LE MATEMATICHE
Vol. LXVI (2011) – Fasc. II, pp. 13–21
doi: 10.4418/2011.66.2.2

APPROXIMATING THE STIELTJES INTEGRAL BY USING THE GENERALIZED TRAPEZOID RULE

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Accurate approximations for the Stieltjes integral by the generalized trapezoid rule. The generalized trapezoid rule is established on the basis of the derivative of function with respect to strictly increasing function, defined in [9].

1. Introduction and the derivative of function with respect to the strictly increasing function

Our aim is to describe the generalized trapezoid rule for the approximation of the Stieltjes integral

$$I = \int_a^b f(x)du(x),\ a < b,\ a, b \in R, \tag{1}$$

where $f(x)$ is a given continuous function on $[a,b]$ and $u(x)$ is a given function of the bounded variation on $[a,b]$.

It is known $[7, p.205]$ that the function $u(x)$ presented in the form

$$u(x) = \varphi(x) - \psi(x),\ x \in [a,b] \tag{2}$$

Entrato in redazione: 15 febbraio 2011

AMS 2010 Subject Classification: 26D15, 26A42

Keywords: Stieltjes integral, Generalized trapezoid rule, The derivative of the function with respect to the strictly increasing function

where $\varphi(x)$ and $\psi(x)$ are the known increasing functions on $[a,b]$.
Different methods of the approximate calculation of the Stieltjes integral in works $[1-6]$ is suggested. Particularly, in 1998 Dragomir and Fedotov [2], in order to approximate the Stieltjes integral (1) with the simpler expression

$$\frac{1}{b-a}[u(b)-u(a)]\int_a^b f(x)dx,$$

they introduced the following error functional

$$D(f,u;a,b) := \int_a^b f(x)dx - \frac{1}{b-a}[u(b)-u(a)]\int_a^b f(x)dx.$$

In this work, for the approximate calculation of the Stieltjes interal (1), the generalized midpoint rule is suggested. This is based on the notion of the derivative of function with respect to the strictly increasing function [9]. The generalized midpoint rule summarizes the midpoint rule [8]. Then we need the concept of the derivative defined in the work [9] and the theorems with proofs connected with it. Apparently the first notion of the derivative, with respect to the strictly increasing function, was introduced in [9].

Definition. The derivative of a function $f(x)$ with respect to $\varphi(x)$ is the function $f'_\varphi(x)$, whose value at $x \in (a,b)$ is the number:

$$f'_\varphi(x) = \lim_{\Delta\to 0}\frac{f(x+\Delta)-f(x)}{\varphi(x+\Delta)-\varphi(x)}, \tag{3}$$

where $\varphi(t)$ is a given strictly increasing continuous function in (a,b).

If the limit in equation (3) exists, we say that $f(x)$ has a derivative (is differentiable) with respect to $\varphi(x)$. The first derivative $f'_\varphi(x)$ may also be a differentiable function with respect to $\varphi(x)$ at every point $x \in (a,b)$. If so, its derivative

$$f''_\varphi(x) = (f'_\varphi(x))'_\varphi,$$

is called the second derivative of $f(x)$ with respect to $\varphi(x)$. The names continue as one can imagine they would, by

$$f^{(n)}_\varphi(x) = (f^{(n-1)}_\varphi(x))'_\varphi$$

denoting the $n-th$ derivative of $f(x)$ with respect to $\varphi(x)$.

Example 1.1. The function $f(x) = |x|$ is nondifferentiable at point $x = 0$. If

$$\varphi(x) = \begin{cases} -|x|^{\frac{1}{3}}, & x < 0, \\ |x|^{\frac{1}{3}}, & x \geq 0, \end{cases}$$

then the function $\varphi(x)$ is the increasing continuous function in $(-\infty,\infty)$. We shall show that the function $f(x)=|x|$ has the continuous derivative by means of $\varphi(x)$ at every point $x\in(-\infty,\infty)$.

