LE MATEMATICHE
Vol. LXV (2010) – Fasc. II, pp. 45–48
doi: 10.4418/2010.65.2.5

REGULARITY OF SOME METHOD OF SUMMATION FOR DOUBLE SEQUENCES

FRANCESCO TULONE

Some generalization of Toeplitz method of summation is introduced for double sequences and condition of regularity of it is obtained.

1. Introduction

Here we introduce a method of summation for double sequences and consider a sufficient condition of regularity of this method which is a generalization of the known one-dimension Toeplitz regularity conditions (see for example [3] and [7]).

In the one dimensional case Toeplitz method of summation of sequence $\{s_i\}$ is defined by an infinite matrix $A = (\alpha_{kj})$ with $k, j = 0, 1, \dots$, satisfying the conditions:

a) $\lim_{k\to\infty} \alpha_{kj} = 0$ for all j;

b) $N = \sup_k N_k < \infty$;

c) $\lim_{k\to\infty} A_k = 1$,

where $N_k = \sum_{j=0}^{\infty} |\alpha_{kj}|$, $A_k = \sum_{j=0}^{\infty} \alpha_{kj}$.

We say that sequence $\{s_i\}$ is *A-summable* to σ if $\sigma_k = \sum_{j=0}^{\infty} \alpha_{kj} s_j$ converges to σ when $k \to \infty$.

Entrato in redazione: 14 febbraio 2011

AMS 2010 Subject Classification: 40B05, 40C05, 40G05.
Keywords: Double sequences, Toeplitz method of summation, conditions of regularity.
Supported by “Potenziamento alla Ricerca”, Dip. di Matematica e Inf., Università di Palermo

We remind that a method of summation is said to be regular if it "sums" every convergent sequences to the same value to which the sequence converges, that is, in our notation, $\lim_{k\to\infty}\sigma_k = \lim_{i\to\infty} s_i$ if the last limit exists and it is finite.

The necessity and sufficiency of the above conditions a) - c) for regularity were proved by O. Toeplitz in the case of triangular matrices in [6], but it is true also for any matrix summation method (see [7]).

2. Main result

In the two dimensional case we consider the following method of summation for a double sequence $\{s_{ij}\}$.

Definition 2.1. Let a four-dimensional sequence $A = \{\alpha_{klij}\}$ be given such that

1) $\lim_{k+l\to\infty}\alpha_{klij} = 0$ for each i and j;

2) $N = \sup_{k,l} N_{kl} < \infty$;

3) $\lim_{k+l\to\infty} A_{kl} = 1$,

where $N_{kl} = \sum_{i,j=0}^{\infty}|\alpha_{klij}|$, $A_{kl} = \sum_{i,j=0}^{\infty}\alpha_{klij}$.

Note that the condition 2) implies that the value of A_{kl} for each fixed k and l does not depend on the way of summation of double series because the series defining A_{kl} is absolutely convergent.

With every double sequence $\{s_{ij}\}_{i,j=0}^{\infty}$ we associate the double sequence $\{\sigma_{kl}\}_{k,l=0}^{\infty}$ given by

$$\sigma_{kl} = \sum_{i,j=0}^{\infty}\alpha_{klij}s_{ij}.$$

provided a finite limit $\lim_{\nu\to\infty}\sum_{i+j\le\nu}^{\infty}\alpha_{klij}s_{ij}$ exists.

We say that a double sequence $\{s_{ij}\}$ is *A-summable* to σ if σ_{kl} converges to σ when $k+l\to\infty$.

This type of convergence of double sequences where the sum of indexes tends to infinity is called in [2] "convergence in the restricted sense".

The following theorem shows that conditions 1) - 3) imply the regularity of the above method of summation.

Theorem 2.2. *Let a double sequence $\{s_{ij}\}$ be such that $\lim_{i+j\to\infty}s_{ij} = s$. Then $\{s_{ij}\}$ is A-summable to s, i.e., $\lim_{k+l\to\infty}\sigma_{kl} = s$ where matrix A is defined as in the Definition 2.1.*

Proof. Put $s_{ij} = s + \varepsilon_{ij}$ where $\lim_{i+j\to\infty}\varepsilon_{ij} = 0$. Then we have $\sigma_{kl} = \sigma'_{kl} + \sigma''_{kl}$ where $\sigma'_{kl} = sA_{kl}$ and $\sigma''_{kl} = \sum_{i,j=0}^{\infty}\alpha_{klij}\varepsilon_{ij}$. We note that, by condition 3), $\lim_{k+l\to\infty}\sigma'_{kl} = s$.

