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PERIODIC SOLUTIONS OF THE FORCED PENDULUM : CLASSICAL VS RELATIVISTIC

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The paper surveys and compares some results on the existence and multiplicity of T-periodic solutions for the forced classical pendulum equation

$$u'' + A\sin u = h(x),$$

the forced p-pendulum equation

$$(|u'|^{p-2}u')' + A\sin u = h(x)$$

and the forced relativistic pendulum equation

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + A\sin u = h(x).$$

1. Introduction

The periodic solutions of the classical forced pendulum equation are the solutions of the problem

$$u'' + A\sin u = h(x), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{1}$$

where $A > 0$, $T > 0$, $h \in L^1 := L^1(0,T)$ are given. We set $\omega := 2\pi/T$. Notice that the case where $A < 0$ is reduced to this one by considering $v = u + \pi$. The

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problem is to find conditions upon the data under which problem (1) has at least one solution, and to discuss the possible multiplicity of the solutions.

Let us first look for *necessary* conditions for the existence of a solution to (1). Given a vector subspace S of L^1, we define

$$\widetilde{S} := \left\{ v \in S \mid \overline{v} := \frac{1}{T}\int_0^T v = 0 \right\}$$

Assuming that problem (1) has a solution, integrating both members of the equation over $[0,T]$ and using the boundary conditions, we see that a necessary condition for existence of a solution to (1) is that $\overline{h} \in [-A,A]$. Consequently, a necessary conditions for existence of a solution to problem (1) for all $A > 0$, and all $T > 0$ is that $h \in \widetilde{L^1}$. A natural question is the sufficiency of this necessary condition. We show in Section 2 that the answer is positive, recall the history of the problem and the various methods used to solve it.

A natural generalization of problem (1) consists in replacing u'' by the p-Laplacian $(|u'|^{p-2}u')'$ with $p > 2$. We show in Section 3 that all the results of Section 2 can be extended to this more general frame. Another generalization consists is replacing $(|u'|^{p-2}u')'$, which is associated to the homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(s) = |s|^{p-2}s$, by the relativistic-type acceleration $\left(\frac{u'}{\sqrt{1-u'^2}}\right)'$, associated to the homeomorphism $\phi : (-1,1) \to \mathbb{R}$ with $\phi(s) = \frac{s}{\sqrt{1-s^2}}$. We describe in Section 4 some results recently obtained in this direction in collaboration with H. Brezis [3]. Finally, we mention that the problem where u'' is replaced by the curvature operator $\left(\frac{u'}{\sqrt{1+u'^2}}\right)'$ associated to the homeomorphism $\phi : \mathbb{R} \to (-1,1)$ with $\phi(s) = \frac{s}{\sqrt{1+s^2}}$ seems to be open.

2. The forced classical pendulum

The first important contribution to problem (1) was given in 1922 by G. Hamel [9], in an issue of the *Mathematische Annalen* dedicated to Hilbert's sixtieth anniversary. In this paper, Hamel considered the special case of (1)

$$u'' + A\sin u = B\cos\omega x, \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{2}$$

and observed that (2) is the Euler-Lagrange equation of the C^1 action functional

$$\mathscr{I}_H(u) := \int_0^T \left(\frac{u'^2}{2} + A\cos u + uB\cos\omega x\right) dx$$

on the space

$$C^1_\# := C^1_\#[0,T] = \{u \in C^1[0,T] \mid u(0) = u(T)\}.$$

Using the direct method of the calculus of variations, Hamel showed that $\mathscr{I}_H$ has a minimum over $C^1_\#$, and hence that problem (2) has at least one solution for all $A > 0$, and $B \in \mathbb{R}$. It is easy to see that Hamel's proof remains valid for $B\cos\omega x$ replaced by any $h \in \widetilde{C^0}$. Hamel's result was rapidly forgotten.

In a more modern and appropriate setting, problem (1) is the Euler-Lagrange equation of the C^1 action functional

$$\mathscr{I}(u) = \int_0^T \left(\frac{u'^2}{2} + A\cos u + uh\right) dx$$

on the Sobolev space

$$H^1_\# = H^1_\#(0,T) := \{u \in H^1(0,T) \mid u(0) = u(T)\}$$

Motivated by a question raised in 1979 by Fučik in [8], Castro [4] used in 1980 a variational Lyapunov-Schmidt reduction method to prove that problem (1) has at least one solution for all $0 < A < \omega^2$, and all $h \in \widetilde{L^2}$. Willem [18] in 1981 and Dancer [6] in 1982 independently showed that for all $A > 0$, and all $h \in \widetilde{L^1}$, $\mathscr{I}$ has a minimum over $H^1_\#$, by proving that $\mathscr{I}$ is weakly sequentially lower semi-continuous and has a bounded minimizing sequence. Consequently, problem (1) has at least one solution for all $A > 0$, and all $h \in \widetilde{L^1}$. All three authors were not aware of the existence of Hamel's paper [9].

