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BOUNDARY BLOW-UP FOR NONLINEAR ELLIPTIC EQUATIONS WITH GENERAL GROWTH IN THE GRADIENT: AN APPROACH VIA SYMMETRIZATION

VINCENZO FERONE

In this paper we give a survey of some recent results obtained via symmetrization methods for solutions of elliptic equations in the form $A(u) = H(x,u,Du)$, where the principal term is a laplacian-type operator and $H(x,u,Du)$ grows with respect to Du at most like $|Du|^q$, $1 \leq q \leq 2$. In particular, it is considered the case where the solution blows up on the boundary and some comparison results are illustrated. Also an isoperimetric inequality for the so-called "ergodic constant" is given and the connections with the homogeneous Dirichlet problem for the quoted equations are discussed.

1. Introduction

We present some recent results obtained for solutions to elliptic equations which contain a main term in the form of a laplacian and a lower-order term which grows at most as a power of the gradient of the unknown function. A typical example of such a problem is the following:

$$\begin{cases} -(a_{ij}u_{x_i})_{x_j} = H(x,Du) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded open set, the coefficients a_{ij} are bounded measurable functions satisfying the ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \qquad \text{a.e.}\, x \in \Omega,\ \forall \xi \in \mathbb{R}^n, \tag{1.2}$$

and $H(x,\xi)$ is a Carathéodory function satisfying for some $q \in [1,2]$ the growth condition:

$$|H(x,\xi)| \leq \theta|\xi|^q + f(x), \qquad \text{a.e.}\, x \in \Omega,\ \forall \xi \in \mathbb{R}^n, \tag{1.3}$$

where θ is a non-negative constant and f is a non-negative bounded function.

It is well known that symmetrization techniques have turned out to be useful in order to study homogeneous problems like (1.1), while our main interest here is to analyze the case of solutions which blow-up on the boundary. However, it could be useful to recall briefly some results concerning the existence for problem (1.1) because, as we will see, some information about such a subject can be obtained from comparison results about boundary blow-up problems.

For instance, in [14] it is shown that one can estimate a bounded solution to problem (1.1) in terms of the bounded solution to the symmetrized problem

$$\begin{cases} -\Delta v = \theta|Dv|^q + f^\# & \text{in } \Omega^\# \\ \\ v = 0 & \text{on } \ \partial\Omega^\#, \end{cases} \tag{1.4}$$

where $\Omega^\#$ is the ball centered at the origin such that $|\Omega^\#| = |\Omega|$ and $f^\#$ is the spherically symmetric decreasing rearrangement of f (see Section 2 for the definition).

The existence of the solution to problem (1.4) can be obtained under a “smallness” assumption on f and, because of the symmetry of problem (1.4), such a condition can be easily determined in an optimal form. A consequence of existence for problem (1.4) is the existence for problem (1.1). Existence results for problems in the form (1.1) can be found, for instance, also in [6], [7], [8], [22], [9], [2], [10], [19], [15].

We remark that here and in the following we restrict ourselves to the case where in the equation a laplacian-like operator appears, but most of the described results hold true also when the laplacian is substituted by a p-laplacian, $p > 1$, and the power in (1.3) is such that $p-1 \leq q \leq p$.

Our main purpose is to illustrate some comparison results obtained for solutions, blowing-up on the boundary, to a class of equations similar to those discussed above (see [12]). For example, one can consider a problem in the form

$$\begin{cases} \Delta u + H(x,u,Du) = g(u) & \text{in } \ \Omega \\ \\ u(x) \to +\infty & \text{as } x \to \partial\Omega, \end{cases} \tag{1.5}$$

where $H(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, for a suitable constant γ, satisfies the condition

$$|H(x,s,\xi)|\le\gamma|\xi|^2$$

and $g:\mathbb{R}\to\mathbb{R}_+$ is a continuous function.

