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THREE NONTRIVIAL SOLUTIONS FOR NEUMANN PROBLEMS RESONANT AT ANY POSITIVE EIGENVALUE

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We consider a semilinear Neumann problem with a parametric reaction which has a concave term and a perturbation which at $\pm\infty$ can be resonant with respect to any positive eigenvalue. Using variational methods based on the critical point theory and Morse theory, we show that there exists a critical parameter value $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$, then the problem has at least three nontrivial smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary. We study the following nonlinear Neumann problem:

$$\begin{aligned} &-\triangle u(z) = \lambda |u(z)|^{q-2} u(z) + f\big(z, u(z)\big) \quad \text{in } \Omega, \\ &\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,\ \lambda > 0, 1 < q < 2. \end{aligned} \tag{P$_\lambda$}$$

We are looking for multiple nontrivial smooth solutions when the equation is resonant with respect to any positive (i.e., nonprincipal) eigenvalue of the negative Neumann Laplacian.

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In the past resonant Neumann problems were investigated by Gupta [5], Iannacci-Nkashama [6, 7], Kuo [9], Mawhin [13], Mawhin-Ward-Willem [14], Rabinowitz [20], Tang [21] and Tang-Wu [22]. Gupta [5] and Iannacci-Nkashama [6, 7] use a sign condition on the reaction term. Kuo [9] uses a kind of Landesman-Lazer type condition, while Mawhin [13] and Mawhin-Ward-Willem [14] use a monotonicity condition. Rabinowitz [20] uses a periodicity condition. Finally, Tang [21] and Tang-Wu [22] employ an anticoercivity condition on $F(z,\cdot)$ (recall $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$). With the exception of Iannacci-Nkashama [6] and Tang [21], all the other works mentioned above, consider problems resonant with respect to the principal eigenvalue $\widehat{\lambda}_0 = 0$. Moreover, only Tang [21] proves multiplicity results. The others have existence theorems. In fact, Tang [21] considers equations resonant with respect to $\widehat{\lambda}_0 = 0$ and also equations resonant with respect to $\widehat{\lambda}_k > 0$, $k \geq 1$. Under different hypotheses in the two cases proves the existence of two nontrivial solutions using variational methods (the local linking theorem and the reduction technique). In his problem the reaction is z-independent, i.e., $f(z,x) = f(x)$ and C^1.

In this paper we use critical point theory and Morse theory to prove a multiplicity theorem establishing the existence of at least three nontrivial smooth solutions for equations which can be resonant at $\pm\infty$ with respect to any positive eigenvalue $\widehat{\lambda}_k > 0$. We point out that the term $\lambda|x|^{q-2}x$, $1<q<2$ is *concave* (sublinear) term. So, our reaction (the right-hand side of the equation) is not C^1 even if $f(z,\cdot)$ is. This is in contrast to Tang [21]. None of the aforementioned works allows the presence of concave terms. Equations with such terms were investigated in the context of Dirichlet problems. In this direction, we mention the works of de Paiva-Massa [4], Li-Wu-Zhou [12], Perera [19] and Wu-Yang [23].

2. Mathematical Background

In this section, for the convenience of the reader, we recall some of the main mathematical tools which we will use in this paper.

We start with critical point theory. Let X be a Banach space and X^* its topological dual. By $\langle \cdot,\cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition* (the C-condition for short), if the following is true:

> "Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1+\|x_n\|)\varphi'(x_n) \longrightarrow 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence".

Using this compactness-type condition, we can derive the following theorem, known in the literature as the *mountain pass theorem*, which gives a minimax characterization of certain critical values of a C^1-functional.

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$ and satisfies the C-condition, $x_0, x_1 \in X$, $\|x_1 - x_0\| > r > 0$*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf[\varphi(x) : \|x - x_0\| = r] = \eta_r$$

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \varphi\big(\gamma(t)\big) \text{ where } \Gamma = \big\{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\big\},$$

then $c \geq \eta_r$ and c is a critical value of φ.

For $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \le c\}, \ K_\varphi = \{x \in X : \varphi'(x) = 0\} \text{ and } K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the $k^{\underline{\text{th}}}$ relative singular homology group with integer coefficients for the pair (Y_1, Y_2). For $k < 0$, $H_k(Y_1, Y_2) = 0$. The critical groups of φ at an isolated critical point $x \in X$ with $\varphi(x) = c$ (i.e., $x \in K_\varphi^c$), are defined by

$$C_k(\varphi, x) = H_k\big(\varphi^c \cap U, \varphi^c \cap U \setminus \{x\}\big) \quad \text{for all } k \geq 0,$$

with U a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$ (see Chang [2] and Mawhin-Willem [15]). The excision property of singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood U of x.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0.$$

The deformation theorem, (valid since φ satisfies the C-condition, see Papageorgiou-Kyritsi [18, p. 274]), implies that this definition of critical groups is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$. If K_φ is finite, then we set

$$M(t, x) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, x) t^k \ \text{ for all } t \in \mathbb{R},\ x \in K_\varphi$$

$$\text{and} \quad P(t, \infty) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k \ \text{ for all } t \in \mathbb{R}.$$

We have the *Morse relation*

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t) Q(t), \tag{1}$$

where $Q(t)=\sum_{k\geq 0}\beta_k t^k$ is a formal series in $t\in\mathbb{R}$ with nonnegative integer coefficients (see [2] and [15]).