Take $\eta > 0$ and choose ν such that $|\varepsilon_{ij}| < \frac{\eta}{2N}$ if $i+j > \nu$. Having fixed such a ν and using property 1) of $\{\alpha_{klij}\}$ we can choose p such that for $k+l > p$ we have $\sum_{i+j\leq\nu} |\alpha_{ijkl}\varepsilon_{ij}| < \frac{\eta}{2}$. Therefore, having in mind property 2), we get

$$|\sigma''_{kl}| = |\sum_{i,j=0}^{\infty} \alpha_{klij}\varepsilon_{ij}| \leq \sum_{i+j\leq\nu} |\alpha_{klij}\varepsilon_{ij}| + (\sum_{i+j>\nu}^{\infty} |\alpha_{klij}|)\frac{\eta}{2N} \leq \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

for all $k+l > p$. It means that $\lim_{k+l\to\infty} \sigma''_{kl} = 0$ and so $\lim_{k+l\to\infty} \sigma_{kl} = s$. $\square$

As an example of a method to which the above theorem is applicable we can take the following version of the arithmetic means method of summation for a double sequence. Put $\alpha_{klij} = \frac{1}{kl}$ if $0 \leq i \leq k-1$ and $0 \leq j \leq l-1$ and $\alpha_{klij} = 0$ if $i \geq k$ or $j \geq l$. We get $A_{kl} = N_{kl} = \sum_{i,j=0}^{\infty} \frac{1}{kl} = \sum_{i,j=0}^{k-1,l-1} \frac{1}{kl} = 1$. So according to Theorem 2.2, if $\lim_{i+j\to\infty} s_{ij} = s$ then

$$\lim_{k+l\to\infty} \frac{\sum_{i,j=0}^{k-1,l-1} s_{ij}}{kl} = s.$$

Remark 2.3. We note that the conditions 1) - 3) do not guarantee regularity if we understand it in the following sense: the A-method of summation defined by $A = \{\alpha_{klij}\}$ is regular if the existence of the limit $\lim_{i,j\to\infty} s_{ij} = s$ implies that $\lim_{k,l\to\infty} \sigma_{kl} = s$ (on regularity for this type of convergence see for example [2] and [4]). So the type of convergence of the sequence $\{s_{ij}\}$ with respect to i and j is essential for our result. For example it can not be replaced by the above mentioned convergence when $i \to \infty$ and $j \to \infty$ independently or by a bounded (or λ-regular) convergence as in [1]. Indeed, it is enough to consider a sequence $\{s_{ij}\}$ putting $s_{0j} = j$, $s_{i0} = i$ and $s_{ij} = 0$ elsewhere. In this example $\lim_{i,j\to\infty} s_{ij} = 0$, but arithmetic means do not go to zero because

$$\sigma_{kl} = \sum_{i,j=0}^{k-1,l-1} \frac{s_{ij}}{kl} = \sum_{j=0}^{l-1} \frac{j}{kl} + \sum_{i=0}^{k-1} \frac{i}{kl} = \frac{(l-1)l}{2kl} + \frac{(k-1)k}{2kl} = \frac{(l-1)}{2k} + \frac{(k-1)}{2l}.$$

In particular in the case $k = l$ they tend to one.

In contrast with this, the convergence, with respect to k and l, of the sequence $\{\sigma_{kl}\}$ as well as convergence in the condition 1) can be replaced by any type of convergence with k and l tending to infinity (for instance k and l can tend to infinity independently, regularly, etc.).

Remark 2.4. If $s = 0$ then the condition 3) can be dropped in the formulation of Theorem 2.2. Moreover if the sequence $\{\alpha_{klij}\}$ is constituted by not negative number, then the condition 2) can be dropped as it is a consequence of the condition 3) in this case.

The method of summation for double sequences considered here will be applied in [5] in the theory of double Haar and Walsh series.

REFERENCES

[1] M. I. Alkhimov, *Bounded summability of double series and sequences*, Ukrainian Mathematical Journal 21 (4) (1969), 445–451.

[2] V. G. Chelidze, *Nekotorye metody summirovaniya dvoinykh ryadov i dvoinykh integralov*, (Russian) Some summability methods for double series and double integrals, Izdat. Tbilis. Univ., Tbilisi, 1977.

[3] G. H. Hardy, *Divergent series*, Oxford, 1949.

[4] O. M. Logunova, *Certain peculiarities of regular methods of double sequences*, Sibirskii Matematicheskii Zurnal 10 (4) (1969), 910–919 (in russian), trans. in Siberian Mathematical Journal (10) (4) (1969), 669–675.

[5] V. A. Skvortsov - F. Tulone, *Two-dimensional dyadic Henstock- and Perron-type integrals in the theory of Haar and Walsh series*, (to be published)

[6] O. Toeplitz, *Über allgemeine lineare Mittelbildungen*, Prace Mat.-Fiz. (22) (1911), 113–119.

[7] A. Zygmund, *Trigonometric series*, Vol. I and II, Third edition, Cambridge Univ. Press, 2002.

FRANCESCO TULONE
Dipartimento di Matematica e Informatica
Università di Palermo
Via Archirafi 34
90123 Palermo, Italy
e-mail: `tulone@math.unipa.it`