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A PATH CROSSING LEMMA AND APPLICATIONS TO NONLINEAR SECOND ORDER EQUATIONS UNDER SLOWLY VARYING PERTURBATIONS

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We present some recent results on the existence of periodic solutions and chaotic like dynamics for second order scalar nonlinear ODEs. The equations under consideration belong to a simple class of perturbed planar Hamiltonian systems with slowly varying periodic coefficients, a typical example being given by the pendulum equation with moving support. Although there is already a broad literature on this subject, our approach, based on the concept of *stretching along the paths*, appears new in this context. In particular, our method is global in nature and stable with respect to small perturbations of the coefficients. Thus it applies even when some small friction terms are inserted into the equations. The main tool on which all our results are based is a topological lemma (that we call *path crossing lemma*) which was already implicitly used by Poincaré (1883-1884) [51], as well as by Butler (1976) [8] and Conley (1975) [12] and subsequently "rediscovered" and applied in many different contexts. For this reason, the first part of this paper is devoted to a detailed exposition of the *Crossing Lemma* and its connections with other topological results.

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1. Introduction

In the present article we are concerned with the existence and multiplicity of periodic solutions to some nonlinear second order differential equations of the form

$$u'' + w(t)g(u) = 0, \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is *locally Lipschitz and globally bounded*, that is

$$\exists M > 0 : \quad |g(x)| \leq M, \ \forall x \in \mathbb{R},$$

and $w : \mathbb{R} \to \mathbb{R}$ is a T-periodic function ($T > 0$) with $w \in L^1([0,T])$. Solutions of (1) are meant in the generalized (Carathéodory) sense (see [20]), namely, $u(\cdot)$ is a solution of (1) if and only if $u : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with u' absolutely continuous and such that $-u''(t) = w(t)g(u(t))$ for almost every $t \in \mathbb{R}$. In the case that the weight function $w(t)$ is continuous, solutions are classical, that is of class C^2.

Equation (1) can be written as the equivalent first order system in the phase plane $(x,y) = (u,u')$

$$\begin{cases} x' = y \\ y' = -w(t)g(x). \end{cases} \tag{2}$$

This is a special case of the planar system

$$z' = Z(t,z), \quad z = (x,y), \tag{3}$$

with $Z : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ locally Lipschitz, T-periodic in the t-variable and satisfying the Carathéodory assumptions (see [20]), and for which we assume there exist $\alpha, \beta \in L^1([0,T],\mathbb{R}^+)$ such that

$$||Z(t,z)|| \leq \alpha(t)||z|| + \beta(t), \text{ for all } x \in \mathbb{R}^2 \text{ and for a.e. } t \in [0,T].$$

The fundamental theory of ODEs guarantees that for any initial point $z_0 = (x_0,y_0) \in \mathbb{R}^2$ and for each $t_0 \in \mathbb{R}$, there exists a unique solution

$$\zeta(\cdot) = \zeta(\cdot\,; t_0, z_0)$$

of (3) which is globally defined on $\mathbb{R}$ and satisfies the initial condition $\zeta(t_0) = z_0$. Hence the Poincaré map

$$\Phi : z_0 \mapsto \zeta(t_0 + T; t_0, z_0),$$

which associates to any point z_0 the value of the solution after the time T, is well defined as a global homeomorphism of $\mathbb{R}^2$ onto itself. In the sequel, we'll constantly take $t_0 = 0$ and write $\zeta(t,z) := \zeta(t;0,z)$, so that

$$\Phi(z) = \zeta(T,z).$$

The search of periodic solutions of (3), looking for fixed points or periodic points of the map Φ, is a classical approach (see [27]). Clearly, $\bar{z} \in \mathbb{R}^2$ is a fixed point for Φ if and only if $\zeta(\cdot,\bar{z})$ is a T-periodic solution of (3) (also called a *harmonic solution*). Moreover, if z_0 is a periodic point of Φ of minimal period $k \geq 2$, that is

$$\Phi^k(z_0) = z_0\,, \quad \text{and } \Phi^i(z_0) \neq z_0\,,\ \forall\, i = 1,\dots,k-1,$$

then $\zeta(t) = \zeta(t,z_0)$ is a kT-periodic solution of (3) with kT minimal among the (positive) integer multiples of T. Clearly, the converse is also true. In this case we say that $\zeta(t)$ is a *subharmonic solution of order k* for (3). Observe also that if z_0 (as above) is a fixed point of Φ^k (of minimal period k) then all the k points $z_0\,, z_1\,,\dots, z_{k-1}$ of the same orbit, with