Let $X=H$ be a Hilbert space, $x\in H$, U a neighborhood of x, and $\varphi\in C^2(U)$. If $x\in K_\varphi$, then its *Morse index* is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(x)$ is negative definite. We say that $x\in K_\varphi$ is *nondegenerate*, if $\varphi''(x)$ is invertible. The critical groups of φ at a nondegenerate critical point $x\in H$ with Morse index d are given by $C_k(\varphi,x)=\delta_{k,d}\mathbb{Z}$ for all $k\geq 0$, where $\delta_{k,d}$ is the Kronecker symbol, i.e., $\delta_{k,d}=\begin{cases}1 & \text{if } k=d\\ 0 & \text{if } k\neq d\end{cases}$.

Next, let us recall some basic facts about the spectrum of the negative Neumann Laplacian. Let $m\in L^\infty(\Omega)$, $m\geq 0$, $m\neq 0$, (a weight function), and consider the following weighted linear eigenvalue problem

$$-\triangle u(z)=\widehat{\lambda}m(z)u(z)\ \text{ in }\Omega,\ \frac{\partial u}{\partial n}=0\ \text{ on }\partial\Omega. \tag{2}$$

Note that $\widehat{\lambda}_0=\widehat{\lambda}_0(m)=0$ is an eigenvalue of (2) with corresponding eigenspace $\mathbb{R}$. Moreover, using the spectral theorem for compact operators, we can show that (2) has a sequence $\{\widehat{\lambda}_k(m)\}_{k\geq 0}$ of distinct eigenvalues such that $\widehat{\lambda}_k(m)\to+\infty$. If $m\equiv 1$, then we simply write $\widehat{\lambda}_k$ for $\widehat{\lambda}_k(1)$.

For every integer $k\geq 0$, by $E\big(\widehat{\lambda}_k(m)\big)$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(m)$. Regularity theory implies that $E\big(\widehat{\lambda}_k(m)\big)\subseteq C^1_n(\overline{\Omega})$ and it has the unique continuation property (UCP for short), namely if $u\in E\big(\widehat{\lambda}_k(m)\big)$ vanishes on a set of positive measure, then $u(z)=0$ for all $z\in\overline{\Omega}$. We set

$$\overline{H}_k=\bigoplus_{i=0}^{k}E\big(\widehat{\lambda}_i(m)\big)\ \text{ and }\ \widehat{H}_k=\overline{H}_k^{\perp}=\overline{\bigoplus_{i\geq k+1}E\big(\widehat{\lambda}_i(m)\big)}.$$

Using these spaces, we have the following variational characterizations of the eigenvalues $\{\widehat{\lambda}_k(m)\}_{k\geq 0}$:

$$0=\widehat{\lambda}_0(m)=\min\Big[\frac{\|Du\|_2^2}{\int_\Omega mu^2\mathrm{d}z}:u\in H^1(\Omega),u\neq 0\Big] \tag{3}$$

and for $k\geq 1$

$$\begin{aligned}\widehat{\lambda}_k(m)&=\max\Big[\frac{\|D\overline{u}\|_2^2}{\int_\Omega m\overline{u}^2\mathrm{d}z}:\overline{u}\in\overline{H}_k,\overline{u}\neq 0\Big]\\ &=\min\Big[\frac{\|D\widehat{u}\|_2^2}{\int_\Omega m\widehat{u}^2\mathrm{d}z}:\widehat{u}\in\widehat{H}_{k-1},\widehat{u}\neq 0\Big].\end{aligned} \tag{4}$$

In (3) the minimum is attained on $E\left(\widehat{\lambda}_0(m)\right) = \mathbb{R}$. In (4) the maximun and the minimum are both realized on $E\left(\widehat{\lambda}_k(m)\right)$. From (4) and the UCP, it is clear that the following monotonicity property is true for the eigenvalues:

"If $m, m' \in L^\infty(\Omega)$, $0 \leq m \leq m'$, $m \neq 0$, $m' \neq m$, then $\widehat{\lambda}_k(m') < \widehat{\lambda}_k(m)$ for all $k \geq 1$".

From Iannizzotto-Papageorgiou [8], we take the following simple lemma:

Lemma 2.2. *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leq 0$ a.e. in Ω and $\vartheta \neq 0$, then there exists $\xi_0 > 0$ such that $\|Du\|_2^2 - \int_\Omega \vartheta(z)u(z)^2 dz \geq \xi_0 \|u\|^2$ for all $u \in H^1(\Omega)$.*

Hereafter by $\|\cdot\|$ we denote the norm in the Sobolev space $H^1(\Omega)$ and by $\|\cdot\|_2$ the norm of $L^2(\Omega)$ and of $L^2(\Omega, \mathbb{R}^N)$. Finally, by $|\cdot|_N$ we denote the Lesesgue measure on $\mathbb{R}^N$ and $2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N > 2 \\ +\infty & \text{if } N \leq 2 \end{cases}$ is the Sobolev critical exponent.