When $g(u)=e^u$, $H=0$, the problem plays an important role in the theory of riemannian surfaces with constant negative curvature ([5], [18]), while when $H\neq 0$ the problem is related to a stochastic control problem ([16], [4], [24]). We are interested in problems in the form (1.5) and we show that the comparison result which can be stated depends on the sign of the term $H(x,s,\xi)$.

We also consider the following problem

$$\begin{cases} -\Delta u+|Du|^q+\kappa_q=f & \text{in } \Omega \\ \\ u(x)\to+\infty & \text{as } x\to\ \partial\Omega, \end{cases} \tag{1.6}$$

where $1<q\le 2$, f is a bounded function in Ω and κ_q is a constant to be determined. Such a problem has been studied in [16] where a stochastic control problem is considered and problem (1.6) comes out in the ergodic limit. In particular, in [16] the existence and uniqueness of the couple (κ_q,u) which solves (1.6) is proved.

Theorem 1.1. (see [16]) *There exists a unique constant (ergodic constant) $\kappa_q=\kappa_q(\Omega,f)$ such that* (1.6) *has a unique solution $u\in W^{2,r}_{loc}(\Omega)$ ($\forall r<\infty$). Moreover, u is unique up to an additive constant.*

From the above result it seems natural to ask if the ergodic constant has properties similar to those of an eigenvalue. Actually, in [13] it is shown that this is the case, i.e., it holds

$$\kappa_q(\Omega^\#,f_\#)\le\kappa_q(\Omega,f), \tag{1.7}$$

where $f_\#$ is the spherically symmetric increasing rearrangement of f (see Section 2 for the definition).

The above result can be used in order to obtain information about problems in the form (1.1). Indeed, in [21] it has been shown that the ergodic constant is related to the existence for problem (1.1).

The paper is organized as follows: in Section 2 we recall some definitions and properties about rearrangements; in Section 3 we give some comparison results for problems in the form (1.5); in Section 4 we give the isoperimetric inequality about the ergodic constant and we discuss some consequences about existence for problems in the form (1.1).

2. Preliminaries about rearrangements

In this section we recall some definitions about rearrangements (for further details see, e.g., [17], [20]).

Let Ω be an open bounded subset of $\mathbb{R}^n$ and $u:\Omega\to\mathbb{R}$ be a measurable function. If one denotes by $|E|$ the Lebesgue measure of a set $E\subset\mathbb{R}^n$ and set $\Omega^*=]0,|\Omega|[$, one can define the distribution function of u, $\nu_u:\mathbb{R}\to\overline{\Omega^*}$, as follows:

$$\nu_u(t)=|\{x\in\Omega: u(x)<t\}|=|\{u<t\}|, \qquad t\geq 0.$$

The function ν_u is increasing and left continuous; moreover, its generalized inverse function is the increasing rearrangement of u, $u_*:\overline{\Omega^*}\to\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$:

$$u_*(s)=\begin{cases}\inf\{t\in\mathbb{R}:\nu_u(t)>s\}, & s\in[0,|\Omega|[\\ \\ \operatorname{ess\,sup}_\Omega u, & s=|\Omega|\end{cases}$$

moreover, $u_*(0)=\operatorname{ess\,inf}_\Omega u$.

The spherically symmetric increasing rearrangement of u is defined by:

$$u_\#(x)=u_*(\omega_n|x|^n), \qquad x\in\Omega^\#,$$

where $\Omega^\#$ is the ball centered at the origin having the same measure as Ω and ω_n is the measure of the unit ball in $\mathbb{R}^n$.