$$z_j = \Phi^j(z_0), \quad \text{for } j = 1,\dots,k-1$$

are pairwise distinct and of minimal period k. The corresponding kT-periodic solutions $\zeta_j(t) = \zeta(t,z_j) = \zeta(t+jT,z_0)$ are different subharmonic solutions of (3) which, however, are obtained one from the other by a shift in time. In this case we say that the k solutions $\zeta_j(\cdot)$ belong to the same *periodicity class*. In the sequel when we produce some results about multiplicity of subharmonic solutions we refer to solutions not belonging to the same periodicity class. We also recall, as a standard fact, that Φ induces a *discrete dynamical system* on $\mathbb{R}^2$. Accordingly, for every $z \in \mathbb{R}^2$, we denote, respectively, by $O^+(z)$ and $O(z)$ the *positive semiorbit* and the *orbit* of z, defined by

$$O^+(z) := \{\Phi^i(z) : i \in \mathbb{N}\}, \quad O(z) := \{\Phi^i(z) : i \in \mathbb{Z}\},$$

being $\mathbb{N} = \mathbb{Z}^+ = \{0,1,2,\dots,n,\dots\}$ the set of nonnegative integers, and using the convention $\Phi^0(z) = z$.
In this setting, the map Φ is an *orientation preserving* homeomorphism of $\mathbb{R}^2$ onto itself. In the particular case of system (3), which is a planar Hamiltonian system, Φ turns out to be also an *area preserving* map (by Liouville theorem), in the sense that $\mu(\Phi^{-1}(A)) = \mu(A)$ for every Borel subset $A \subset \mathbb{R}^2$ where μ is the Lebesgue measure. Such relevant properties of the Poincaré map allow to apply some deep results of planar dynamical systems like the Brouwer lemma on translation arcs or, for the case of system (3), the Poincaré-Birkhoff fixed point theorem and its generalizations (see for instance [5, 9, 13, 19] and the references therein).

In this paper we wish to present a recent approach which permits to prove the existence of periodic points and to detect the presence of complex dynamics for Φ. Our method, which extends to $\mathbb{R}^N$ and even to the case of compact maps in topological cylinders in normed spaces [45, 50], can be exposed, in the planar setting considered here, by using elementary concepts. From this point of view, the present work is in part expository, in the sense that in Section 2 we give a detailed presentation of all the tools which are needed for our approach. Such tools are based only on elementary plane topology and classical facts like rotation numbers and index with respect to a curve. The main result in this section is the so-called *Crossing Lemma*, a result which, although classical from some point of view, represents our principal tool for the proofs of all our subsequent theorems. We also show the links between this lemma and several other well known theorems for planar maps. In Section 3 we introduce the method of *stretching along the paths* and show how this method can be applied for the existence of fixed points, periodic points and chaotic-like dynamics of planar maps. We also derive a corollary which is particularly suited for the applications to *linked twist maps*. Finally, in Section 4 we present a list of recent and new applications to (1), with special emphasis to the case of differential equations with slowly varying coefficients.

Before starting with the presentation of our results, we describe the kind of dynamics we are interested to detect for the Poincaré map. With this respect, the following definition plays a crucial role in all the subsequent applications of Section 3 and Section 4. It could be formulated in the more general setting of a continuous map (not necessarily a homeomorphism) defined on a suitable domain of a metric space (see [47, Definition 1.1]).

Definition 1. Let $\mathscr{K}_0,\dots,\mathscr{K}_{m-1}\subset\mathbb{R}^2$ (for $m\geq 2$) be m nonempty compact pairwise disjoint sets. We say that Φ *induces chaotic dynamics on m symbols in the set*

$$\mathscr{K} := \bigcup_{i=0}^{m-1}\mathscr{K}_i$$

if for every two-sided sequence of m symbols

$$(s_i)_{i\in\mathbb{Z}}\in\Sigma_m := \{0,\dots,m-1\}^{\mathbb{Z}}$$

there exists a point $z\in\mathbb{R}^2$ such that

$$\Phi^i(z)\in\mathscr{K}_{s_i}\,,\quad \forall\, i\in\mathbb{Z} \tag{4}$$

and, whenever $(s_i)_i$ is a k-periodic sequence (that is, $s_{i+k}=s_i\,,\forall\, i\in\mathbb{Z}$) for some $k\geq 1$, there exists a k-periodic point $z\in\mathscr{K}_{s_0}$ satisfying (4).

