provide s_i units of a certain product and the j^{th} demand node has a demand for d_i units. The fractional transportation of products from the i^{th} supply node to the j^{th} demand node carries a cost of c_{ij} and l_{ij} per unit of product transported. The problem is to determine a feasible way of transporting the available amounts, to satisfy demand that minimizes the total transportation cost with the maximum profit.

Let x_{ij} be the number of units transported from supply i to demand j. The mathematical description of the conventional transportation problem is

$$\text{Min} \frac{\sum_i \sum_j c_{ij} x_{ij}}{\sum_i \sum_j l_{ij} x_{ij}} \tag{1}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq s_i,\ i = 1,2,...,m,$$
$$\sum_{i=1}^{m} x_{ij} = d_j,\ j = 1,2,...,n,$$
$$x_{ij} \geq 0 \quad \forall i,\ j.$$

The first set of constraints stipulates that the sum of the shipments from a source cannot exceed its supply; the second set requires that the sum of the shipments to a destination must satisfy its demand. The above problem implies that the total supply $\sum_i s_i$ must be greater than or equal to total demand $\sum_j d_j$.

Suppose the value of supply s_i and the value of demand d_j are varying, and they can be represented by $\hat{s}_i$ and $\hat{d}_j$ with the lower and upper bounds $[\underline{S}_i, \overline{S}_i]$ and $[\underline{D}_j, \overline{D}_j]$, respectively. The fractional transportation problem with varying supply and demand has the following mathematical form:

$$\text{Min} \frac{\sum_i \sum_j c_{ij} x_{ij}}{\sum_i \sum_j l_{ij} x_{ij}} \tag{2}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq \hat{s}_i,\ i = 1,2,...,m,$$
$$\sum_{i=1}^{m} x_{ij} = \hat{d}_j,\ j = 1,2,...,n,$$
$$x_{ij} \geq 0 \quad \forall i,\ j,$$

where $\hat{s}_i \in [\underline{S}_i, \overline{S}_i]$ and $\hat{d}_j \in [\underline{D}_j, \overline{D}_j]$ $\forall i, j$.

Intuitively, if any of the parameters is varying, the objective value should be varying as well. In next section, we shall develop the solution procedure for the fractional transportation problem with varying supply and demand quantities.

3. Solution Procedure

Suppose we are interested in deriving the lower and upper bounds of the total transportation cost. The major difficulty lies in how to deal with the varying supply and demand quantities

Let $G = \{(\hat{s},\hat{d}) \mid \underline{S}_i \leq \hat{s}_i \leq \overline{S}_i, \underline{D}_j \leq \hat{d}_j \leq \overline{D}_j, i = 1,2,...,m,\ j = 1,2,...,n\}$.

For each $(\hat{s},\hat{d}) \in G$, we denote $Z(\hat{s},\hat{d})$ to be the objective value of Model (2). Let $\underline{Z}$ and $\overline{Z}$ represents respectively the minimum and the maximum of $Z(\hat{s},\hat{d})$ on G. Then

$$\underline{Z} = \text{Min}\left\{Z(\hat{s},\hat{d}) \mid (\hat{s},\hat{d}) \in G\right\} \tag{3}$$

$$\overline{Z} = \text{Max}\left\{Z(\hat{s},\hat{d}) \mid (\hat{s},\hat{d}) \in G\right\} \tag{4}$$

From Eqs. (3) and (4), we obtain the equivalent pair of two-level mathematical programs:

$$\underline{Z} = \underset{(\hat{s},\hat{d})\in G}{\text{Min}}\ \underset{x}{\text{Min}} \frac{\sum_i \sum_j c_{ij}x_{ij}}{\sum_i \sum_j l_{ij}x_{ij}} \tag{5a}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq \hat{s}_i,\ i = 1,2,...,m,$$
$$\sum_{i=1}^{m} x_{ij} = \hat{d}_j,\ j = 1,2,...,n,$$
$$x_{ij} \geq 0 \quad \forall i,j.$$

$$\overline{Z} = \underset{(\hat{s},\hat{d})\in G}{\text{Max}}\ \underset{x}{\text{Min}} \frac{\sum_i \sum_j c_{ij}x_{ij}}{\sum_i \sum_j l_{ij}x_{ij}} \tag{5b}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq \hat{s}_i,\ i = 1,2,...,m,$$
$$\sum_{i=1}^{m} x_{ij} = \hat{d}_j,\ j = 1,2,...,n,$$
$$x_{ij} \geq 0 \quad \forall i,j.$$