Let $x<0$, then from definition,we obtain

$$f'_\varphi(x)=\lim_{\Delta x\to 0}\frac{|x+\Delta x|-|x|}{\varphi(x+\Delta x)-\varphi(x)}=\lim_{\Delta x\to 0}\frac{\left(|x+\Delta x|^{\frac{1}{3}}\right)^3-\left(|x|^{\frac{1}{3}}\right)^3}{-\left(|x+\Delta x|^{\frac{1}{3}}-|x|^{\frac{1}{3}}\right)}=$$

$$=-\lim_{\Delta x\to 0}\frac{\left(|x+\Delta x|^{\frac{1}{3}}-|x|^{\frac{1}{3}}\right)\left(|x+\Delta x|^{\frac{2}{3}}+|x+\Delta x|^{\frac{1}{3}}|x|^{\frac{1}{3}}+|x|^{\frac{2}{3}}\right)^3}{\left(|x+\Delta x|^{\frac{1}{3}}-|x|^{\frac{1}{3}}\right)}=-3|x|^{\frac{2}{3}}.$$

If $x>0$, then $f'_\varphi(x)=3|x|^{\frac{2}{3}}$.
Let $x=0$ and $\Delta x>0$. Then

$$\lim_{\Delta x\to 0}\frac{f(\Delta x)-f(0)}{\varphi(\Delta x)-\varphi(0)}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{|\Delta x|^{\frac{1}{3}}}=0.$$

If $x=0$ and $\Delta x<0$, then

$$\lim_{\Delta x\to 0}\frac{f(\Delta x)-f(0)}{\varphi(\Delta x)-\varphi(0)}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{-|\Delta x|^{\frac{1}{3}}}=0,$$

i.e.

$$f'_\varphi(0)=0.$$

Then we obtain

$$f'_\varphi(x)=\begin{cases}-3|x|^{\frac{2}{3}}, & x<0,\\ 3|x|^{\frac{2}{3}}, & x\geq 0.\end{cases}$$

It is clear that the function $f'_\varphi(x)$ is the continuous function in $(-\infty,\infty)$.

Theorem 1.2. *Let function $f(x)$ be the continuous function in $[a,b]$, $\varphi(x)$ is strictly increasing continuous function in $[a,b]$ and*

$$F(x)=\int_a^x f(t)d\varphi(t),\ \ x\in[a,b].$$

Then

$$F'_\varphi(x)=\left(\int_a^x f(t)d\varphi(t)\right)'_\varphi=f(x),\ \ x\in[a,b],$$

where

$$F'_\varphi(a)=\lim_{\Delta x\to 0+}\frac{F(a+\Delta x)-F(a)}{\varphi(a+\Delta x)-\varphi(a)},\ \ F'_\varphi(b)=\lim_{\Delta x\to 0-}\frac{F(b+\Delta x)-F(b)}{\varphi(b+\Delta x)-\varphi(b)}$$

Proof. From the definition of $F'_{\varphi}(x)$, we obtain $F'_{\varphi}(x) =$

$$= \lim_{\Delta x \to 0} \left(f(x) \int_x^{x+\Delta x} d\varphi(t) - \int_x^{x+\Delta x} (f(x) - f(t)) d\varphi(t) \right) / [\varphi(x+\Delta x) - \varphi(x)]$$

$$= f(x) - \lim_{\Delta x \to 0} \psi(x, \Delta x),$$

where

$$\psi(x, \Delta x) = \left(\int_x^{x+\Delta x} (f(x) - f(t)) d\varphi(t) \right) / [\varphi(x+\Delta x) - \varphi(x)].$$

Then

$$|\psi(x, \Delta x)| \leq \left[\omega_f(\Delta x) \left(\int_x^{x+\Delta x} d\varphi(t) \right) \right] / [\varphi(x+\Delta x) - \varphi(x)] = \omega_f(\Delta x),$$

where

$$\omega_f(\delta) = \sup_{|t-x| \leq \delta} |f(x) - f(t)|,$$

and $\lim_{\delta \to 0} \omega_f(\delta) = 0$. Therefore

$$\lim_{\Delta x \to 0} |\psi(x, \Delta x)| \leq \lim_{\Delta x \to 0} \omega_f(|\Delta x|) = 0.$$

Hence, $F'_{\varphi}(x) = f(x)$. Analogously, the other cases will be proven. Theorem 1.2 is therefor proven. □

Corollary 1.3. *Let $F_0(x) = f(x) \in C[a,b]$, $\varphi(x)$ be the strictly increasing continuous function on $[a,b]$ and*

$$F_i(x) = \int_a^x F_{i-1}(t) d\varphi(t), x \in [a,b], i = 1, ..., n.$$

Then $F_n(x) \in C_{\varphi}^{(n)}[a,b]$, where $C_{\varphi}^{(n)}[a,b]$ is the linear space of all continuous functions $v(x)$ defined in $[a,b]$ such that $v_{\varphi}^{(n)}(x) \in C[a,b]$.