3. Three Nontrivial Solutions

The hypotheses on $f(z,x)$ are:

H: $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \leq \alpha(z) + c|x|^{r-2}$ for a.a $z \in \Omega$, all $x \in \mathbb{R}$, with $\alpha \in L^\infty(\Omega)_+$, $c > 0$, $2 < r < 2^*$;

(ii) there exist an integer $m \geq 1$, a function $\eta_0 \in L^\infty(\Omega)$, $\beta_0 > 0$ and $\mu \in (q, 2]$ such that

$$\eta_0(z) \leq \liminf_{|x| \to \infty} \frac{f(z,x)}{x} \leq \limsup_{|x| \to \infty} \frac{f(z,x)}{x} \leq \widehat{\lambda}_m,$$

uniformly for a.a. $z \in \Omega$

$$\eta_0(z) \geq \widehat{\lambda}_{m-1} \text{ for a.a. } z \in \Omega, \text{ and if } m = 1, \text{ then } \eta_0 \neq \widehat{\lambda}_{m-1},$$

$$\beta_0 \leq \liminf_{|x| \to \infty} \frac{f(z,x)x - 2F(z,x)}{|x|^\mu} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) $f'_x(z,0) = \lim_{x \to 0} \frac{f(z,x)}{x} \leq 0$ uniformly for a.a. $z \in \Omega$ and $f'_x(\cdot,0) \neq 0$.

Remark 3.1. Hypothesis H(ii) implies that at $\pm\infty$ we can have resonance with respect to any positive eigenvalue. In fact double resonance is possible for $m \geq 2$.

Example 3.2. The following function $f(x)$ satisfies hypotheses H (for the sake of simplicity we drop the z-dependence):

$$f(x) = \begin{cases} \lambda_m x - |x|^{\tau-2}x - \frac{\xi}{|x|} & \text{if} \quad x < -1 \\ -x & \text{if} \ -1 \leq x \leq 1 \\ \lambda_m x - x^{\tau-1} + \frac{\xi}{x} & \text{if} \quad 1 < x, \end{cases}$$

with $m \geq 2$, $1 < q < \tau < 2$ and $\xi = (2-\tau)/2 > 0$.

Let $\varphi_\lambda : H^1(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F\big(z, u(z)\big)\mathrm{d}z \quad \text{for all } u \in H^1(\Omega)$$

(recall that $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$). Evidently $\varphi_\lambda \in C^1\big(H^1(\Omega)\big) \cap C^2\big(H^1(\Omega) \setminus \{0\}\big)$.

Proposition 3.3. *If hypotheses H hold and $\lambda > 0$, then φ_λ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$\varphi_\lambda(u_n) \longrightarrow c \in \mathbb{R} \ \text{ as } n \to \infty \tag{5}$$

$$\text{and} \quad (1 + \|u_n\|)\varphi_\lambda'(u_n) \longrightarrow 0 \ \text{ in } H^1(\Omega)^* \ \text{ as } n \to \infty. \tag{6}$$

From (6) we have

$$\begin{aligned} &|\langle \varphi_\lambda'(u_n), h\rangle| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h \in H^1(\Omega) \text{ with } \varepsilon_n \longrightarrow 0^+, \\ \Rightarrow &\Big| \langle A(u_n), h\rangle - \lambda \int_\Omega |u_n|^{q-2}u_n h\mathrm{d}z - \int_\Omega f(z,u_n)h\mathrm{d}z \Big| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \end{aligned} \tag{7}$$

for all $n \geq 1$, where $A \in \mathscr{L}\big(H^1(\Omega), H^1(\Omega)^*\big)$ is defined by

$$\langle A(u), y\rangle = \int_\Omega (Du, Dy)_{\mathbb{R}^N}\mathrm{d}z$$

for all $u, y \in H^1(\Omega)$. In (7) we choose $h = u_n \in H^1(\Omega)$ and have

$$\Big| \|Du_n\|_2^2 - \lambda\|u_n\|_q^q - \int_\Omega f(z,u_n)u_n\mathrm{d}z \Big| \leq \varepsilon_n \quad \text{for all } n \geq 1.$$

Since $\varepsilon_n \longrightarrow 0^+$, given $\varepsilon > 0$ we can find $n_0 = n_0(\varepsilon) \geq 1$ such that

$$-\varepsilon \leq -\|Du_n\|_2^2 + \lambda\|u_n\|_q^q + \int_\Omega f(z,u_n)u_n\mathrm{d}z \leq \varepsilon \quad \text{for all } n \geq n_0. \tag{8}$$

Also, by virtue of (5), we can find $n_1 = n_1(\varepsilon) \geq n_0$ such that

$$2c-\varepsilon \leq \|Du_n\|_2^2 - \frac{2}{q}\lambda\|u_n\|_q^q - \int_\Omega 2F(z,u_n)\mathrm{d}z \leq 2c+\varepsilon \quad \text{for all } n \geq n_1. \tag{9}$$

Adding (8) and (9) we obtain

$$\begin{aligned}&\int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\mathrm{d}z \leq 2(c+\varepsilon) + \lambda\Big(\frac{2}{q}-1\Big)\|u_n\|_q^q \\ &\qquad \text{for all } n \geq n_1, \qquad (10)\\ &\Rightarrow \limsup_{n\to\infty} \frac{1}{\|u_n\|^\mu}\int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\mathrm{d}z \leq 0\end{aligned}$$

(recall that $q < \mu \leq 2$).

Claim: $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ is bounded.