In an analogous way, one can define the decreasing rearrangement u^* of u as

$$u^*(s)=\sup\{t\geq 0: |\{u<t\}|>s\}, \qquad s\in[0,|\Omega|].$$

The spherically symmetric decreasing rearrangement of u is defined by:

$$u^\#(x)=u^*(\omega_n|x|^n), \qquad x\in\Omega^\#.$$

A brief list of well known properties of rearrangement follows:

- $\displaystyle\int_0^{|E|}u_*(s)\,ds\leq\int_E u\,dx\leq\int_0^{|E|}u^*(s)\,ds, \qquad E\subset\Omega.$
- $\displaystyle\int_{u>t}u\,dx=\int_0^{|\{u>t\}|}u^*(s)\,ds.$
- $\displaystyle\int_\Omega|uv|\,dx\leq\int_0^{|\Omega|}|u|^*(s)|v|^*(s)\,ds=\int_0^{|\Omega|}|u|_*(s)|v|_*(s)\,ds.$

  (Hardy-Littlewood inequality)
- $\|u\|_{L^p(\Omega)}=\|u^\#\|_{L^p(\Omega^\#)} \qquad 1\leq p\leq\infty.$

- $|u|^*(s) \le |v|^*(s),\ s \in (0,|\Omega|) \Rightarrow \|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^\#)} \quad 1 \le p \le \infty.$

Finally, we recall that the following Pólya-Szegö inequality holds true.

Theorem 2.1. *Let $u \in W^{1,p}(\mathbb{R}^n)$ be a non-negative function, then:*

$$\int_{\mathbb{R}^n} |Du^\#|^p dx \le \int_{\mathbb{R}^n} |Du|^p dx. \tag{2.1}$$

3. Comparison results for blow-up solutions

We start by considering a problem in the form (1.5) with H positive, namely,

$$\begin{cases} \operatorname{div}(a(x,u,Du)) + H(x,u,Du) = g(u) & \text{in } \Omega \\ \\ u(x) \to \infty & \text{as } x \to \partial\Omega, \end{cases} \tag{3.1}$$

where $a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the following ellipticity condition:

$$a(x,s,\xi)\xi \ge |\xi|^2, \quad \text{a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.2}$$

$H(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying, for some positive constant γ, the inequality

$$a(x,s,\xi)\xi \le H(x,s,\xi) \le \gamma|\xi|^2, \quad \text{a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^n, \tag{3.3}$$

and $g : \mathbb{R} \to \mathbb{R}_+$ is a continuous function.

We say that $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution to problem (3.1) if $a(x,u,Du) \in \left(L^2(\Omega)\right)^n$, $g(u) \in L^2_{loc}(\Omega)$ and

$$-\int_{\Omega'} a(x,u,Du) D\psi\, dx + \int_{\Omega'} H(x,u,Du)\psi\, dx = \int_{\Omega'} g(u)\psi\, dx, \tag{3.4}$$

for every $\psi \in W^{1,2}_0(\Omega') \cap L^\infty(\Omega')$, with $\Omega' \subset\subset \Omega$ and $\lim_{x\to\partial\Omega} u(x) = +\infty$.

In [12] the following comparison result between solutions to problem (3.1) and the solution to the symmetrized problem

$$\begin{cases} \Delta v + |Dv|^2 = g(v) & \text{in } \Omega^\# \\ \\ v(x) \to \infty & \text{as } x \to \partial\Omega^\#. \end{cases} \tag{3.5}$$

Theorem 3.1. *Let $u \in W^{1,2}_{loc}(\Omega)$ be a weak solution to problem* (3.1)*. If $\beta(s) = sg(\log s)$, $s > 0$, satisfies*

i) $\beta(s)$ is a continuous increasing function such that $\beta(0) = 0$ and $\beta(s) > 0$, $\forall s > 0$;

ii) (Keller condition)

$$\int^{\infty} \frac{1}{(\int_0^s \beta(\tau)d\tau)^{\frac{1}{2}}} ds < +\infty,$$

and $v \in W^{1,2}_{loc}(\Omega^{\#})$ is the radial solution to problem (3.5)*, then*

$$\operatorname*{ess\,inf}_{x\in\Omega} u(x) \geq \operatorname*{ess\,inf}_{x\in\Omega^{\#}} v(x). \tag{3.6}$$

Remark 3.2. In the proof of the above theorem it is contained the comparison result

$$u_*(s) \leq w_*(s), \quad \forall s \in [0, |B|],$$

where w is the solution to the symmetrized problem (3.5) in a ball B such that

$$\operatorname*{ess\,inf}_{x\in\Omega} u = \operatorname*{ess\,inf}_{x\in B} w.$$

In some cases it is possible to write explicitly the solution to the symmetrized problem. Thus, it is possible to obtain an explicit bound for the minimum of the solution to problem (3.1).

