LE MATEMATICHE
Vol. LXV (2010) – Fasc. II, pp. 69–78
doi: 10.4418/2010.65.2.9

A BIFURCATION–TYPE THEOREM FOR THE POSITIVE SOLUTIONS OF A NONLINEAR NEUMANN PROBLEM WITH CONCAVE AND CONVEX TERMS

D. KRAVVARITIS - N. S. PAPAGEORGIOU - G. SMYRLIS

We consider a nonlinear elliptic Neumann problem driven by the p-Laplacian with a reaction that involves the combined effects of a "concave" and of a "convex"terms. The convex term (p-superlinear term) need not satisfy the Ambrosetti-Rabinowitz condition. Employing variational methods based on the critical point theory together with truncation techniques, we prove a bifurcation type theorem for the equation.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with a C^2-boundary $\partial\Omega$.

We consider the following nonlinear Neumann problem:

$$\begin{cases} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \lambda |u(z)|^{q-2}u(z) + f(z,u(z)) \\ \text{a.e. in } \Omega, \quad u > 0, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \\ \beta \in L^\infty(\Omega)_+ \setminus \{0\}, \quad \lambda > 0, \quad 1 < q < p < \infty. \end{cases} \tag{1}$$

Entrato in redazione: 25 ottobre 2010

AMS 2010 Subject Classification: 35J20, 35J65

Keywords: C-condition, nonlinear maximum principle, local minimizers, p-Laplacian, bifurcation type theorem.

Here $\Delta_p u = \text{div}\,(||Du||^{p-2}Du)$.

Note that the term $x \to \lambda|x|^{q-2}x$ is $(p-1)$–*sublinear* near $+\infty$ (“concave” term).

The Carathéodory function $f(z,x)$, $z \in \Omega$, $x \in \mathbb{R}$ is supposed to be $(p-1)$–*superlinear* near $+\infty$ (“convex” perturbation).

The aim of this work is to establish a *bifurcation-type* result for the positive smooth solutions of (1), with respect to the parameter $\lambda > 0$.

Particular case. The right hand side term of (1) has the form

$$x \to \lambda|x|^{q-2}x + |x|^{r-2}x,$$

with $1 < q < p < r < p^*$ (= the critical Sobolev exponent). This particular case is what we mostly encounter in the literature and only in the context of Dirichlet problems. In this direction we mention the semilinear (i.e., $p = 2$) work of Ambrosetti-Brezis-Cerami [1], which was the first to consider problems with concave and convex terms. The above work was extended to nonlinear problems driven by the p-Laplacian, by Garcia Azorero-Manfredi-Peral Alonso [4] and by Guo-Zhang [5], for $p \geq 2$.

For Dirichlet problems driven by the p-Laplacian and with reactions of more general form we refer to the following works:

- Boccardo-Escobedo-Peral [3], where the reaction is $\lambda g(x) + x^{r-1}$, $x \geq 0$, $g : \mathbb{R}_+ \to \mathbb{R}$ continuous with $(q-1)$-polynomial growth with $1 < q < p < r < p^*$ and the function $x \to \lambda g(x) + x^{r-1}$ is nondecreasing on $\mathbb{R}_+$. In their work, they prove the existence of only one positive solution for $\lambda > 0$ suitably small.

- Hu-Papageorgiou [6], where the “convex” ($(p-1)$–superlinear) term is a more general Caratheodory function $f(z,x)$ satisfying the well-known *Ambrosetti-Rabinowitz (AR) condition.*

To the best of our knowledge, no bifurcation-type results exist for the *Neumann* problem. We mention only the work of Wu-Chen [11], where the reaction is of the form $\lambda f(z,x)$, $\lambda > 0$ and $f(\cdot,\cdot)$ is $(p-1)$–sublinear near infinity in $x \in \mathbb{R}$. In [11], the authors also impose the extra restrictive conditions that $\operatorname*{essinf}_{\Omega} \beta > 0$ and that $N < p$. They produce three solutions for all $\lambda > 0$ in an open interval. The obtained solutions are not positive.