The above definition follows one of the classical paradigms about the concept of *chaos*, that is the possibility of reproducing through the dynamics of a given map (in our case the Poincaré map Φ) all the possible outcomes of a coin

flipping experiment [57]. More in detail, assume for a moment that $m = 2$ and thus we have two nonempty compact sets $\mathscr{K}_0$ and $\mathscr{K}_1$ with $\mathscr{K}_0 \cap \mathscr{K}_1 = \emptyset$. We can imagine to attach a label like H=*Head* to $\mathscr{K}_0$ and T=*Tail* to $\mathscr{K}_1$. In this case Definition 1 reads as follows: given any sequence of "Heads" and "Tails" there is a corresponding orbit $O(z)$ for Φ whose points jump from $\mathscr{K}_0$ to $\mathscr{K}_1$ according to the preassigned sequence of "Heads" and "Tails". For instance, if we fix at the beginning the sequence $HHTHT\dots$, we claim that there exists a point $z \in \mathscr{K}_0$ such that $\Phi(z) \in \mathscr{K}_0$, $\Phi^2(z) \in \mathscr{K}_1$, $\Phi^3(z) \in \mathscr{K}_0$, and so on. Besides this aspect, our definition is enhanced also from the fact that we can take two-sided sequences of symbols (and corresponding full orbits for Φ) and, moreover, from the special relevance that we give to the presence of periodic points. To explain a little more such aspect, consider again the case when $m = 2$ as above. According to Definition 1, if we have a periodic sequence of "Heads" and "Tails" we require not only that the corresponding sequence of the iterates of Φ jumps from $\mathscr{K}_0$ to $\mathscr{K}_1$ in a periodic fashion, but we also require that the orbit of Φ is made by periodic points (having the same fundamental period). For instance, if we assign the periodic sequence $HHTHHT\dots$, then we claim the existence of a point $z \in \mathscr{K}_0$ of minimal period $k = 3$ such that $\Phi(z) \in \mathscr{K}_0$, $\Phi^2(z) \in \mathscr{K}_1$ and $\Phi^3(z) = z$. In particular, for a map Φ satisfying our definition with respect to m sets $\mathscr{K}_0, \dots, \mathscr{K}_{m-1}$ we have also the existence of a fixed point for Φ in each of the $\mathscr{K}_i$'s (clearly, fixed points correspond to constant sequences of symbols).
From this point of view, if we prove that a given Poincaré map Φ fulfills the requirements of Definition 1 then we have immediately a result on the existence and multiplicity of harmonic and subharmonic solutions. Once more we stress the special role of the periodic points in our definition, since the existence of periodic points is not always guaranteed for general systems exhibiting chaotic dynamics of the *coin tossing* type (see the example at page 369 in [10] and Example 10 in [25]). Of course, the *chaotic dynamics* according to Definition 1 is not our "discovery". The reader may find in [32, 37, 58, 67, 68] some previous results in which the authors considered chaotic dynamics in which the periodic points play the same role as in our definition.

As a last remark we observe that for a homeomorphism Φ satisfying Definition 1 there exists a nonempty compact set

$$\Lambda \subset \mathscr{K}$$

which is invariant for Φ (i.e., $\Phi(\Lambda) = \Lambda$) and such that $\Phi|_\Lambda$ is semiconjugate by a map $g : \Lambda \to \Sigma_m$ to the two-sided Bernoulli shift σ on m symbols

$$\sigma : \Sigma_m \to \Sigma_m, \quad \sigma\big((s_i)_{i\in\mathbb{Z}}\big) = (s_{i+1})_{i\in\mathbb{Z}},$$

in the sense that g is *continuous and surjective* and the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\Phi"] \arrow[d, "g"'] & \Lambda \arrow[d, "g"] \\ \Sigma_m \arrow[r, "\sigma"'] & \Sigma_m \end{tikzcd}$$

commutes. In particular, the map $\Phi|_\Lambda$ has positive topological entropy [63]. Moreover, the subset of Λ consisting of the periodic points of Φ is dense in Λ and the counterimage (by the semiconjugacy g) of any periodic sequence in Σ_m contains a periodic point of Φ with the same minimal period (see [47] for the details).