The existence of a second geometrically distinct solution of (1) (i.e. of a solution not differering by a multiple of 2π) was first proved in 1984 for all $h \in \widetilde{L^1}$ [12], more than sixty years after Hamel's first solution. The authors showed that, for such h, $\mathscr{I}$ is bounded from below on $H^1_\#$, satisfies Palais-Smale type conditions $(PS)_c$ and (BPS) [13] for all $c \in \mathbb{R}$ (so that $\mathscr{I}$ has a minimum on $H^1_\#$ at some u_0), and has the geometry of a generalized mountain pass lemma with respect to the two minimums u_0 and $u_0 + 2\pi$. As a consequence, problem (1) has at least two solutions for all $A > 0$, and all $h \in \widetilde{L^1}$. Notice that such a multiplicity result is sharp because, for $A \leq \omega^2$, and $h \equiv 0$, problem (1) has exactly the two T-periodic solutions $u_0 \equiv \pi$, and $u_1 \equiv 0$.

Alternate proofs of this multiplicity results were given independently in 1988-89 by Rabinowitz [15], K.C. Chang [5], J. Franks [7] and the author [10] (see also [13]). Franks' proof is symplectic and based upon an generalized Poincaré-Birkhoff theorem. The idea underlying the three other proofs is that for all $h \in \widetilde{L^1}$, and $u \in H^1_\#$, one has $\mathscr{I}(u + 2\pi) = \mathscr{I}(u)$, so that $\mathscr{I}$ can be viewed as defined on $S^1 \times \widetilde{H^1_\#}$, where it is of class C^1, bounded from below, and satisfies the Palais-Smale condition [13]. By Palais' generalization of the Lusternik-Schnirel'man theorem [14], $\mathscr{I}$ has at least $cat_{S^1 \times \widetilde{H^1_\#}}(S^1 \times \widetilde{H^1_\#})$ critical points, where $cat_M(M)$ denotes the Lusternik-Schnirel'man category of the manifold M [13]. Now, one can show that

$$cat_{S^1 \times \widetilde{H^1_\#}}(S^1 \times \widetilde{H^1_\#}) = cat_{S^1}(S^1) = 2,$$

and the multiplicity result follows.

Remarks.

1. The function $A \sin u$ can be replaced by a Carathéodory function $g(x,u)$ periodic in u and such that $\int_0^{2\pi} g(x,u)\, du = 0$ for a.e. $x \in [0,T]$.

2. Extensions have been made to systems, and in particular to the problem

$$u'' + \nabla_u F(x,u) = h(x), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

where $h \in (\widetilde{L}^1)^n$ and F, besides of natural regularity assumptions, is T_i-periodic in each component u_i of u.

3. All known proofs of the results described in this Section are variational or symplectic (Morse theory [13] can be used as well).

4. All known existence results based upon degree theory require restrictions upon A and T, but cover situations where $\overline{h}$ can be different from 0.

For more informations and references, the reader can consult the recent survey [11].

3. The forced 'p-pendulum' ($p > 1$)

In order to describe with more details some of the techniques mentioned in Section 2, let us consider the more general problem of the periodic solutions of the forced 'p-pendulum equation'

$$(|u'|^{p-2}u')' + A \sin u = h(x), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{3}$$

where $p > 1$, $A > 0$, $T > 0$, and $h \in L^1$. A solution of (3) is a function $u \in C^1_\#$, such that $|u'|^{p-2}u'$ is absolutely continuous on $[0,T]$ and (3) is satisfied. Again, a necessary condition for the existence of a solution to (3) is that $\overline{h} \in [-A,A]$, so that a necessary condition for the existence of a solution for all $A > 0$, and all $T > 0$ is that $h \in \widetilde{L^1}$. We show that such a condition is sufficient for the existence of at least two geometrically distinct solutions.

It is standard to prove that problem (3) is the Euler-Lagrange equation of the action functional

$$\mathscr{I}_p(u) := \int_0^T \left(\frac{|u'|^p}{p} + A\cos u + uh\right) dx$$

It is easy to show that $\mathscr{I}_p$ is of class C^1 on the Sobolev space

$$W_{\#}^{1,p} := W_{\#}^{1,p}(0,T) := \{u \in W^{1,p}(0,T) \mid u(0) = u(T)\}$$

and that, for all $h \in \widetilde{L^1}$, and all $u \in W_{\#}^{1,p}$, $\mathscr{I}_p(u+2\pi) = \mathscr{I}_p(u)$. Hence, for all $h \in \widetilde{L^1}$, $\mathscr{I}_p$ can be viewed as defined on $S^1 \times \widetilde{W_{\#}^{1,p}}$.