We argue by contradiction. So, we assume that $\|u_n\| \longrightarrow \infty$ and set $y_n = \frac{u_n}{\|u_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so (at least for a subsequence), we have

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \longrightarrow y \text{ in } L^2(\Omega). \tag{11}$$

From (7) we have

$$\begin{aligned}&\Big|\langle A(y_n),h\rangle - \frac{\lambda}{\|u_n\|^{2-q}}\int_\Omega |y_n|^{q-2}y_n h\mathrm{d}z - \int_\Omega \frac{f(z,u_n)}{\|u_n\|}h\mathrm{d}z\Big| \\ &\leq \varepsilon_n \frac{\|h\|}{(1+\|u_n\|)\|u_n\|} \quad \text{for all } n \geq 1.\end{aligned} \tag{12}$$

Choose $h = y_n - y \in H^1(\Omega)$ and pass to the limit as $n \to \infty$. Using (11) and because $\left\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|}\right\} \subseteq L^2(\Omega)$ is bounded (see H(i), (ii)), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0,\\ &\Rightarrow \|Dy_n\|_2 \longrightarrow \|Dy\|_2 \text{ as } n \to \infty.\end{aligned}$$

From (11) we also have $Dy_n \xrightarrow{w} Dy$ in $L^2(\Omega,\mathbb{R}^N)$ and so from the Kadec-Klee property of Hilbert spaces, we have that $Dy_n \longrightarrow Dy$ in $L^2(\Omega,\mathbb{R}^N)$. Therefore

$$y_n \longrightarrow y \text{ in } H^1(\Omega) \text{ and } \|y\| = 1. \tag{13}$$

Since $\left\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|} = g_n(\cdot)\right\}_{n\geq 1} \subseteq L^2(\Omega)$ is bounded, we may assume that

$$g_n \xrightarrow{w} g \text{ in } L^2(\Omega). \tag{14}$$

Using hypothesis H(ii) and reasoning as in Motreanu-Motreanu-Papageorgiou [16, see the proof of Proposition 5], we show that

$$g(z) = \xi(z)y(z) \text{ a.e. in } \Omega, \text{ with } \eta_0(z) \le \xi(z) \le \widehat{\lambda}_m \text{ a.e. in } \Omega. \tag{15}$$

So, if in (12) we pass to the limit as $n \to \infty$, recalling that $q \in (1,2)$ and using (13) through (15), we obtain

$$\begin{aligned} &\langle A(y), h\rangle = \int_\Omega \xi y h \mathrm{d}z \quad \text{for all } h \in H^1(\Omega), \\ &\Rightarrow A(y) = \xi y, \\ &\Rightarrow -\triangle y(z) = \xi(z) y(z) \text{ a.e. in } \Omega, \ \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{16}$$

Recall that $\eta_0(z) \le \xi(z) \le \widehat{\lambda}_m$ a.e. in Ω (see (15)). If $m = 1$, then $\xi \ge 0$, $\xi \ne 0$ and if $\xi \ne \widehat{\lambda}_1$, then from (16) we have $y = 0$, since $\widehat{\lambda}_1(\widehat{\lambda}_1) = 1 < \widehat{\lambda}_1(\xi)$. But this contradicts (15). If $m \ge 2$ and $\xi \ne \widehat{\lambda}_{m-1}$, $\xi \ne \widehat{\lambda}_m$, then again by virtue of the monotonicity of the eigenvalues with respect to the weight (see Section 2), we have $\widehat{\lambda}_{m-1}(\xi) < \widehat{\lambda}_{m-1}(\widehat{\lambda}_{m-1}) = 1$ and $\widehat{\lambda}_m(\xi) > \widehat{\lambda}_m(\widehat{\lambda}_m) = 1$. This fact together with (16) implies that $y = 0$, a contradiction to (15). So, suppose that $\xi = \widehat{\lambda}_{m-1}$ or $\xi = \widehat{\lambda}_m$. Then $y \in E(\widehat{\lambda}_m)$ and so by the UCP $y(z) \ne 0$ a.e. in Ω (recall that $y \ne 0$, see (15)). This implies that $|u_n(z)| \longrightarrow +\infty$ for a.a. $z \in \Omega$. Hence by virtue of hypothesis H(ii) we have

$$0 < \beta_0 \le \liminf_{n\to\infty} \frac{f\big(z, u_n(z)\big)u_n(z) - 2F\big(z, u_n(z)\big)}{|u_n(z)|^\mu} \text{ for a.a. } z \in \Omega. \tag{17}$$

Using Fatou's lemma, we have

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{\|u_n\|^\mu} \int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\mathrm{d}z \\ &\liminf_{n\to\infty} \int_\Omega \frac{f(z,u_n)u_n(z) - 2F(z,u_n)}{|u_n|^\mu} |y_n|^\mu \mathrm{d}z \ge \beta_0 \|y\|_\mu^\mu > 0 \end{aligned} \tag{18}$$

(see (17) and (15)).

Comparing (10) and (18) we reach a contradiction. This proves the Claim.

Because of the Claim, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \longrightarrow u \text{ in } L^2(\Omega) \text{ as } n \to \infty. \tag{19}$$

In (7) we choose $h = u_n - u$, pass to the limit as $n \to \infty$ and use (19) and use the fact that $\left\{\frac{f\left(\cdot,u_n(\cdot)\right)}{\|u_n\|}\right\}_{n\geq 1} \subseteq L^2(\Omega)$ is bounded. We obtain

$$\begin{aligned}
&\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,\\
\Rightarrow\ & u_n \longrightarrow u \text{ in } H^1(\Omega) \text{ as } n\to\infty \quad \text{(as before)},\\
\Rightarrow\ & \varphi_\lambda \text{ satisfies the C-condition.}
\end{aligned}$$

□

Proposition 3.4. *If hypotheses H hold, then there exists $\lambda^* > 0$ such that for every $\lambda \in (0,\lambda^*)$, we can find $\rho_\lambda \in (0,1)$ small such that* $\inf[\varphi_\lambda(u) : \|u\| = \rho_\lambda] = m_\lambda > 0$.