Theorem 1.4. *Let $\varphi(x)$ be the strictly increasing continuous function on $[a,b]$ and $f'_{\varphi}(x), g'_{\varphi}(x) \in C[a,b]$. Then*

$$\int_a^b f(x) g'_{\varphi}(x) d\varphi(x) = [f(x)(g(x)+c)]|_a^b - \int_a^b f'_{\varphi}(x)(g(x)+c) d\varphi(x),$$

where c is the arbitrary constant.

Proof. Integrating the following formula

$$[f(x)(g(x)+c)]'_{\varphi} = f'_{\varphi}(x)(g(x)+c) + f(x) g'_{\varphi}(x),$$

we obtain the desired formula. Theorem 1.4 is therefor proven. □

2. The generalized trapezoid rule

Let

$$x_i = a + ih, i = 0, 1, ..., n, h = \frac{b-a}{n},$$

then with the approximate value of the integral (1) we can define the formula

$$A_n = \frac{1}{2}\sum_{i=1}^{n}[f(x_i) + f(x_{i-1})][u(x_i) - u(x_{i-1})] = \frac{1}{2}\sum_{i=1}^{n}[f(x_i) +$$

$$f(x_{i-1})][\varphi(x_i) - \varphi(x_{i-1})] - \frac{1}{2}\sum_{i=1}^{n}[f(x_i + f(x_{i-1})][\psi(x_i) - \psi(x_{i-1})]. \quad (4)$$

Theorem 2.1. *Let $\varphi(x)$ and $\psi(x)$ be the strictly increasing continuous functions on $[a,b], f''_{\varphi}(x), f''_{\psi}(x) \in C[a,b]$. Then*

$$|I - A_n| \leq \frac{M_0}{12}(\varphi(b) - \varphi(a))(\omega_{\varphi}(h))^2 + \frac{M_1}{12}(\psi(b) - \psi(a))(\omega_{\psi}(h))^2, \quad (5)$$

where

$$\begin{cases} M_0 = \sup\limits_{x\in[a,b]} |f''_{\varphi}(x)|, M_1 = \sup\limits_{x\in[a,b]} |f''_{\psi}(x)|, \\ \omega_{\varphi}(h) = \sup\limits_{|x-y|\in h} |\varphi(x) - \varphi(y)|, \omega_{\psi}(h) = \sup\limits_{|x-y|\in h} |\psi(x) - \psi(y)|. \end{cases} \quad (6)$$

Proof. Let us introduce the notations:

$$\begin{cases} P_i = \int\limits_{x_{i-1}}^{x_i} f(x)d\varphi(x),\ Q_i = \int\limits_{x_{i-1}}^{x_i} f(x)d\psi(x), \\ M_i = \frac{1}{2}[f(x_i) + f(x_{i-1})][\varphi(x_i) - \varphi(x_{i-1})], \\ N_i = \frac{1}{2}[f(x_i) + f(x_{i-1})][\psi(x_i) - \psi(x_{i-1})],\ i = 1, 2, ..., n, \end{cases} \quad (7)$$

$$\alpha = -\frac{1}{2}(\varphi(x_i) + \varphi(x_{i-1}), \beta = -\frac{1}{8}[\varphi(x_i) - \varphi(x_{i-1})]^2,\ i = 1, 2, ..., n. \quad (8)$$

On the strength of Theorem 1.4 from (7) we obtain

$$P_i = \int_{x_{i-1}}^{x_i} f(x)d\varphi(x) = \int_0^h f(t + x_{i-1})d\varphi(t + x_{i-1}) =$$

$$f(t + x_{i-1})[\varphi(t + x_{i-1}) + \alpha]|_0^h - \int_0^h f'_{\varphi}(t + x_{i-1})[\varphi(t + x_{i-1}) + \alpha]d\varphi(t + x_{i-1}) =$$

$$f(t+x_{i-1})[\varphi(t+x_{i-1})+\alpha]|_0^h - f'_\varphi(t+x_{i-1})\left[\frac{[\varphi(t+x_{i-1})+\alpha]^2}{2}+\beta\right]|_0^h +$$

$$\int_0^h f''_\varphi(t+x_{i-1})\left[\frac{[\varphi(t+x_{i-1})+\alpha]^2}{2}+\beta\right]d\varphi(t+x_{i-1}). \tag{9}$$