2. A path crossing lemma

2.1. Poincaré-Miranda Theorem in the planar case

In 1817 the Czech philosopher and mathematician Bernard Bolzano gave the first proof of the *intermediate value theorem* for a continuous function defined on a compact interval $[a,b] \subset \mathbb{R}$. In 1883-1884 Henri Poincaré (see [35, 36]) obtained an extension of Bolzano's result to the n-dimensional case:

> Soient $X_1, X_2, \dots, X_n$ n fonctions continues des n variables x_1, x_2, $\dots$, x_n. Supposons que X_i soit toujours positif pour $x_i = a_i$ et toujours négatif pour $x_i = -a_i$. Il existera au moins un système de valeurs des x qui satisfera aux inégalités
>
> $$-a_1 < x_1 < a_1,\ -a_2 < x_2 < a_2, \dots, -a_n < x_n < a_n$$
>
> et aux équations
>
> $$X_1 = X_2 = \cdots = X_n = 0.$$

This result was published on the *Bulletin Astronomique* in a paper [51] concerning the three-body problem applied to celestial mechanics; Poincaré showed that the initial conditions of the periodic solutions of a differential system in $\mathbb{R}^n$ must satisfy the hypothesis of this generalization of the intermediate value theorem. But his work remained unknown to the most part of the mathematicians.

This result is now known as Poincaré-Miranda theorem due to the fact that, in 1940, the Italian mathematician Carlo Miranda gave the first proof of its equivalence with the Brouwer fixed point theorem We will present now the precise statement and a possible proof of this theorem in the case $n = 2$.

Theorem 2.1 (Poincaré-Miranda Theorem). *If $f = (f_1, f_2) : R \to \mathbb{R}^2$ is a continuous function defined on the square $R = [-1,1]^2$ such that*

$$f_1(-1,y) \le 0 \le f_1(1,y), \ \ \forall y \in [-1,1] \tag{5}$$

$$f_2(x,-1) \le 0 \le f_2(x,1), \ \ \forall x \in [-1,1] \tag{6}$$

then there exists a point $(x_0,y_0) \in R$ such that $f(x_0,y_0) = 0$.

The main tool used by the proof is the concept of *rotation number of a vector field along a curve* whose definition and properties we are briefly recalling.

Definition 2. We say that $\gamma : [a,b] \to \mathbb{R}^2$ is a *curve* (or a *path*) if γ is a continuous function. We denote by $\bar{\gamma}$ the support of the curve, that is the set $\gamma([a,b])$. A path γ is *closed* (also called a *loop*) if $\gamma(a) = \gamma(b)$.

Definition 3. Let $P \in \mathbb{C} \simeq \mathbb{R}^2$ and $\gamma : [a,b] \to \mathbb{R}^2 - \{P\}$ a closed curve. The rotation number (or *winding number*) of the curve γ about the point P is

$$w(\gamma,P) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-P} .$$

In plain words, w_γ computes the numbers of turns of γ around P. The main properties of this topological tool are the following: [1]

- If $\gamma : [a,b] \to \mathbb{C} - \{P\}$ is a closed path that avoids P, then the path $\gamma - P : [a,b] \to \mathbb{C}^*$ avoids the origin; with the change of variable $u := z - P$ we have
$$w(\gamma,P) = w(\gamma - P, 0) = \frac{1}{2\pi i} \int_{\gamma - P} \frac{du}{u} .$$
- If $\gamma_0, \gamma_1 : [a,b] \to \mathbb{C} - \{P\}$ are two homotopically equivalent closed paths that is there exists a homotopy $h_\lambda(\cdot)$ with $h_0 = \gamma_0$, $h_1 = \gamma_1$ and $h_\lambda : [a,b] \to \mathbb{C} - \{P\}$ for every $\lambda \in [0,1]$ (namely h_λ is an admissible homotopy), then
$$w(\gamma_0,P) = w(\gamma_1,P) .$$
- If $\bar{\gamma}$ is contained in a simply-connected subset $D \subset \mathbb{C} - \{P\}$ then $w(\gamma,P) = 0$.