Lemma 3.1. *$\mathscr{I}_p$ is bounded from below and satisfies the Palais-Smale condition on $S^1 \times \widetilde{W_{\#}^{1,p}}$.*

Proof. Writing $u = \overline{u} + \widetilde{u} \in S^1 \times \widetilde{W_{\#}^{1,p}}$, we have, using Sobolev inequality

$$\begin{aligned} \mathscr{I}_p(u) &\geq \frac{\|u'\|_p^p}{p} - AT - \|h\|_1 \|\widetilde{u}\|_\infty \\ &\geq \frac{\|u'\|_p^p}{p} - AT - \|h\|_1 T^{\frac{p-1}{p}} \|u'\|_p \end{aligned}$$

so that

$$\mathscr{I}_p(u) \to +\infty \quad as \quad \|u'\|_p \to \infty,$$

and hence is bounded from below.

Now let (u_n) be a sequence in $S^1 \times \widetilde{W_{\#}^{1,p}}$ such that $|\mathscr{I}_p(u_n)| \leq C$ for some $C > 0$ and $\mathscr{I}_p'(u_n) \to 0$ as $n \to \infty$. By the first part of the proof, (u_n') is bounded in L^p and so (u_n) is bounded in $S^1 \times \widetilde{W_{\#}^{1,p}}$. Hence, up to a subsequence, we can assume that there exists $u \in S^1 \times \widetilde{W_{\#}^{1,p}}$ such that

$$u_n \to u \quad in \quad C[0,T], \quad u_n \rightharpoonup u \quad in \quad S^1 \times \widetilde{W_{\#}^{1,p}}.$$

Consequently,

$$\langle \mathscr{I}_p'(u_n) - \mathscr{I}_p'(u), u_n - u \rangle \to \quad as \quad n \to \infty.$$

Now

$$\begin{aligned} &\langle \mathscr{I}_p'(u_n) - \mathscr{I}_p'(u), u_n - u \rangle \\ = &\int_0^T (|u_n'|^{p-2}u_n' - |u'|^{p-2}u')(u_n' - u)\, dx \\ - &A \int_0^T (\sin u_n - \sin u)(u_n - u)\, dx, \end{aligned}$$

so that

$$
\begin{aligned}
& 2AT\|u_n - u\|_\infty + \langle \mathscr{I}_p'(u_n) - \mathscr{I}_p'(u), u_n - u\rangle \\
\geq\ & \int_0^T (|u_n'|^{p-2}u_n' - |u'|^{p-2}u')(u_n' - u)\,dx \\
\geq\ & \|u_n'\|_p^p - \int_0^T |u_n'|^{p-1}|u'|\,dx - \int_0^T |u'|^{p-1}|u_n'|\,dx + \|u'\|_p^p \\
\geq\ & \|u_n'\|_p^p - \|u_n'\|_p^{p-1}\|u'\|_p - \|u'\|_p^{p-1}\|u_n'\|_p + \|u'\|_p^p \\
=\ & (\|u_n'\|_p^{p-1} - \|u'\|_p^{p-1})(\|u_n'\|_p - \|u'\|_p) \geq 0.
\end{aligned}
$$

Consequently, if $n \to \infty$, $\|u_n'\|_p \to \|u'\|_p$ and hence $u_n' \to u'$ in L^p, so that $u_n \to u$ in $S^1 \times \widetilde{W_\#^{1,p}}$. □

Theorem 3.2. *For any $h \in \widetilde{L^1}$, problem (3) has at least two geometrically distinct solutions.*

Proof. By Palais' version of Lusternik-Schnirel'mann theorem, $\mathscr{I}_p$ has at least $cat_{S^1\times\widetilde{W_\#^{1,p}}}(S^1 \times \widetilde{W_\#^{1,p}})$ critical points. Now,

$$cat_{S^1\times\widetilde{W_\#^{1,p}}}(S^1 \times \widetilde{W_\#^{1,p}}) = cat_{S^1}(S^1) = 2$$

and hence (3) has at least two solutions for all $A > 0$ and all $h \in \widetilde{L^1}$. □

Remarks.