For example, when $n > 2$, it is possible to choose

$$g(s) = e^{\frac{4}{n-2}s}.$$

The symmetrized problem becomes:

$$\begin{cases} \Delta v + |Dv|^2 = e^{\frac{4}{n-2}v} & \text{in } \Omega^{\#} \\ \\ v(x) \to +\infty & \text{as } x \to \partial\Omega^{\#}. \end{cases}$$

In such a case the hypotheses of the theorem are satisfied and a straightforward calculation gives the following inequality (R si the radius of $\Omega^{\#}$):

$$\operatorname*{ess\,inf}_{x\in\Omega} u \geq \frac{n-2}{2} \log\left(\left[n(n-2)\right]^{\frac{1}{2}} \frac{1}{R}\right).$$

Remark 3.3. We observe that, in order to prove the comparison result in Theorem 3.1 it is necessary to know that for the problem

$$\begin{cases} \Delta w = \beta(w) & \text{in } B \\ \\ w(x) \to \infty & \text{as } x \to \partial B, \end{cases}$$

there is a unique solution and the minimum of such a solution decreases as the radius of B increases. Every time g is such that the above properties are satisfied one can prove Theorem 3.1. The assumptions we have made on g are an example of a sufficient condition.

We now consider a problem in the form (1.5) with H negative, namely,

$$\begin{cases} \operatorname{div}(a(x,u,Du)) = G(x,u,Du) + g(u) & \text{in } \Omega \\ \\ u(x) \to +\infty & \text{as } x \to \partial\Omega, \end{cases} \tag{3.7}$$

where $a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the ellipticity condition (3.2), $G(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the condition

$$|G(x,s,\xi)| \le |\xi|^2, \ \text{ a.e. } x \in \Omega,$$

and $g : \mathbb{R} \to \mathbb{R}_+$ is a continuous function.

In this case we consider the symmetrized problem

$$\begin{cases} \Delta v = |Dv|^2 + g(v) & \text{in } \Omega^\# \\ \\ v \to +\infty & \text{as } x \to \partial\Omega^\#. \end{cases} \tag{3.8}$$

Theorem 3.4. *Let $u \in W^{1,2}_{loc}(\Omega)$ be a weak solution to problem* (3.7) *and let $F(r) = r\,g(\log r^{-1})$, $r > 0$, be a decreasing function such that $\lim_{r\to 0^+} F(r) < +\infty$. If $v \in W^{1,2}_{loc}(\Omega^\#)$ is the solution to problem* (3.8) *then*

$$\operatorname*{ess\,inf}_{x\in\Omega} u(x) \ge \operatorname*{ess\,inf}_{x\in\Omega^\#} v(x). \tag{3.9}$$

Furthermore, if $F(r) \in C^2(]0,+\infty[)$, then

$$\int_0^s g(u_*(r))\, e^{-u_*(r)}\, dr \ge \int_0^s g(v_*(r))\, e^{-v_*(r)}\, dr, \quad r \in [0,|\Omega|[.$$

Remark 3.5. Let us observe that in the proof of Theorem 3.4 it is contained the comparison result

$$u_*(s) \le w_*(s), \quad \forall s \in [0, |B|],$$

where w is the solution to the problem (3.8) in a ball B such that

$$\operatorname*{ess\,inf}_{x\in\Omega} u = \operatorname*{ess\,inf}_{x\in B} v.$$