Definition 4. Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field and $\gamma : [a,b] \to D$ a closed path such that $f(\gamma(t)) \neq P \ \forall t \in [a,b]$. Then $\sigma := f \circ \gamma$ is a path in $\mathbb{R}^2 - \{P\}$. Hence there exists its winding number about the point P according with Definition 3. The *rotation number of f along γ* is defined as

$$\mathrm{rot}_\gamma(f,P) := w(\sigma,P) = w(f \circ \gamma, P) .$$

[1] The reader can find more details in [11, 24]

Theorem 2.2 (Normalization)**.** *If D is a Jordan domain (see Definition 6) whose boundary is anticlockwise oriented, then*

$$\mathrm{rot}_{\partial D}(I-P,P)=1 \quad \textit{for all} \quad P\in \mathring{D}\,.$$

Theorem 2.3 (Kronecker Theorem)**.** *Let γ be a simple closed curve whose support is the boundary of a Jordan domain D. Let $f : D \to \mathbb{R}^2$ be a vector field which never vanishes on ∂D and let $P \in \mathring{D}$. Then we have*

$$\mathrm{rot}_{\gamma}(f,P)\neq 0 \Rightarrow \exists z^{*}\in \mathring{D} \textit{ such that } f(z^{*})=0\,.$$

Given these definitions and properties, we can now expose a proof of Theorem 2.1.

Proof. We can assume $f(x,y)\neq 0$ for all $(x,y)\in\partial R$, otherwise the proof is already complete. This assumption insures that the rotation number $\mathrm{rot}_{\partial R}(f,0)$ is well defined. Consider the homotopy $h_{\lambda}(x,y):=h(x,y,\lambda)$ defined by

$$(x,y,\lambda)\mapsto(1-\lambda)f(x,y)+\lambda(x,y)\,. \tag{7}$$

Clearly h is a well-defined continuous function, but we must also check its admissibility on the boundary ∂R. Consider for example a point $P\in\{-1\}\times[-1,1]$ (the other cases are similar) so that $P=(-1,y)$. If $\lambda=0$

$$h_0(P)=f(-1,y)\neq 0$$

because P lies on ∂R. Similarly, if $\lambda=1$

$$h_1(P)=P\neq 0\,.$$

Fix now a value $\lambda\in(0,1)$. If $h_{\lambda}(P)=0$, from definition (7) follows that

$$(1-\lambda)f(P)+\lambda P=0\Rightarrow f(-1,y)=-\frac{\lambda}{1-\lambda}(-1,y)$$

where $\frac{\lambda}{1-\lambda}$ is a strictly positive constant. But this is clearly a contradiction with the assumption $f_1(-1,y)\leq 0$. Thus we have proved that h is an admissible homotopy and we can conclude that

$$\mathrm{rot}_{\partial R}(f,0)=\mathrm{rot}_{\partial R}(h_1,0)=\mathrm{rot}_{\partial R}(I,0)=1\neq 0\,.$$

As the last step, the thesis comes from Kronecker theorem. (For a more general proof in the case $R\subset\mathbb{R}^n$ using Brouwer degree see, for instance, [62]; for a combinatorial proof of this theorem see [28].) $\square$

This theorem can be easily generalized, indeed the same conclusions hold also if we reverse the inequalities in (5) and (6); moreover we can prove that the result holds also for a general rectangle $R = [a,b] \times [c,d] \subset \mathbb{R}^2$, simply slightly modifying the homotopy h of the proof.

Poincaré-Miranda theorem plays a crucially important role in many contexts related to the topology of $\mathbb{R}^n$ also because it is equivalent to the well-known Brouwer fixed point theorem:

Theorem 2.4 (Brouwer Theorem)**.** *Every continuous function f of the closed unit ball $\mathscr{B} \subset \mathbb{R}^n$ into itself has a fixed point.*

Also in this case we focus our attention only on the planar case. In this setting, a two-dimensional "ball" will be named as a "disc". Now it is very easy to prove that the two-dimensional version of Brouwer theorem can be seen as a corollary of Theorem 2.1 using the following argument. Let $f : R := [-1,1]^2 \to R$ be a continuous function and define a new map

$$g(x,y) = (x,y) - f(x,y)\,.$$

With this definition g satisfies the inequalities (5) and (6), therefore we can conclude that there exists a point in R such that $g(x,y) = 0$. Hence f has a fixed point in the square. Moreover, since the fixed-point property is invariant for homeomorphisms, we conclude that any closed disc and, more generally, every Jordan domain has the fixed point property.