1. The function $A\sin u$ can be replaced by a Carathéodory function $g(x,u)$ periodic in u with $\int_0^{2\pi} g(x,u)\,du = 0$ for a.e. $x \in [0,T]$.
2. Extensions can be given to systems of the form

   $$(\|u'\|^{p-2}u')' + \nabla_u F(x,u) = h(x), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

   with F like in Section 2.
3. Existence and multiplicity results using fixed point theory and degree theory have been given in [1] under some conditions upon A and T. They do not contain Theorem 3.2.
4. It should be possible replace the p-Laplacian $\left(|u'|^{p-2}u'\right)'$ by more general operators $(\varphi(u'))'$ for suitable classes of homeomorphisms $\varphi : \mathbb{R} \to \mathbb{R}$.

4. The forced 'relativistic' pendulum

A suitable approximation of the problem of periodic solutions of the forced pendulum in special relativity is given by

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + A\sin u = h(x), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{4}$$

where $A > 0$, $T > 0$, and $h \in L^1$. A solution of (4) is a function $u \in C^1_{\#}[0,T]$ such that $\|u'\|_\infty < 1$, $\frac{u'}{\sqrt{1-u'^2}}$ is absolutely continuous on $[0,T]$ and verifies (4).

Again, a necessary condition for the existence of a solution to (4) is that $\overline{h} \in [-A, A]$, so that a necessary condition for existence for all $A > 0$, and all $T > 0$ is that $h \in \widetilde{L}^1$. We describe the results of the recent work [3] showing that this condition is sufficient for the existence of at least one solution.

The action functional associated to problem (4) is given by

$$\mathscr{I}_r(u) := \int_0^T (-\sqrt{1-u'^2} + A\cos u + uh)\,dx.$$

To obtain the set where $\mathscr{I}_r$ is defined, let us introduce the space

$$Lip_{\#} := Lip_{\#}(0,T) := \{u : [0,T] \to \mathbb{R} \mid u \textit{ Lipschitzian},\ u(0) = u(T)\}.$$

If $[u]_{0,1} := \sup_{x \neq y \in [0,T]} \frac{|u(x)-u(y)|}{|x-y|}$, then $Lip_{\#}$ is Banach space with norm

$$\|u\|_{0,1} := \|u\|_\infty + [u]_{0,1}.$$

Furthermore, if $u \in Lip_{\#}$, then u' exists a.e., and $\|u'\|_\infty = [u]_{0,1}$.

$\mathscr{I}_r$ is defined on the closed convex set

$$K = \{u \in Lip_{\#} \mid [u]_{0,1} \leq 1\} = \{u \in Lip_{\#} \mid \|u'\|_\infty \leq 1\}$$

and of class C^1 on $\{u \in Lip_{\#} \mid \|u'\|_\infty < 1\}$.

With respect to the situations considered in Sections 2 and 3, one must notice the following new features :

1. $Lip_{\#}$ is not reflexive.

2. $\mathscr{I}_r$ is only defined on a closed convex subset of $Lip_{\#}$.

Consequently, (4) needs not *a priori* be the Euler-Lagrange equation of $\mathscr{I}_r$, and finding a critical point of $\mathscr{I}_r$ is not sufficient to have a solution of (4). It will be the case if such a critical point satisfies the condition $\|u'\|_\infty < 1$.

We first consider the problem of minimizing $\mathscr{I}_r$ over K. The simple proof of the following lemma can be found in [3].

Lemma 4.1. *For any sequence (u_j) in K converging in $C[0,T]$ to some $u \in K$, one has*

$$\liminf_{j\to\infty} \int_0^T (-\sqrt{1-u_j'^2})\,dx \geq \int_0^T (-\sqrt{1-u'^2})\,dx.$$

Proposition 4.2. *For any $h \in \widetilde{L^1}$, $\mathscr{I}_r$ has a minimum over K.*

Proof. By the 2π-periodicity of $\mathscr{I}_r$, it is equivalent to minimize $\mathscr{I}_r$ on the bounded closed convex set

$$\widehat{K} := \{u \in Lip_\# \mid \overline{u} \in [0,2\pi],\ \|u'\|_\infty \leq 1\}.$$

$\widehat{K}$ is equicontinuous and Ascoli-Arzelá's theorem implies that, up to a subsequence, any minimizing sequence in $\widehat{K}$ converges uniformly to some $u^* \in K$, which, using Lemma 4.1, minimizes $\mathscr{I}_r$ on K. □

We now show that any minimizer of $\mathscr{I}_r$ on K satisfies a variational inequality.