As for the previuos problem, in some cases, it is possible to write the solution to problem (3.8) and then to make explicit the lower bound in (3.9). In the case $n = 2$ we can choose

$$g(s) = e^{s-e^{-s}}. \tag{3.10}$$

We then consider the problem (3.8) in the form

$$\begin{cases} \Delta v - |Dv|^2 = e^{v-e^{-v}} & \text{in } B_R \\ \\ v(x) \to \infty & \text{as } x \to \partial B_R, \end{cases} \tag{3.11}$$

where B_R is the ball centered at the origin with radius R. Clearly $F(r) = r\,g(-\log r) = e^{-r}$ and the assumptions of Theorem 3.4 are satisfied. If we put $L_R = 4 + 2\sqrt{4 + 2R^2}$, the function

$$v(x) = -\log\left(2\log\frac{L_R^2 - 8|x|^2}{8L_R}\right),$$

is the radial solution to problem (3.11). Hence, for a solution u to problem (3.7) with $g(u)$ as in (3.10), in any domain Ω with $|\Omega| = |B_R|$, the following inequality holds:

$$\operatorname*{ess\,inf}_{x\in\Omega} u \ge -\log\left(2\log\frac{L_R}{8}\right).$$

We conclude this section giving a sketch of the proof of Theorem 3.1. The proof of Theorem 3.4 uses similar arguments with suitable modifications.

Sketch of the proof of Theorem 3.1. Let $u \in W^{1,2}_{loc}(\Omega)$ be a weak solution to problem (3.1). Let us consider, for $t > 0$, the following function belonging to $W_0^{1,2}(\Omega') \cap L^\infty(\Omega')$, for some $\Omega' \subset\subset \Omega$:

$$\varphi(x) = \begin{cases} e^u h, & \bar{u} < t - h \\ e^u (t - \bar{u}), & t - h \le \bar{u} < t \\ 0, & \bar{u} \ge t, \end{cases}$$

where $h > 0$ and $\bar{u} = e^u$. If we use $\varphi(x)$ as test function in (3.4), using the ellipticity condition (3.2) and inequality (3.3), we have

$$\begin{aligned}\int_{t-h\le\bar{u}<t} |D\bar{u}|^2\,dx &\le \int_{t-h\le\bar{u}<t} a(x,u,Du)Du\,e^{2u}\,dx\\ &\le \int_{\bar{u}<t-h} g(u)\,e^u h\,dx + \int_{t-h\le\bar{u}<t} g(u)\,e^u(t-\bar{u})\,dx.\end{aligned}$$

Dividing through by h and letting h go to zero, in a standard way (see, for example, [2], [23]) we obtain

$$\frac{d}{dt}\int_{\bar{u}<t} |D\bar{u}|^2 dx \le \int_{\bar{u}<t} \beta(\bar{u})\,dx. \tag{3.12}$$

The left-hand side in (3.12) can be estimated from below using Fleming–Rishel coarea formula and, by the properties of rearrangements given in Section 2, we derive the following inequality, a.e. in $[0,|\Omega|]$:

$$U'(s)\left(n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}}\right)^2 \le \int_0^s \beta(U(\sigma))\,d\sigma, \tag{3.13}$$

where we have put $U(s) = \bar{u}_*(s) = e^{u_*(s)}$. After an integration, from (3.13) it is immediate to obtain

$$U(s) \le M + c_n \int_0^s \tau^{\left(\frac{2}{n}-2\right)} \int_0^\tau \beta(U(\sigma))d\sigma\,d\tau, \quad \forall s \in [0,|\Omega|[. \tag{3.14}$$

where $M = \operatorname{ess\,inf}_{x\in\Omega} \bar{u}(x) = e^{u_*(0)}$ and $c_n = \left(n\omega_n^{\frac{1}{n}}\right)^{-2}$.