2. The hypotheses on the perturbation.

$(\mathbf{H})$: The Carathéodory function $f(z,x)$, $z \in \Omega$, $x \in \mathbb{R}$ has $(r-1)$–polynomial growth with respect to x, where $p < r < p*$. Moreover

(i) $\lim\limits_{x\to 0^+} \dfrac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$

(ii) there exists $\delta_0 > 0$ such that $f(z,x) \geq 0$ for a.a. $z \in \Omega$, all $x \in [0, \delta_0]$

and $\forall \theta > 0$, $\exists\, \hat{\xi}_\theta > 0$ such that for a.a. $z \in \Omega$,

$$x \to f(z,x) + \hat{\xi}_\theta x^{p-1} \quad \text{is increasing on } [0,\theta].$$

(iii) if $F(z,x) = \int_0^x f(z,s)ds$, then

$$\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty, \quad \eta_0 \leq \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^\tau},$$

uniformly for a.a. $z \in \Omega$, where

$$\tau \in \left((r-p)\max\left\{1, \frac{N}{p}\right\},\ p^* \right), \quad q < \tau, \quad \eta_0 > 0.$$

Remark 2.1. In order to express the " $(p-1)$–superlinearity" of $f(z,x)$ with respect to x near $+\infty$, instead of the usual in such cases AR-condition, we employ the much weaker conditions H(iii).

3. Some function spaces

In the study of our problem we will use the following two function spaces

$$C_n^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}, \quad W_n^{1,p}(\Omega) = \overline{C_n^1(\overline{\Omega})}^{||\cdot||},$$

where $||\cdot||$ denotes the Sobolev norm of $W^{1,p}(\Omega)$.

Note that $C_n^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C_n^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

4. The Euler functional

Let $\varphi_\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ be the Euler functional for problem (1) defined by

$$\varphi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \frac{\lambda}{q}||u^+||_q^q - \int_\Omega F(z,u)dz,$$

where $F(z,x) = \int_0^x f(z,s)ds.$

Proposition 4.1. *Under hypotheses (H), $\varphi_\lambda \in C^1\left(W_n^{1,p}(\Omega)\right)$ and each nontrivial critical point of φ_λ is a positive smooth solution of (1).*

The proof is mainly based on the nonlinear regularity theory and also on the nonlinear maximum principle combined with hypothesis H(ii).

Proposition 4.2. *Under hypotheses (H), φ_λ satisfies the Cerami condition (***C***–condition): " Every sequence $\{x_n\}_{n\geq 1} \subseteq X = W_n^{1,p}(\Omega)$ such that*

$$\sup_n |\varphi_\lambda(x_n)| < \infty, \quad (1+||x_n||)\varphi'_\lambda(x_n) \to 0 \ \text{in} \ X^* \ \text{as} \ n \to \infty,$$

has a strongly convergent subsequence. "

Some ideas of the proof may be found in [9, proof of Proposition 1]. We note that hypothesis H(iii) is crucially used.

5. The bifurcation -type result

Theorem 5.1. *If hypotheses (H) hold and $\beta \in L_+^\infty(\Omega) \setminus \{0\}$, then there exists $\lambda^* > 0$ such that*

(a) *for $\lambda \in (0, \lambda^*)$ problem (1) has at least two positive smooth solutions*

(b) *for $\lambda = \lambda^*$ problem (1) has at least one positive smooth solution*

(c) *for $\lambda > \lambda^*$ problem (1) has no positive solution*

The proof of Theorem 1 may be divided into two parts:

Part I. We prove that the set

$$S = \{\lambda > 0 : \text{ problem (1) has a positive smooth } \lambda \text{ -solution}\}$$

is nonempty and bounded from above.

Part II. We prove that $\lambda^* = \sup S$ has the desired properties.

Sketch of the proof of Part I.