Proposition 4.3. *If $\mathscr{I}_r(u) = \min_K \mathscr{I}_r$, then, for all $v \in K$, one has*

$$\int_0^T \left[-\sqrt{1-v'^2} + \sqrt{1-u'^2} + (-A\sin u + h)(v-u)\right] dx \geq 0.$$

Proof. It consists in starting from the inequality

$$\mathscr{I}_r(u) \leq \mathscr{I}_r[u + \lambda(v-u)]$$

for all $v \in K$ and all $\lambda \in (0,1]$, using the convexity of the function $-\sqrt{1-s^2}$, and letting $\lambda \to 0+$. □

To show that $\|u'\|_\infty < 1$ for any minimizer, we introduce the following auxiliary problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' - u = f(x), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{5}$$

where $f \in L^1$.

Lemma 4.4. *For any $f \in L^1$, problem (5) has a unique solution $\widehat{u}_f$, and*

$$\|\widehat{u}_f'\|_\infty < 1.$$

Proof. Existence is a special case of a result in [2] based upon fixed point theory and degree arguments, and the uniqueness is proved in a standard way. □

A direct consequence of Lemma 4.4 is the following

Corollary 4.5. *For any $f \in L^1$, $\widehat{u}_f \in K$ and, for all $v \in K$, one has*

$$\int_0^T \left[-\sqrt{1-v'^2}+\sqrt{1-\widehat{u}_f'^2}+(\widehat{u}_f+f)(v-\widehat{u}_f)\right]dx \geq 0.$$

One can now state and proof the existence result for (4).

Theorem 4.6. *For any $h \in \widetilde{L^1}$, and for any $A > 0$ problem (4) has at least one solution minimizing $\mathscr{I}_r$ over K.*

Proof. Let $u \in K$ be a minimizer of $\mathscr{I}_r$ over K. Writing the differential equation in (4) in the form

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)'-u=-A\sin u-u+h(x), \quad u(0)=u(T), \quad u'(0)=u'(T)$$

and letting, for any $w \in K$, $f_w := -A\sin w - w + h \in L^1$, we see that u is a solution of the variational inequality

$$\int_0^T \left[-\sqrt{1-v'^2}+\sqrt{1-u'^2}+(u+f_u)(v-u)\right]dx \geq 0$$

for all $v \in K$. Now, for any $w \in K$, the unique solution $\widehat{u}_{f_w}$ of

$$\left(\frac{u'}{\sqrt{1-u'2}}\right)'-u=f_w, \quad u(0)=u(T), \quad u'(0)=u'(T)$$

satisfies the variational inequality

$$\int_0^T \left[-\sqrt{1-v'^2}+\sqrt{1-\widehat{u}_{f_w}'^2}+(\widehat{u}_{f_w}+f_w)(v-\widehat{u}_{f_w})\right]dx \geq 0$$

for all $v \in K$. This easily implies $u = \widehat{u}_{f_u}$ and hence that $\|u'\|_\infty < 1$. $\square$

Remarks.

1. $A\sin u$ can be replaced by any Carathéodory function $g(x,u)$ periodic in u and such that $\int_0^{2\pi} g(x,u)\,du = 0$.
2. $-\sqrt{1-u'^2}$ can be replaced by $\Phi(u')$ with $\Phi \in C[-a,a] \cap C^1(-a,a)$ strictly convex and such that $\phi = \Phi' : (-a,a) \to \mathbb{R}$ is a homeomorphism with $\phi(0) = 0$.

3. The problem of finding a pure *variational* proof of Theorem 4.6 is open, as well as that of the existence of a second geometrically distinct solution.

4. Existence and multiplicity results have been obtained by fixed point theory and degree techniques for $\overline{h}$ possibly different from zero, but under restrictions upon A and T, for the more general equation

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + f(u)u' + A\sin u = h(x)$$

(see [1, 16, 17]). They do not contain Theorem 4.6.

A 'dual' problem of (4) consists in the study of the T-periodic solutions of the forced 'curvature' pendulum equation

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' + A\sin u = h(x), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{6}$$

The corresponding action functional, given by

$$\mathscr{I}_c(u) = \int_0^T (\sqrt{1+u'^2} + A\cos u + uh)\,dx,$$

is defined over $W^{1,1}_{\#} := W^{1,1}_{\#}(0,T) := \{u \in W^{1,1}(0,T) \mid u(0) = u(T)\}$, and its critical points are the solutions of (6). It is easy to see that $\mathscr{I}_c$ is coercive for all $h \in \widetilde{L^1}$ such that $\|H\|_\infty < 1$, where $H' = h$, and $\overline{H} = 0$. Because of the non-reflexivity of $W^{1,1}_{\#}$, this does not imply the existence of a minimum of $\mathscr{I}_c$ over $W^{1,1}_{\#}$, and the problem of the existence of a solution of (6) under those restrictions remains open.

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