Model (5a and 5b) is feasible if and only if $\sum_{i=1}^{m} \hat{s}_i \geq \sum_{j=1}^{n} \hat{d}_j$. Since $\hat{s}_i$ and $\hat{d}_j$ are allowed to vary within the ranges of $[\underline{S}_i, \overline{S}_i]$ and $[\underline{D}_j, \overline{D}_j]$, respectively, it is necessary that the constraint $\sum_{i=1}^{m} \hat{s}_i \geq \sum_{j=1}^{n} \hat{d}_j$ be imposed in the first level of Model (5a and 5b) to ensure the fractional transportation problem to be feasible in the second level of Model (5a and 5b). Therefore, Model (5a and 5b) becomes:

$$\underline{Z} = \underset{\substack{(\hat{s},\hat{d})\in G \\ \sum_{i=1}^{m} \hat{s}_i \geq \sum_{j=1}^{n} \hat{d}_j}}{\text{Min}} \ \underset{x}{\text{Min}} \frac{\sum_i \sum_j c_{ij} x_{ij}}{\sum_i \sum_j l_{ij} x_{ij}} \tag{6a}$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &\leq \hat{s}_i, \ i = 1,2,...,m, \\ \sum_{i=1}^{m} x_{ij} &= \hat{d}_j, \ j = 1,2,...,n, \\ x_{ij} &\geq 0 \quad \forall i,j. \end{aligned}$$

$$\overline{Z} = \underset{\substack{(\hat{s},\hat{d})\in G \\ \sum_{i=1}^{m} \hat{s}_i \geq \sum_{j=1}^{n} \hat{d}_j}}{\text{Max}} \ \underset{x}{\text{Min}} \frac{\sum_i \sum_j c_{ij} x_{ij}}{\sum_i \sum_j l_{ij} x_{ij}} \tag{6b}$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &\leq \hat{s}_i, \ i = 1,2,...,m, \\ \sum_{i=1}^{m} x_{ij} &= \hat{d}_j, \ j = 1,2,...,n, \\ x_{ij} &\geq 0 \quad \forall i,j. \end{aligned}$$

Since Model (6a) is to find the minimum value against the best possible value on G, one can directly insert the constraints of level 1 into level 2 and simplify the two-level mathematical program to the conventional one-level mathematical program as follows:

$$\underline{Z} = \underset{x}{\text{Min}} \frac{\sum_i \sum_j c_{ij} x_{ij}}{\sum_i \sum_j l_{ij} x_{ij}} \tag{7}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq \hat{s}_i,\ i = 1,2,...,m,$$
$$\sum_{i=1}^{m} x_{ij} = \hat{d}_j,\ j = 1,2,...,n,$$
$$x_{ij} \geq 0 \quad \forall i,j,$$
$$\sum_{i=1}^{m} \hat{s}_i \geq \sum_{j=1}^{n} \hat{d}_j$$
$$\underline{S}_i \leq \hat{s}_i \leq \overline{S}_i, \quad \underline{D}_j \leq \hat{d}_j \leq \overline{D}_j, \quad \forall i,j.$$

Model (7) is a linear fractional program that can be solved easily. If the lower bound of the total supply is greater than or equal to the upper bound of total demand, i.e., $\sum_{i=1}^{m} \underline{S}_i \geq \sum_{j=1}^{n} \overline{D}_j$, the constraint $\sum_{i=1}^{m} \hat{s}_i \geq \sum_{j=1}^{n} \hat{d}_j$ can be deleted from Model (7). In this case the benefit of network structure would be manifested.