Proposition 5.2. *There exists* $\hat{\lambda} > 0$ *such that for every* $\lambda \in (0, \hat{\lambda})$ *we can find* $\rho_\lambda > 0$ *for which we have*

$$\inf[\, \varphi_\lambda(u) : ||u|| = \rho_\lambda \,] = \eta_\lambda > 0.$$

For the proof of Prop. 5.2 one needs to work in a similar way as in the proof of Lemma 2.1 (i) of [7], taking into account hypothesis H(i) combined with the fact that $q < p$.

Proposition 5.3. *We have*

$$\varphi_\lambda(tu) \to -\infty \quad \textit{as } t \to +\infty,$$

for each $u \in C_+ \setminus \{0\}$ *with* $||u||_p = 1$.

The proof of Prop. 5.3 is based on the p-superlinearity of $F(z,x)$ with respect to x near $+\infty$ (H(iii)) and also on the fact that $q < p$.

Now Prop. 4.1, 4.2, 5.2, 5.3 via Mountain Pass Theorem yield

Proposition 5.4. *If* $\hat{\lambda}$ *is as postulated in Prop. 5.2, then* $(0, \hat{\lambda}) \subseteq S$ *. Hence,* $S \neq \varnothing$.

To proceed, we prove that S is bounded from above. We begin with a comparison result stated below:

Lemma 5.5. *Let* $\beta \in L^\infty(\Omega)_+ \setminus \{0\}$ *,* $u, \widetilde{u} \in$ *int* C_+ *and* $R > 0$ *such that for a.a.* $z \in \Omega$,

$$-\Delta_p u(z) + \beta(z)u(z)^{p-1} + R \leq -\Delta_p \widetilde{u}(z) + \beta(z)\widetilde{u}(z)^{p-1}.$$

Then $u < \widetilde{u}$ *on* $\overline{\Omega}$.

The proof of the above lemma is mainly based on the monotonicity properties of the operator $T : X \to X^*$ $(X = W_n^{1,p}(\Omega))$ induced by the differential operator $u \to -\Delta_p u + \beta(\cdot)|u|^{p-2}u$.

Proposition 5.6. *The set S is bounded from above.*

Proof. The $(p-1)$–superlinearity of $f(z,x)$ with respect to x near $+\infty$ combined with hypothesis H(ii) enables us to choose $\overline{\lambda} > 0$ large such that

$$\overline{\lambda} x^{q-1} + f(z,x) \geq ||\beta||_\infty x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0.$$

Claim. $\overline{\lambda}$ is an upper bound of S.

Indeed, suppose that for some $\lambda > \overline{\lambda}$ our problem has a λ -solution $u \in$ int C_+. Let $m = \min_{\overline{\Omega}} u > 0$. Then for a.a. $z \in \Omega$,

$$\begin{aligned} -\Delta_p u(z) + \beta(z)u(z)^{p-1} &\geq ||\beta||_\infty u(z)^{p-1} + (\lambda - \overline{\lambda})u(z)^{q-1} \\ &\geq -\Delta_p m + \beta(z)m^{p-1} + (\lambda - \overline{\lambda})m^{q-1} \end{aligned}$$

which implies (see Lemma 5.5) that $u > m$ on $\overline{\Omega}$ (false!). □

Sketch of the proof of Part II.

Lemma 5.7. *Let* $0 < \lambda < \widetilde{\lambda}$ *and* $\widetilde{u} \in int\, C_+$ *be a* $\widetilde{\lambda}$ *-solution. Then there exists a* λ*-solution* $u_0 \in int\, C_+$ *such that* $0 < u_0 < \widetilde{u}$ *on* $\overline{\Omega}$, $\varphi_\lambda(u_0) < 0$.