Proof. Hypotheses H(i) and (iii) imply that given $\varepsilon > 0$, we can find $c_1 = c_1(\varepsilon) > 0$ such that

$$F(z,x) \leq \frac{1}{2}\big(f'_x(z,0)+\varepsilon\big)x^2 + c_1|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{20}$$

Recall that $f'_x(z,0) \leq 0$ a.e. in Ω and $f'_x(\cdot,0) \neq 0$. For every $u \in H^1(\Omega)$ we have:

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F\big(z,u(z)\big)\mathrm{d}z\\
&\geq \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{q}\|u\|_q^q - \frac{1}{2}\int_\Omega f'_x(z,0)u^2\mathrm{d}z - \frac{\varepsilon}{2}\|u\|^2 - c_2\|u\|^r\\
&\text{for some } c_2 > 0 \quad \text{(see (20))}\\
&\geq \frac{\xi_0-\varepsilon}{2}\|u\|^2 - \frac{\lambda}{q}c_3\|u\|^q - c_2\|u\|^r \text{ for some } c_3 > 0
\end{aligned}\tag{21}$$

(see Lemma 2.2). Choosing $\varepsilon \in (0,\xi_0)$, we obtain

$$\begin{aligned}
&\varphi_\lambda(u) \geq c_4\|u\|^2 - \frac{\lambda}{q}c_3\|u\|^q - c_2\|u\|^r \quad \text{with } c_4 = \frac{\xi_0-\varepsilon}{2} > 0,\\
\Rightarrow\ &\varphi_\lambda(u) \geq \big(c_4 - \frac{\lambda}{q}c_3\|u\|^{q-2} - c_2\|u\|^{r-2}\big)\|u\|^2 \quad \text{for all } u \in H^1(\Omega).
\end{aligned}\tag{22}$$

We introduce the function

$$\sigma_\lambda(t) = \frac{\lambda}{q}c_3t^{q-2} + c_2t^{r-2},\ t > 0.$$

Evidently σ_λ is continuous on $(0,+\infty)$ and since $q < 2 < r$, we have

$$\sigma_\lambda(t) \longrightarrow +\infty \text{ as } t \to 0^+ \text{ and } \sigma_\lambda(t) \longrightarrow +\infty \text{ as } t \to +\infty.$$

Therefore, we can find $t_0 \in (0,+\infty)$ such that

$$
\begin{aligned}
& \sigma_\lambda(t_0) = \inf_{(0,+\infty)} \sigma_\lambda, \\
\Rightarrow\ & \sigma'_\lambda(t_0) = \frac{\lambda}{q}(q-2)c_3 t_0^{q-3} + c_2(r-2)t_0^{r-3} = 0, \\
\Rightarrow\ & \frac{\lambda}{q}(2-q)c_3 t_0^{q-3} = c_2(r-2)t_0^{r-3}, \\
\Rightarrow\ & t_0 = t_0(\lambda) = \Big[\frac{\lambda c_3(2-q)}{qc_2(r-2)}\Big]^{\frac{1}{r-q}}.
\end{aligned}
$$

We consider $\sigma_\lambda(t_0)$ and we see that we can find $\lambda^* > 0$ such that

$$\lambda \in (0,\lambda^*) \Rightarrow \sigma_\lambda(t_0) < c_4 \quad \text{(see (22))}.$$

Therefore, from (22) it follows that

$$\inf[\varphi_\lambda(u) : \|u\| = \rho_\lambda] = m_\lambda > 0 \ \text{ where } \rho_\lambda = t_0(\lambda) > 0.$$

$\square$

With the next proposition, we produce the full mountain pass geometry for the functional φ_λ.

Proposition 3.5. *If hypotheses H hold and $\lambda > 0$, then $\varphi_\lambda(\xi) \longrightarrow -\infty$ as $|\xi| \to \infty$, $\xi \in \mathbb{R}$.*

Proof. Hypothesis H(ii) implies that

$$\eta_0(z) \le \liminf_{|x|\to\infty} \frac{2F(z,x)}{x^2} \le \limsup_{|x|\to\infty} \frac{2F(z,x)}{x^2} \le \widehat{\lambda}_m \text{ uniformly for a.a. } z \in \Omega \tag{23}$$

(see, for example, Aizicovici-Papageorgiou-Staicu [1]). Then (23) together with hypothesis H(i), imply that given $\varepsilon > 0$, we can find $c_5 = c_5(\varepsilon) > 0$ such that

$$F(z,x) \ge \frac{1}{2}\big(\eta_0(z) - \varepsilon\big)x^2 - c_5 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{24}$$

Then for $\xi \in \mathbb{R}$ we have

$$
\begin{aligned}
\varphi_\lambda(\xi) &= -\frac{\lambda}{q}|\xi|^q|\Omega|_N - \int_\Omega F(z,\xi)\mathrm{d}z \\
&\le -\frac{\lambda}{q}|\xi|^q|\Omega|_N + \frac{\xi^2}{2}\int_\Omega \big(\varepsilon - \eta_0(z)\big)\mathrm{d}z + c_5|\Omega|_N \quad \text{(see (24))}.
\end{aligned} \tag{25}
$$

Since $\eta_0 \geq 0$, $\eta_0 \neq 0$ (see H(ii)), we see that if $\varepsilon \in (0, \frac{1}{|\Omega|_N}\int_\Omega \eta_0 dz)$, then from (25) it follows that

$$\varphi_\lambda(\xi) \longrightarrow -\infty \text{ as } |\xi| \to \infty, \xi \in \mathbb{R}.$$

□

From Motreanu-Motreanu-Papageorgiou [17], we know that the concave term leads to the triviality of the critical groups of φ_λ, $\lambda > 0$, at $u = 0$.