Model (6b) is to find the maximum value among the best possible objective values over all decision variables. The well-known duality theorem of linear fractional programming indicates that the primal model and the dual model have the same objective value [2]. Consequently, to derive the upper bound of total transportation cost in Model (6b), the dual of the level 2 problem is formulated to become a maximization problem to be consistent with the maximization operation of the level 1. Hence Model (6b) is reformulated as

$$\underset{\substack{(\hat{s},\hat{d})\in G \\ \sum_{i=1}^{m} \hat{s}_i \geq \sum_{j=1}^{n} \hat{d}_j}}{\text{Max}} \quad \text{Max } y_0 \tag{8}$$

subject to

$$-\sum_{i=1}^{m} \hat{s}_i v_i + \sum_{j=1}^{n} \hat{d}_j w_j \geq 0,$$
$$l_{ij} y_0 + v_i - w_j \geq c_{ij},\ i = 1,2,...,m,\ j = 1,2,...,n,$$
$$v_i \geq 0,\ i = 1,2,...,m, w_j \text{ is unrestricted in sign}, j = 1,2,...,n,$$

where y_0 denotes a vector containing $n+m-1$ components

$$y_0, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n.$$

Since both levels 1 and 2 problems perform the same maximization operation, their constraints can be combined together to become the one-level mathematical program as follows:

$$\overline{Z} = \underset{y_0}{\text{Max}}\ y_0 \tag{9}$$

subject to

$$-\sum_{i=1}^{m} \hat{s}_i v_i + \sum_{j=1}^{n} \hat{d}_j w_j \geq 0,$$
$$l_{ij} y_0 + v_i - w_j \geq c_{ij},\ i = 1,2,...,m,\ j = 1,2,...,n,$$
$$\sum_{i=1}^{m} \hat{s}_i \geq \sum_{j=1}^{n} \hat{d}_j,$$
$$\underline{S}_i \leq \hat{s}_i \leq \overline{S}_i,\ \underline{D}_j \leq \hat{d}_j \leq \overline{D}_j,\ \forall i,j,$$
$$v_i \geq 0,\ i = 1,2,...,m,\ w_j \text{ is unrestricted in sign},\ j = 1,2,...,n.$$

Model (9) is a **linear program**.There are several effective and efficient methods for solving this problem [2].

4. Numerical Example

To illustrate the proposed approach, consider a fractional transportation problem with varying supply and demand quantities. The problem has the following form:

$$\text{Min}\ \frac{\sum_i \sum_j c_{ij} x_{ij}}{\sum_i \sum_j l_{ij} x_{ij}} = \frac{35x_{11} + 30x_{12} + 10x_{13} + 5x_{21} + 25x_{22} + 40x_{23}}{13x_{11} + 25x_{12} + 12x_{13} + 7x_{21} + 15x_{22} + 26x_{23}} \tag{10}$$

subject to

$$\begin{aligned}
x_{11} + x_{12} + x_{13} &\leq \hat{s}_1,\\
x_{21} + x_{22} + x_{23} &\leq \hat{s}_2,\\
x_{11} + x_{21} &= \hat{d}_1,\\
x_{12} + x_{22} &= \hat{d}_2,\\
x_{13} + x_{23} &= \hat{d}_3,\\
x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} &\geq 0,\\
\sum_{i=1}^{m} \sum_{j=1}^{n} l_{ij} x_{ij} &> 0,
\end{aligned}$$

where $\hat{s}_1 \in [60,120]$, $\hat{s}_2 \in [75,150]$, $\hat{d}_1 \in [45,90]$, $\hat{d}_2 \in [30,60]$ and $\hat{d}_3 \in [60,120]$.

According to Models (7) and (9), the lower and upper bounds of the total transportation cost $[\underline{Z}, \overline{Z}]$ can be solved as

$$\underline{Z} = \operatorname*{Min}_{x} \frac{\sum_i \sum_j c_{ij} x_{ij}}{\sum_i \sum_j l_{ij} x_{ij}} = \frac{35x_{11} + 30x_{12} + 10x_{13} + 5x_{21} + 25x_{22} + 40x_{23}}{13x_{11} + 25x_{12} + 12x_{13} + 7x_{21} + 15x_{22} + 26x_{23}} \tag{11}$$