Proof. We consider the following truncation of the reaction:

$$g_\lambda(z,x) = \begin{cases} 0, & \text{if} \quad x \leq 0 \\ \lambda x^{q-1} + f(z,x), & \text{if} \quad 0 < x < \widetilde{u}(z) \\ \lambda\, \widetilde{u}(z)^{q-1} + f(z,\widetilde{u}(z)), & \text{if} \quad \widetilde{u}(z) \leq x. \end{cases}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds$ and consider the C^1-functional

$$\psi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \int_\Omega G_\lambda(z,u)dz.$$

By using suitable test functions we may show that each critical point of ψ_λ lies in the interval $[0,\widetilde{u}]$ and it is also a critical point of the Euler functional φ_λ.

Moreover, ψ_λ is coercive and weakly lower semicontinuous.

Employing hypothesis H(ii) in conjunction with Lemma 5.5 and with Proposition 4.1, we may show that each global minimizer u_0 of ψ_λ satisfies the conclusions of Lemma 5.7. □

To proceed, set $\lambda^* = \sup S$.

Proposition 5.8. *If* $\lambda \in (0,\lambda^*)$, *then problem (1) has least two smooth positive solutions*

$$u_0\,,\ \hat{u} \in \text{int}\, C_+\,, \quad u_0 \neq \hat{u}, \quad u_0 \leq \hat{u}, \quad \varphi_\lambda(u_0) < 0.$$

Proof. Let $\lambda \in (0,\lambda^*)$. Choose $\widetilde{\lambda} \in (\lambda,\, \lambda^*) \cap S$ and a $\widetilde{\lambda}$ -solution $\widetilde{u} \in$ int C_+.

By view of Lemma 5.7, we may find a λ -solution $u_0 \in$ int C_+ such that

$$0 < u_0 < \widetilde{u}, \quad \varphi_\lambda(u_0) < 0.$$

Next, consider the following truncation of the reaction:

$$\hat{f}_\lambda(z,x) = \begin{cases} \lambda u_0(z)^{q-1} + f(z,u_0(z)), & \text{if } x \leq u_0(z) \\ \lambda x^{q-1} + f(z,x), & \text{if } u_0(z) < x. \end{cases}$$

Let $\hat{F}_\lambda(z,x) = \displaystyle\int_0^x \hat{f}_\lambda(z,s)ds$ and consider the C^1-functional

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \int_\Omega \hat{F}_\lambda(z,u)dz.$$

By using suitable test functions we may show that for each critical point w of $\hat{\varphi}_\lambda$, we have $u_0 \leq w$ and that w is also a critical point of the Euler functional φ_λ.

Evidently, $\hat{\varphi}_\lambda \mid_{[0,\,\widetilde{u}]}$ is coercive and weakly lower semicontinuous, so, it possesses a minimizer $\widetilde{u}_0 \in [0,\, \widetilde{u}]$. It is known that in general, $\widetilde{u}_0$ lies in the normal cone of $[\,0,\, \widetilde{u}\,]$ at $\widetilde{u}_0$, i.e.,

$$0 \leq \int_\Omega ||D\widetilde{u}_0||^{p-2}(D\widetilde{u}_0,\, Dy - D\widetilde{u}_0)_{\mathbb{R}^N} dz + \int_\Omega \beta \widetilde{u}_0^{p-1}\,(y - \widetilde{u}_0\,) - \int_\Omega \hat{f}_\lambda(z, \widetilde{u}_0\,)(y - \widetilde{u}_0\,)dz, \tag{2}$$

for all $y \in [0,\, \widetilde{u}]$.

Let $h \in W_n^{1,p}(\Omega)$ and $\delta > 0$ and define

$$y(z) = \begin{cases} 0, & \text{if} \quad z \in \{\, \widetilde{u}_0 + \delta h \leq 0 \,\} \\ \widetilde{u}_0(z) + \delta h(z), & \text{if} \quad z \in \{\, 0 < \widetilde{u}_0 + \delta h < \widetilde{u} \,\} \\ \widetilde{u}(z), & \text{if} \quad z \in \{\, \widetilde{u} \leq \widetilde{u}_0 + \delta h \,\}. \end{cases}$$

Evidently $y \in [0,\, \widetilde{u}]$, so we may use it as a test function in (2). Arguing in a similar way as in [10, proof of Proposition 9] and then taking the limit as $\delta \to 0^+$, we may show that $\varphi_\lambda'(\widetilde{u}_0) = 0$. Hence, $\widetilde{u}_0$ is a positive smooth λ-solution to our problem.