Proposition 3.6. *If hypotheses H hold and* $\lambda > 0$, *then* $C_k(\varphi_\lambda, 0) = 0$ *for all* $k \geq 0$.

Next we compute the critical groups of φ_λ at infinity.

Proposition 3.7. *If hypotheses H hold and* $\lambda > 0$, *then* $C_k(\varphi_\lambda, \infty) = \delta_{k,d_{m-1}}\mathbb{Z}$ *for all* $k \geq 0$, *where* $d_{m-1} = \dim \overline{H}_{m-1}$.

Proof. Let $\tau \in (\lambda_{m-1}, \lambda_m)$ and consider the C^2-functional $\sigma : H^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\sigma(u) = \frac{1}{2}\|Du\|_2^2 - \frac{\tau}{2}\|u\|_2^2 \quad \text{for all } u \in H^1(\Omega).$$

We consider the homotopy $\widehat{h} : [0,1] \times H^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{h}(t,u) = (1-t)\varphi_\lambda(u) + t\sigma(u) \quad \text{for all } (t,u) \in [0,1] \times H^1(\Omega).$$

Claim : There exist $\alpha \in \mathbb{R}$ and $\delta > 0$ such that for every $t \in [0,1]$

$$\widehat{h}(t,u) \leq \alpha \Rightarrow (1 + \|u\|)\|\widehat{h}'_u(t,u)\| \geq \delta.$$

We argue indirectly. So, suppose that the Claim is not true. Then we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$\begin{gathered} t_n \longrightarrow t \in [0,1],\ \|u_n\| \longrightarrow \infty,\ \widehat{h}(t,u_n) \longrightarrow -\infty \\ \text{and } (1 + \|u_n\|)\widehat{h}'_u(t_n,u_n) \longrightarrow 0 \text{ as } n \to \infty \end{gathered} \tag{26}$$

(recall that $\widehat{h}$ is bounded, i.e., maps bounded sets to bounded ones).

From the last convergence in (26), we have

$$\begin{aligned} &\left| \langle A(u_n), h\rangle - \lambda(1-t_n)\int_\Omega |u_n|^{q-2}u_n h dz - (1-t_n)\int_\Omega f(z,u_n)h dz - t_n\tau\int_\Omega u_n h dz \right| \\ &\leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \end{aligned} \tag{27}$$

for all $h \in H^1(\Omega)$ with $\varepsilon_n \to 0^+$.

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \longrightarrow y \text{ in } L^2(\Omega) \text{ as } n \to \infty. \tag{28}$$

Multiplying (27) with $\frac{1}{\|u_n\|}$, we obtain

$$\Big| \langle A(y_n), h\rangle - \frac{\lambda(1-t_n)}{\|u_n\|^{2-q}} \int_\Omega |y_n|^{q-2} y_n h dz - (1-t_n) \int_\Omega \frac{f(z,u_n)}{\|u_n\|} h dz - t_n \tau \int_\Omega y_n h dz \Big| \leq \frac{\varepsilon_n \|h\|}{(1+\|u_n\|)\|u_n\|} \tag{29}$$

Recall that

$$\frac{f(\cdot, u_n(\cdot))}{\|u_n\|} \xrightarrow{w} g = \xi y \text{ in } L^2(\Omega) \text{ with } \eta_0(z) \leq \xi(z) \leq \lambda_m \text{ a.e. in } \Omega \tag{30}$$

(see (14), (15)).

If in (29) we choose $h = y_n - y \in H^1(\Omega)$ and pass to the limit as $n \to \infty$, then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0 \quad \text{(see (28), (30))}, \\ &\Rightarrow y_n \longrightarrow y \text{ in } H^1(\Omega) \text{ and so } \|y\| = 1. \end{aligned} \tag{31}$$

Passing to the limit as $n \to \infty$ in (29) and using (30) and (31) and the fact that $q < 2$, we obtain

$$\begin{aligned} &\langle A(y), h\rangle = \big((1-t)\xi + t\tau\big)y, \\ &\Rightarrow -\triangle y(z) = \xi_t(z) y(z) \text{ a.e. in } \Omega, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \end{aligned} \tag{32}$$

with $\xi_t(z) = (1-t)\xi(z) + t\tau$.

Note that $\widehat{\lambda}_{m-1} \leq \xi_t(z) \leq \widehat{\lambda}_m$ a.e. in Ω.

If $t \in (0,1]$, then the two inequalities are strict and so as before exploiting the nonotonicity of the eigenvalues on the weight function, we infer that $y = 0$, which contradicts (31).