On the other hand, one can consider the solution V to the equality

$$V(s) = M + c_n \int_0^s \tau^{\left(\frac{2}{n}-2\right)} \int_0^\tau \beta(V(\sigma))d\sigma\,d\tau, \quad \forall s \in [0,s_0[, \tag{3.15}$$

where s_0 is a suitable number such that $0 < s_0 \le |\Omega|$. Actually, if $z(x) = V(\omega_n|x|^n)$, we have that $z(x)$ is the solution to problem

$$\begin{cases} \Delta z = \beta(z) & \text{in } B \\ \\ z(x) \to \infty & \text{as } x \to \partial B, \end{cases} \tag{3.16}$$

where B is the ball such that $|B| = s_0$.

Skipping here some technicalities which can be found in [12], using (3.14) and (3.15), one can compare $U(s)$ with $V(s)$ obtaining

$$U(s) \le V(s), \quad \forall s \in [0,s_0[. \tag{3.17}$$

Set $w(x) = \log(z(x))$, $x \in B$, $w(x)$ is the unique solution to problem

$$\begin{cases} \Delta w + |Dw|^2 = g(w) & \text{in } B \\ \\ w(x) \to +\infty & \text{as } x \to \partial B. \end{cases} \tag{3.18}$$

Let us observe that $V(x) = e^{w(x)}$ so $V(s) = e^{w_*(s)}$. Then, being $U(s) = e^{u_*(s)}$, from (3.17) it follows $u_*(s) \leq w_*(s)$, for all $s \in [0, |B|]$, where $|B| = s_0 \leq |\Omega|$.

Finally, considering the radial solution $v(x)$ to problem (3.5), the inequality (3.6) follows from the fact that the minimum of the solution to (3.18) decreases as the radius of B increases. The theorem is thus proved.

4. Isoperimetric inequality for the ergodic constant

Let us recall some well known properties for nonlinear eigenvalue problems. Let us consider the eigenvalue problem for the p-laplacian operator ($\Delta_p w = \operatorname{div}(|Dw|^{p-2}Dw)$), $p > 1$,

$$\begin{cases} -\Delta_p w + m|w|^{p-2}w = \lambda |w|^{p-2}w & \text{in } \Omega \\ \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where m is a bounded function in Ω.

It is well known that in the linear case ($p = 2$) and in the nonlinear case the first eigenvalue can be obtained from a minimum problem and it is simple. For the general problem (4.1) such properties can be found, for example, in [11].

Theorem 4.1. *Let us define*

$$\lambda_p(\Omega, m) = \inf \left\{ \int_\Omega (|D\varphi|^p + m|\varphi|^p)\, dx : \varphi \in W_0^{1,p}(\Omega), \int_\Omega |\varphi|^p dx = 1 \right\}. \tag{4.2}$$

Then $\lambda_p(\Omega, m) > -\infty$ is the lowest eigenvalue of problem (4.1). *Moreover, $\lambda_p(\Omega, m)$ is the only eigenvalue associated to a positive eigenfunction and it is simple.*

As a consequence of Pólya-Szegö inequality and of the properties of rearrangements, it is not difficult to prove the following result (see also [1]), which is a generalization of the classical Faber-Krahn inequality.

Theorem 4.2. *If $\lambda_p(\Omega, m)$ is the first eigenvalue of problem* (4.1) *then*

$$\lambda_p(\Omega, m) \geq \lambda_p(\Omega^\#, m_\#), \tag{4.3}$$

where $\lambda_p(\Omega^\#, m_\#)$ is the first eigenvalue of the problem

$$\begin{cases} -\Delta_p v + m_\# |v|^{p-2} v = \lambda |v|^{p-2} v & \text{in } \Omega^\# \\ \\ v = 0 & \text{on } \partial\Omega^\#. \end{cases} \tag{4.4}$$

Let us consider now problem (1.6) given in the introduction, namely,

$$\begin{cases} -\Delta u + |Du|^q + \kappa_q = f & \text{in } \Omega \\ \\ u(x) \to +\infty & \text{as } x \to \partial\Omega, \end{cases} \tag{4.5}$$

where $1 < q \leq 2$ and f is a bounded function in Ω.