subject to

$$\begin{aligned}
x_{11} + x_{12} + x_{13} &\leq \hat{s}_1, \\
x_{21} + x_{22} + x_{23} &\leq \hat{s}_2, \\
x_{11} + x_{21} &= \hat{d}_1, \\
x_{12} + x_{22} &= \hat{d}_2, \\
x_{13} + x_{23} &= \hat{d}_3, \\
\hat{s}_1 + \hat{s}_2 &\geq \hat{d}_1 + \hat{d}_2 + \hat{d}_3, \\
x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} &\geq 0, \\
\sum_{i=1}^{m} \sum_{j=1}^{n} l_{ij} x_{ij} &> 0, \\
60 \leq \hat{s}_1 &\leq 120, \, 75 \leq \hat{s}_2 \leq 150, \\
45 \leq \hat{d}_1 \leq 90, \; 30 \leq \hat{d}_2 &\leq 60 \text{ and } 60 \leq \hat{d}_3 \leq 120, \\
x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} &\geq 0, \; \sum_{i=1}^{m} \sum_{j=1}^{n} l_{ij} x_{ij} > 0.
\end{aligned}$$

$$\overline{Z} = \operatorname*{Max}_{y_0} y_0 \tag{12}$$

subject to

$$\begin{aligned}
-60v_1 - 75v_2 + 45w_1 + 30w_2 + 60w_3 &\leq 0, \\
13y_0 + v_1 - w_1 &\leq 35, \\
25y_0 + v_1 - w_2 &\leq 30, \\
12y_0 + v_1 - w_3 &\leq 10, \\
7y_0 + v_1 - w_1 &\leq 5, \\
15y_0 + v_2 - w_2 &\leq 25, \\
26y_0 + v_3 - w_3 &\leq 40, \\
60 \leq \hat{s}_1 &\leq 120, 75 \leq \hat{s}_2 \leq 150, \\
45 \leq \hat{d}_1 \leq 90, 30 \leq \hat{d}_2 &\leq 60 \text{ and } 60 \leq \hat{d}_3 \leq 120, \\
v_1, v_2 &\geq 0, \; w_1, w_2, w_3 \text{ unrestricted in sign.}
\end{aligned}$$

A fractional programming solver WinGULF [2] is used to solve the above mathematical programs. The lower bound of $\underline{Z}^* = 1.06$ occurs at $x_{13}^* = 60, x_{21}^* =$

$45, x^*_{22} = 30, x^*_{11} = x^*_{12} = x^*_{23} = 0$ with $\hat{s}_1 = 60, \hat{s}_2 = 75, \hat{d}_1 = 45, \hat{d}_2 = 30$ and $\hat{d}_3 = 60$. At the other extreme end, the upper bound of $\overline{Z}^* = 1.33$ occurs at $x^*_{13} = 60, x^*_{21} = 45, x^*_{22} = 45, x^*_{23} = 60, x^*_{11} = x^*_{12} = 0$ with $\hat{s}_1 = 60, \hat{s}_2 = 150, \hat{d}_1 = 45, \hat{d}_2 = 45$ and $\hat{d}_3 = 120$. In solving the upper bound of the objective value , the initial solution provided to WinGULF is their upper bounds of the supply and demand, i.e., $s^0 = [120\,150]^T$ and $d^0 = [90\,60\,120]^T$. With this initial solution, the optimal solution $Z^* = 1.33$ is derived. We can improve the routs using more-for-less paradox. In more-for-less paradox, there is a possibility of shipping more total goods for less or at least an equal total shipping cost, when compared with the optimal cost of transportation from origin to destination, with all shipment costs being non-negative. In other words, it is possible to reduce the total transportation cost by adjusting the warehouse stocking level or changing manufacturing strategy in each supply without sacrificing the demand quantities of customers.

5. Conclusion

In this paper, we described a method to calculate the lower and upper bounds of the total fractional transportation cost when the supply and demand quantities are varying. A set of two-level transportation problems is transformed into the one-level mathematical programs to find out objective value. The derived result is also in range, where the total transportation cost would appear. This paper allowing for simultaneous changes in supply and demand quantities, and the bounds of the total transportation cost are calculated directly. Due to the structure of the transportation problem, the largest total transportation cost may not occur at the highest total quantity shipped. By fine-tuning the production strategy or warehouse stocking level, the total transportation cost could be reduced.

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