If $t = 0$, then $\xi_0 = \xi$. As in the proof of Proposition 3.3, the cases $m = 1$, $\xi \neq \widehat{\lambda}_1$ and $m \geq 2$, $\xi \neq \widehat{\lambda}_{m-1}$, $\xi \neq \widehat{\lambda}_m$, lead to $y = 0$, a contradiction to (31). So, we assume that $\xi = \widehat{\lambda}_{m-1}$ or $\xi = \widehat{\lambda}_m$. Then $y(z) \neq 0$ a.e. in Ω by the UCP and so $|u_n(z)| \longrightarrow \infty$ for a.a. $z \in \Omega$ and this by virtue of hypothesis H(ii) implies that

$$0 < \beta_0 \leq \liminf_{n\to\infty} \frac{f\big(z, u_n(z)\big)u_n(z) - 2F\big(z, u_n(z)\big)}{|u_n(z)|^\mu} \quad \text{for a.a. } z \in \Omega. \tag{33}$$

From the third convergence in (26), we see that we can find $n_2 \geq 1$ such that

$$\|Du_n\|_2^2 - (1-t_n)\frac{2\lambda}{q}\|u_n\|_q^q - (1-t_n)\int_\Omega 2F(z,u_n)\mathrm{d}z - t_n\tau\|u_n\|_2^2 \leq 0 \qquad (34)$$

for all $n \geq n_2$.

Similarly, from (27) and if $h = u_n \in H^1(\Omega)$, then given $\varepsilon > 0$, we can find $n_3 = n_3(\varepsilon) \geq n_2$ such that

$$-\|Du_n\|_2^2 + \lambda(1-t_n)\|u_n\|_q^q + (1-t_n)\int_\Omega f(z,u_n)u_n\mathrm{d}z + t_n\tau\|u_n\|_2^2 \leq \varepsilon \qquad (35)$$

for all $n \geq n_3$.

Since $t_n \longrightarrow t = 0$, we may assume that $1 - t_n > 0$ for all $n \geq n_3$. Adding (34) and (35) we obtain

$$\int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\mathrm{d}z \leq \frac{\varepsilon_n}{1-t_n} + \lambda\left(\frac{2}{q} - 1\right)\|u_n\|_q^q \quad \text{for all } n \geq n_3,$$

$$\Rightarrow \limsup_{n\to\infty}\int_\Omega \frac{\big(f(z,u_n)u_n - 2F(z,u_n)\big)}{|u_n|^\mu}\,\mathrm{d}z \leq 0 \qquad (36)$$

(recall that $q < \mu$ and $\varepsilon > 0$ was arbitrary).

On the other hand from (33) and Fatou's lemma, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{\|u_n\|^\mu}\int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\mathrm{d}z \\
&\geq \int_\Omega \liminf_{n\to\infty}\frac{f(z,u_n)u_n(z) - 2F(z,u_n)}{|u_n|^\mu}|y_n|^\mu\mathrm{d}z \\
&\geq \beta_0\|y\|_\mu^\mu > 0.
\end{aligned} \qquad (37)$$

Comparing (36) and (37) we reach a contradiction. This proves the Claim.

Note that $\widehat{h}(0,\cdot) = \varphi_\lambda$ satisfies the C-condition (see Proposition 3.3). Similarly since $\tau \in (\lambda_{m-1}, \lambda_m)$, $\widehat{h}(1,\cdot) = \sigma$ too satisfies the C-condition. So, because of the Claim we can apply from Liang-Su [10, Proposition 3.2] and have

$$\begin{aligned}
&C_k\big(\widehat{h}(0,\cdot),\infty\big) = C_k\big(\widehat{h}(1,\cdot),\infty\big) \quad \text{for all } k \geq 0, \\
&\Rightarrow C_k(\varphi_\lambda,\infty) = C_k(\sigma,\infty) \quad \text{for all } k \geq 0.
\end{aligned} \qquad (38)$$

Since $\tau \in (\lambda_{m-1}, \lambda_m)$, $u = 0$ is the only critical point of σ and it is a nondegenerate critical point of Morse index $d_{m-1} = \dim \overset{m-1}{\underset{i=0}{\oplus}} E(\widehat{\lambda}_i)$. Hence

$$\begin{aligned}
&C_k(\sigma,\infty) = C_k(\sigma,0) = \delta_{k,d_{m-1}}\mathbb{Z} \quad \text{for all } k \geq 0, \\
&\Rightarrow C_k(\varphi_\lambda,\infty) = \delta_{k,d_{m-1}}\mathbb{Z} \quad \text{for all } k \geq 0 \quad (\text{see (38)}).
\end{aligned}$$

$\square$

Now we are ready for the "three solutions theorem" for problem (P_λ).

Theorem 3.8. *If hypotheses H hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem* (P_λ) *has at least three nontrivial smooth solutions $\widehat{u}, u_0, y_0 \in C^1(\overline{\Omega})$.*

Proof. Let $\lambda^* > 0$ be as in Proposition 3.4. By virtue of Propositions 3.3, 3.4 and 3.5, we can apply Theorem 2.1 (the mountain pass theorem), and obtain $\widehat{u} \in H^1(\Omega)$ a critical point of φ_λ. Then $\widehat{u}$ solves (P_λ) and $\widehat{u} \in C^1(\overline{\Omega})$ (by the regularity theory).

Moreover, from Li-Li-Liu [11, Theorem 2.7], we have

$$C_k(\varphi_\lambda, \widehat{u}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{39}$$

From Proposition 3.6, we know that

$$C_k(\varphi_\lambda, 0) = 0 \quad \text{for all } k \geq 0. \tag{40}$$

Comparing (39) and (40), we infer that $\widehat{u} \neq 0$ (alternatively the nontriviality of $\widehat{u}$ results from Theorem 2.1, since $\varphi_\lambda(0) = 0 < m_\lambda \leq \varphi_\lambda(\widehat{u})$ (see Proposition 3.4)).