It should be evident that, in view of Theorem 1.1, the ergodic constant κ_q satisfies properties which are similar to those of an eigenvalue. Furthermore, as observed in [16] and [21], if $q = 2$, it holds

$$\kappa_2(\Omega, f) = \lambda_2(\Omega, f), \tag{4.6}$$

that is, the ergodic constant is the first eigenvalue of the operator $-\Delta + f$ with homogeneous Dirichlet conditions. Thus, it is natural to ask if for κ_q a Faber-Krahn inequality holds true. In [13] we prove the following result.

Theorem 4.3. *It holds*

$$\kappa_q(\Omega^\#, f_\#) \leq \kappa_q(\Omega, f), \tag{4.7}$$

where $\kappa_q(\Omega^\#, f_\#)$ is the ergodic constant for the symmetrized problem

$$\begin{cases} -\Delta v + |Dv|^q + \kappa_q = f_\# & \text{in } \Omega^\# \\ \\ v(x) \to +\infty & \text{as } x \to \partial\Omega^\#, \end{cases} \tag{4.8}$$

The equality in (4.7) *holds if and only if $\Omega = \Omega^\#$ and $f = f_\#$, modulo translations.*

We point out that a key step in the proof of Theorem 4.3 is an inequality involving the rearrangement of the solution to (4.5) obtained in the same spirit of the comparison results given in Section 3.

Theorem 4.4. *Let $u \in W^{1,2}_{loc}(\Omega)$ be a weak solution to problem* (4.5) *with $1 < q \leq 2$. Then, for a.e. $s \in (0, |\Omega|)$, it holds*

$$u_*'(s) \leq \frac{1}{n^2 \omega_n^{2/n} s^{2-2/n}} \int_0^s \exp\left(\int_\sigma^s \frac{(u_*'(\tau))^{q-1}}{(n\omega_n^{1/n} \tau^{1-1/n})^{2-q}} \, d\tau \right) (\kappa_q - f_*(\sigma)) \, d\sigma.$$

We conclude this section making an observation about a consequence of Theorem 4.3 on the existence of a solution for a model problem of the type (1.1). Let us consider, for $\mu > 0$, the following problem

$$\begin{cases} -\Delta u + \mu u + |Du|^q = f \quad \text{in } \Omega \\ \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{4.9}$$

where $1 < q \leq 2$. By classical results (see, e.g., [3], [7]) there exists a solution to the above problem and it is a natural question to understand what happens to the solutions of (4.9) when μ goes to zero. In particular one can ask if the solution to problem (4.9) converges to a solution of the following one:

$$\begin{cases} -\Delta \varphi + |D\varphi|^q = f \qquad \text{in } \Omega \\ \\ \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{4.10}$$

Such a question is addressed in [21] and the following result is proven.

Theorem 4.5. *Assume that $1 < q \leq 2$, and $f \in L^\infty(\Omega)$. For $\mu > 0$, let u_μ be the solution of* (4.9) *and let $\kappa_q(\Omega, f)$ be the ergodic constant of problem* (4.5). *If $\kappa_q(\Omega, f) > 0$ then $u_\mu \to \varphi$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, where φ is the unique solution to problem* (4.10)

For the sake of completeness in the same paper it is also proven that in the case $\kappa_q(\Omega, f) \leq 0$ the sequence u_μ does not converge and its asymptotic behaviour is described.

Here we only observe that, in view of Theorems 4.3, 4.5, a sufficient condition to have a solution to (4.10) is that

$$\kappa_q(\Omega^\#, f_\#) > 0. \tag{4.11}$$

On the other hand, the ergodic constant on a ball is proportional to the first eigenvalue for a problem in the form (4.4) and condition (4.11) can be seen as a condition on the negative part of f, obtaining an existence result for problem (4.10) which improves a similar result contained in [14].

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VINCENZO FERONE
Dipartimento di Matematica e Applicazioni "R.Caccioppoli"
Università di Napoli "Federico II"
Complesso Univ. Monte S. Angelo Via Cintia - 80126 Napoli-Italy
e-mail: ferone@unina.it