If $\widetilde{u}_0 \neq u_0$, we are done. Suppose that $\widetilde{u}_0 = u_0$. Since $u_0 \in (0, \widetilde{u})$, we infer that u_0 is a local $C_n^1(\overline{\Omega})$ – minimizer of $\hat{\varphi}_\lambda$ and thus, a local $W_n^{1,p}(\Omega)$ – minimizer of $\hat{\varphi}_\lambda$ (see for example [2, Proposition 3.1]). Without loss of generality, we may assume that u_0 is an isolated critical point and local minimizer of the functional $\hat{\varphi}_\lambda$.Then there exists $r > 0$ such that

$$\hat{\varphi}_\lambda(u_0) < \inf[\, \hat{\varphi}_\lambda(u) : ||u - u_0|| = r\,] \quad \text{(see [8, proof of Prop. 6])}.$$

Moreover, we may show that $\hat{\varphi}_\lambda$ satisfies the conclusions of Propositions 4.2 and 5.3. Now Mountain Pass Theorem gives rise to some critical point $\hat{u}$ of $\hat{\varphi}_\lambda$ such that $\hat{u} \neq u_0$. It follows that $u_0 \leq \hat{u}$ and that $\hat{u}$ is a nontrivial critical point of φ_λ. Thus, $\hat{u}$ is a second positive smooth λ-solution to our problem.

□

Regarding the extremal case $\lambda = \lambda^*$, we show that problem (1) has at least one smooth positive solution.

We begin with an interesting lemma:

Lemma 5.9. *Let $S' \subseteq S$ be nonempty and bounded from below with* $\inf S' > 0$ *and $B \subseteq$ int C_+ be $||\cdot||_\infty$ -bounded. Then there exists $w \in$ int C_+ such that for each $\lambda \in S'$ and for each λ-solution $u \in B$, we have $w \le u$.*

Proof. Set $\lambda_0 = \inf S'$, $\theta = \sup\{||u|| : u \in B\}$ and let $\hat{\xi}_\theta > 0$ be as postulated in hypothesis H(ii). Choose $\rho \in (0, \lambda_0)$ and consider the following Neumann problem:

$$\begin{cases} -\Delta_p w(z) + (\beta(z) + \hat{\xi}_\theta) w(z)^{p-1} = \rho w(z)^{q-1} \quad \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 \quad \text{on a.e. in } \partial\Omega, \quad w > 0. \end{cases} \tag{3}$$

It turns out that (3) has at least one solution $w \in \text{int}\, C_+$. Indeed, we may show that the corresponding Euler functional is coercive (recall that $q < p$) and also that all of its nontrivial critical points lie in int C_+ (see Prop. 4.1).

Now choose $\lambda \in S'$ and let $u \in B$ be a λ -solution. Set

$$t = \min\left\{ \frac{u(z)}{w(z)} : z \in \overline{\Omega} \right\} > 0.$$

It suffices to show that $t \ge 1$. Suppose on the contrary that $0 < t < 1$. Then for a.a. $z \in \Omega$, we have

$$\begin{aligned} -\Delta_p u(z) + (\beta(z) + \hat{\xi}_\theta) u(z)^{p-1} &\ge \lambda u(z)^{q-1} \ge \lambda t^{q-1} w(z)^{q-1} \\ &\ge \lambda_0 t^{p-1} w(z)^{q-1} \text{ (recall: } 0 < t < 1,\ q < p) \\ &= \rho t^{p-1} w(z)^{q-1} + (\lambda_0 - \rho) t^{p-1} w(z)^{q-1} \\ &\ge -\Delta_p (tw)(z) + (\beta(z) + \hat{\xi}_\theta)(tw(z))^{p-1} + R, \end{aligned}$$

where $R = (\lambda_0 - \rho) t^{p-1} \left(\min_{\overline{\Omega}} w \right)^{q-1} > 0.$