Taking into account (8), we obtain

$$f(t+x_{i-1})[\varphi(t+x_{i-1})+\alpha]|_0^h = f(x_i)\frac{\varphi(x_i)-\varphi(x_{i-1})}{2} - f(x_{i-1})\frac{\varphi(x_{i-1})-\varphi(x_i)}{2}$$

$$= \frac{1}{2}[f(x_i)+f(x_{i-1})][\varphi(x_i)-\varphi(x_{i-1})],\ i=1,2,...,n, \tag{10}$$

$$f'_\varphi(t+x_{i-1})\left[\frac{(\varphi(t+x_{i-1})+\alpha)^2}{2}+\beta\right]|_0^h = f'_\varphi(x_i)\left[\frac{(\varphi(x_i)-\varphi(x_{i-1}))^2}{8}+\beta\right] -$$

$$f'_\varphi(x_{i-1})\left[\frac{(\varphi(x_{i-1})-\varphi(x_i))^2}{8}+\beta\right] = 0,\ i=1,2,...,n. \tag{11}$$

Then on the strength of (10) and (11) from (9) we obtain

$$P_i = M_i + \int_0^h f''_\varphi(t+x_{i-1})q_{i1}(t)d\varphi(t+x_{i-1}),\ i=1,2,...,n, \tag{12}$$

where

$$q_{i1}(t) = \frac{1}{2}[\varphi(t+x_{i-1}) - \frac{1}{2}(\varphi(x_i)+\varphi(x_{i-1}))]^2 - \frac{1}{8}(\varphi(x_i)-\varphi(x_{i-1}))^2. \tag{13}$$

Analogously, we obtain the following formula

$$Q_i = N_i + \int_0^h f''_\psi(t+x_{i-1})q_{i2}(t)d\psi(t+x_{i-1}),\ i=1,2,...,n, \tag{14}$$

where

$$q_{i2}(t) = \frac{1}{2}[\psi(t+x_{i-1}) - \frac{1}{2}(\psi(x_i)+\psi(x_{i-1}))]^2 - \frac{1}{8}(\psi(x_i)-\psi(x_{i-1}))^2. \tag{15}$$

On the strength of (13) and (15) we obtain

$$q_{ij}(0)=0,\ q_{ij}(h)=0,\ q_{ij}(t)<0, \tag{16}$$

for all $t\in(0,h),\ i=1,2,...,n,\ j=1,2$. From (16) we have

$$|q_{ij}(t)| = -q_{ij}(t),\ t\in[0,h],\ i=1,2,...,n,\ j=1,2. \tag{17}$$

Taking into account (6), (13) and (17) from (12) we obtain

$$\begin{aligned}
&|P_i - M_i| \le \int_0^h |f''_\varphi(t+x_{i-1})||q_{i1}(t)|d\varphi(t+x_{i-1}) \\
&\le M_0 \int_0^h \left\{ \frac{1}{8}(\varphi(x_i) - \varphi(x_{i-1}))^2 \right. \\
&\left. -\frac{1}{2}[\varphi(t+x_{i-1}) - \frac{1}{2}(\varphi(x_i)+\varphi(x_{i-1}))]^2 \right\} d\varphi(t+x_{i-1}) \\
&= \frac{M_0}{8}(\varphi(x_i)-\varphi(x_{i-1}))^2(\varphi(x_i)-\varphi(x_{i-1})) - \frac{M_0}{6}[\varphi(t+x_{i-1}) - \frac{1}{2}(\varphi(x_i) \\
&+(\varphi(x_{i-1}))]^3|_0^h = \frac{M_0}{8}(\varphi(x_i)-\varphi(x_{i-1}))^3 - \frac{M_0}{24}(\varphi(x_i)-\varphi(x_{i-1}))^3 \\
&= \frac{M_0}{12}(\varphi(x_i)-\varphi(x_{i-1}))^3,\ i = 1,2,...,n \qquad (18)
\end{aligned}$$

Analogously taking into account (6), (15) and (17) from (14) we have

$$|Q_i - N_i| \le \frac{M_1}{12}(\psi(x_i)-\psi(x_{i-1}))^3, i = 1,2,...,n. \qquad (19)$$