Exploiting the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$, we can easily check that φ_λ is sequentially weakly lower semicontinuous. Since $\overline{B}_{\rho_\lambda} = \{u \in H^1(\Omega) : \|u\| \leq \rho_\lambda\}$ ($\rho_\lambda > 0$ as in Proposition 3.4) is weakly compact, by the Weierstrass theorem, we can find $u_0 \in \overline{B}_{\rho_\lambda}$ such that

$$\varphi_\lambda(u_0) = \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda.$$

For $\xi \in \mathbb{R}$, $\|\xi\| \leq \rho_\lambda$ small, since $q < 2$, we see that $\varphi_\lambda(\xi) < 0 = \varphi_\lambda(0)$ (see hypothesis H(iii)). Therefore

$$\varphi_\lambda(u_0) < 0 = \varphi_\lambda(0), \Rightarrow u_0 \neq 0.$$

Let $\sigma = \inf_{\partial B_{\rho_\lambda}} \varphi_\lambda - \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda > 0$ (see Proposition 3.4). Let $\varepsilon \in (0, \sigma)$. Invoking the Ekeland variational principle (see, for example, Papageorgiou-Kyritsi [18, p. 89]), we can find $u_\varepsilon \in \overline{B}_{\rho_\lambda}$ such that

$$\varphi_\lambda(u_\varepsilon) \leq \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda + \varepsilon \tag{41}$$

$$\text{and } \varphi_\lambda(u_\varepsilon) \leq \varphi_\lambda(y) + \varepsilon\|y - u_\varepsilon\| \quad \text{for all } y \in \overline{B}_{\rho_\lambda}. \tag{42}$$

From (41) and since $\varepsilon < \sigma$, we see that

$$\varphi_\lambda(u_\varepsilon) < \inf_{\partial B_{\rho_\lambda}} \varphi_\lambda, \Rightarrow u_\varepsilon \in B_{\rho_\lambda}.$$

Let $\psi_\lambda^\varepsilon(y) = \varphi_\lambda(y) + \varepsilon\|y - u_\varepsilon\|$ for all $y \in H^1(\Omega)$. Clearly this is a locally Lipschitz function and from (42) we see that u_ε is a minimizer of ψ_λ^ε on $\overline{B}_{\rho_\lambda}$. So, if by $\partial\psi_\lambda^\varepsilon(u_\varepsilon)$ we denote the generalized subdifferential of ψ_λ^ε at u_ε and by $(\psi_\lambda^\varepsilon)^0(u_\varepsilon;\cdot)$ the generalized directional derivative of ψ_λ^ε at u_ε (see Clarke [3]), we have

$$\begin{aligned}
&0 \in \partial\psi_\lambda^\varepsilon(u_\varepsilon) \quad (\text{recall } u_\varepsilon \in B_{\rho_\lambda}),\\
\Rightarrow\quad &0 \le (\psi_\lambda^\varepsilon)^0(u_\varepsilon;h) \quad \text{for all } h \in H^1(\Omega),\\
\Rightarrow\quad &-\varepsilon\|h\| \le \left\langle \varphi_\lambda'(u_\varepsilon), h\right\rangle \quad \text{for all } h \in H^1(\Omega).
\end{aligned}$$

Invoking from Papageorgiou-Kyritsi [18, Lemma 4.1.44, p. 287], we can find $u^* \in H^1(\Omega)^*$, $\|u^*\|_* \le 1$ such that

$$\begin{aligned}
&\varepsilon\langle u^*, h\rangle \le \left\langle \varphi_\lambda'(u_\varepsilon), h\right\rangle \quad \text{for all } h \in H^1(\Omega)\\
&\Rightarrow \varphi_\lambda'(u_\varepsilon) = \varepsilon u^*.
\end{aligned} \tag{43}$$

Let $\varepsilon_n = \frac{1}{n}$ and set $u_n = u_{\varepsilon_n} \in B_{\rho_\lambda}$. Then

$$\begin{aligned}
&\varphi_\lambda(u_n) \longrightarrow \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda \quad (\text{see (41)})\\
\text{and}\quad &\varphi_\lambda'(u_n) \longrightarrow 0 \text{ in } H^1(\Omega)^* \quad (\text{see (43)}).
\end{aligned}$$

Invoking Proposition 3.3, we have

$$\begin{aligned}
&u_n \longrightarrow u_0 \text{ in } H^1(\Omega),\\
\Rightarrow\quad &\varphi_\lambda(u_0) = \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda \text{ and } \varphi_\lambda'(u_0) = 0.
\end{aligned}$$

Therefore u_0 is a solution of (P_λ), $u_0 \in C^1(\overline{\Omega})$ (by regularity theory), and $u_0 \neq \{\widehat{u}, 0\}$.

Suppose that $\{0, \widehat{u}, u_0\}$ are the only critical points of φ_λ. From Proposition 3.7, we have

$$c_k(\varphi_\lambda, \infty) = \delta_{k,d_{m-1}}\mathbb{Z} \quad \text{for all } k \ge 0. \tag{44}$$

Also since u_0 is a local minimizer of φ_λ, we have

$$c_k(\varphi_\lambda, u_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \ge 0. \tag{45}$$

From (39), (40), (44), (45), and the Morse relation (1) with $t = -1$, we obtain

$$(-1)^0 + (-1)^1 = (-1)^{d_{m-1}}$$

a contradiction. So, φ_λ has one more critical point $y_0 \notin \{0, \widehat{u}, u_0\}$. Then y_0 solves problem (P_λ) and by regularity theory $y_0 \in C^1(\overline{\Omega})$. □

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