Then Lemma 5.5 implies that $u(z) > tw(z)$, $z \in \overline{\Omega}$, which is a contradiction. □

Proposition 5.10. *For $\lambda = \lambda^*$, problem (1) has at least one smooth positive solution.*

Proof. Choose a nondecreasing sequence $(\lambda_n) \subseteq S$ such that $\lambda_n \uparrow \lambda^*$. By view of Prop.5.8, we may find $\{u_n\}_{n \ge 1} \subseteq \text{int}\, C_+$ such that

$$\varphi'_{\lambda_n}(u_n) = 0, \quad \varphi_{\lambda_n}(u_n) < 0, \quad \text{for all } n \ge 1.$$

Arguing in a similar way as in [9, proof of Proposition 1], we may show (by passing to subsequences) that

$$u_n \to u_*, \text{ strongly in } W_n^{1,p}(\Omega).$$

Then nonlinear regularity theory guarantees that $\sup_n ||u_n||_\infty < \infty$ and that u_* is a smooth λ^*-solution.

Now Lemma 5.9 asserts that for some $w \in \text{ int } C_+$, we have $w \leq u_n$, $n \geq 1$. Thus, $w \leq u_*$, so $u_* \in \text{ int } C_+$. □

REFERENCES

[1] A. Ambrosetti - H. Brezis - G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. 122 (1994), 519–543.

[2] G. Barletta - N. S. Papageorgiou, *A multiplicity theorem for the Neumann p-Laplacian with an asymmetric nonsmooth potential*, J. Glob. Optim. 39 (2007), 365–392.

[3] L. Boccardo - M. Escobedo - I. Peral, *A Dirichlet problem involving critical exponents*, Nonlinear Anal. 24 (1995), 1639–1648.

[4] J. Garcia Azorero - J. Manfredi - I. Peral Alonso, *Sobolev versus Hölder minimizers and global multiplicity for some quasilinear elliptic equations*, Commun. Contemp. Math. 2 (2000), 385–404.

[5] Z. Guo - Z. Zhang, *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. 286 (2003), 32–50.

[6] S. Hu - N. S. Papageorgiou, *Multiplicity of solutions for parametric p-Laplacian equations with nonlinearity concave near the origin*, Tohoku Math. J. 62 (2010), 1–26.

[7] S. Li - S. Wu - H.-S. Zhou, *Solutions to Semilinear Elliptic Problems with Combined Nonlinearities*, J. Diff. Equ. 185 (2002), 200–224.

[8] D. Motreanu - V. Motreanu - N. S. Papageorgiou, *A degree theoretic approach for multiple solutions of constant sign for nonlinear elliptic equations*, Manuscripta Math. 124 (2007), 507–531.

[9] N. S. Papageorgiou - G. Smyrlis, *A multiplicity theorem for Neumann problems with asymmetric nonlinearity*, Annali di Matematica Pura ed Applicata 189 (2010), 253–272.

[10] N. S. Papageorgiou - G. Smyrlis, *Multiple solutions for nonlinear Neumann problems with the p-Laplacian and a nonsmooth crossing potential*, Nonlinearity 23 (2010), 529–548.

[11] X. Wu - L. Chen, *Existence and multiplicity of solutions for elliptic equations involving the p- Laplacian*, NoDEA 15 (2008), 745–755.

DIMITRIE KRAVVARITIS
Department of Mathematics
National Technical University of Athens
e-mail: dkrav@math.ntua.gr

NIKOLAOS S. PAPAGEORGIOU
Department of Mathematics
National Technical University of Athens
e-mail: npapg@math.ntua.gr

GEORGE SMYRLIS
Department of Mathematics
Technological Educational Institute of Athens
e-mail: gsmyrlis@teiath.gr