Taking into account the notations (2) and (7) from (1) and (4) we obtain

$$\begin{cases} I = \int\limits_a^b f(x)du(x) = \sum\limits_{i=1}^n (P_i - Q_i), \\ \\ A_n = \sum\limits_{i=1}^n (M_i - N_i) \end{cases} \qquad (20)$$

Then taking into account (18) and (19), from (20) we have

$$|I - A_n| \le \sum_{i=1}^n [|P_i - M_i| + |Q_i - N_i|] \le \frac{M_0}{12}\sum_{i=1}^n (\omega_\varphi(h))^2(\varphi(x_i)-\varphi(x_{i-1})+$$

$$+\frac{M_0}{12}\sum_{i=1}^n (\omega_\psi(h))^2(\psi(x_i)-\psi(x_{i-1})) =$$

$$\frac{M_0}{12}(\varphi(b)-\varphi(a))(\omega_\varphi(h))^2 + \frac{M_1}{12}(\psi(b)-\psi(a))(\omega_\psi(h))^2.$$

Theorem 2.1 is therefor proven. □

Corollary 2.2. *Let $\varphi(x)$ be the strictly increasing continuous function on $[a,b]$, $\psi(x) = 0$ for all $x \in [a,b]$ and $f''_\varphi(x) \in C[a,b]$. Then*

$$|I - A'_n| \le \frac{M_0}{12}(\varphi(b)-\varphi(a))(\omega_\varphi(h))^2, \qquad (21)$$

where

$$A'_n = \frac{1}{2}\sum_{i=1}^{n}[f(x_i)+f(x_{i-1})][\varphi(x_i)-\varphi(x_{i-1})]. \tag{22}$$

Corollary 2.3. *Let $\varphi(x)$ and $\psi(x)$ be the strictly increasing continuous functions on $[a,b]$, $f''_{\varphi}(x), f''_{\psi}(x) \in C[a,b]$, $\varphi(x) \in C^{\alpha}[a,b], 0 < \alpha \leq 1, \psi(x) \in C^{\beta}[a,b]$, $0 < \beta \leq 1$, i.e. for all $x,y \in [a,b]$*

$$|\varphi(x)-\varphi(y)| \leq c_0|x-y|^{\alpha}, |\psi(x)-\psi(y)| \leq c_1|x-y|^{\beta}, c_0 > 0, c_1 > 0.$$

Then

$$|I-A_n| \leq \frac{M_0c_0^2}{12}(\varphi(b)-\varphi(a))h^{2\alpha} + \frac{M_1c_1^2}{12}(\psi(b)-\psi(a))h^{2\beta}.$$

Corollary 2.4. *Let $\varphi(x)$ be the strictly increasing continuous function on $[a,b], \psi(x) = 0$ for all $x \in [a,b], f''_{\varphi}(x) \in C[a,b]$ and $\varphi(x) \in C^{\alpha}[a,b], 0 < \alpha \leq 1$. Then*

$$|I-A'_n| \leq \frac{M_0c_0^2}{12}(\varphi(b)-\varphi(a))h^{2\alpha}.$$

Theorem 2.5. *Let $u(x)$ be the function of the bounded variation in $[a,b]$ and $f(x) \in C^{\alpha}[a,b], 0 < \alpha \leq 1$, i.e. for all $x,y \in [a,b]$*

$$|f(x)-f(y)| \leq c_2|x-y|^{\alpha}.$$

Then

$$|I-A_n| \leq c_2h^{\alpha}[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)],$$

where A_n is defined by the formula (4).

Proof. Taking into account (2) and (7) from (1) and (4) we can obtain

$$\begin{aligned}
|I-A_n| &= |\sum_{i=1}^{n}[(P_i-M_i)-(Q_i-N_i)]| \\
&\leq \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} |[f(x)-\frac{1}{2}(f(x_i)+f(x_{i-1}))]|d\varphi(x) \\
&+ \int_{x_{i-1}}^{x_i} |[f(x)-\frac{1}{2}(f(x_i)+f(x_{i-1}))]|d\psi(x)| \\
&\leq c_2h^{\alpha}[\varphi(b)-\varphi(a)+\psi(b)-\psi(a)].
\end{aligned}$$

Theorem 2.5 is therefore proven. □

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