Come back now to the original work by Poincaré [51]. In his article the author, besides giving a proof of the theorem for the zeros of vector fields in the n-dimensional case (based on Kronecker's degree theory), also exposed a simple and intuitive argument, for the case in which $f = (X_1, \dots, X_n)$ has only two components:

> Pour faire comprendre comment on peut démontrer ce théorème, supposons que nous n'ayons que deux variables x_1 et x_2, que nous regarderons comme les coordonnées d'un point dans un plan. Alors les inégalités
>
> $$-a_1 < x_1 < a_1\,, \quad -a_2 < x_2 < a_2\,, \dots, -a_n < x_n < a_n\,,$$
>
> signifient que ce point est à l'intérieur d'un certain carré $ABCD$ dont les côtés ont pour équations
>
> $$AB : x_1 = a_1\,,\; CD : x_1 = -a_1\,,\; BC : x_2 = a_2\,,\; DA : x_2 = -a_2\,.$$
>
> La courbe $X_2 = 0$ part alors d'un point du côté AB pour aboutir à un point de CD; de même la courbe $X_1 = 0$, partant d'un point de BC

pour aboutir à un point de DA, doit forcément recontrer la première à l'intérieur du carré.

We can explain Poincaré's suggestion as follows: given a rectangle, imagine to draw two lines crossing it: the first links the left side with the right one, and the second goes from the top to the bottom of the rectangle. Everybody would say that the two lines intersect at a point in the rectangle. But this fact is not so obvious as it seems, (see Section 2.3.2 for a classical example) so we need to formalize the concept of "crossing a rectangle" in order to obtain sufficient conditions for a nonempty intersection.

2.2. Oriented rectangles

Definition 5. $J \subset \mathbb{R}^2$ is a *Jordan curve* if it is homeomorphic to $\mathscr{S}^1 := \{x \in \mathbb{R}^2 : ||x|| = 1\}$. We can equivalently say that J is a Jordan curve if it is the support of a simple closed curve.

In the sequel we also denote by $\mathscr{B}$ the closed unit disc.

Jordan theorem and Schoenflies theorem, that we recall below, are the two fundamental results about Jordan curves which play a crucial role in our approach. Even if their statements are intuitively clear, a rigorous proof is not elementary. The interested reader can find all the details in [38]. We assume these two results (without proof) as a starting point for our exposition.

Theorem 2.5 (Jordan Theorem)**.** *Every Jordan curve J splits the plane in two connected components, of which it is the common boundary.*

Therefore $\mathbb{R}^2 - J = A_i \cup A_e$, where A_i, A_e are open connected sets such that $A_i \cap A_e = \emptyset$ and $\partial A_i = \partial A_e = J$; moreover A_i is a bounded set, while A_e is unbounded.

Theorem 2.6 (Schoenflies Theorem)**.** *Given a Jordan curve J and a homeomorphism $\eta : \mathscr{S}^1 \to J$, there exists a homeomorphism $\tilde{\eta} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\tilde{\eta}_{|\mathscr{S}^1} = \eta$. As a consequence, we have $\tilde{\eta}(\mathscr{B}) = \overline{A_i} = A_i \cup J$ and $\tilde{\eta}(\mathbb{R}^2 - \mathscr{B}) = A_e$.*

Definition 6. We say that $\mathscr{D} \subset \mathbb{R}^2$ is a *Jordan domain* if it is homeomorphic to $\mathscr{B}$. Equivalently, $\mathscr{D} = \overline{A_i}$, where A_i is the internal part of a Jordan curve.

In our approach the sets under consideration are Jordan domains, but we prefer to think about them by starting with a planar homeomorphism defined on the unit square, instead of the unit disc. In this manner we can more easily introduce a concept of orientation for these sets.

Definition 7. A set $R \subset \mathbb{R}^2$ is a *generalized rectangle* if it is homeomorphic to the square $Q := [0, 1]^2$.

From this definition, given a generalized rectangle R, there exists a homeomorphism $\eta : Q \to \mathbb{R}^2$ such that $\eta(Q) = R$; therefore the boundary of the rectangle ∂R is a Jordan curve. We want now to introduce the concept of the sides of a rectangle, and in particular we are interested in the definition of *pairs of opposite sides*.

Definition 8. An *oriented rectangle* is a pair $(R, R^-) := \widetilde{R}$ such that R is a generalized rectangle and $R^- \subset \partial R$ is the union of two disjoint arcs:

$$R^- = R_l^- \cup R_r^-\,, \quad R_l^- := \eta(\{0\} \times [0,1])\,, \quad R_r^- := \eta(\{1\} \times [0,1])\,.$$

We say that R_l and R_r are respectively the left and the right-hand side of the rectangle R. In the same way we define the top and the bottom sides of R:

$$R^+ = R_b^+ \cup R_t^+\,, \quad R_b^+ := \eta([0,1] \times \{0\})\,, \quad R_t^+ := \eta([0,1] \times \{1\})\,.$$

We can always assume that the sequence of the arcs we meet moving along the boundary is "bottom-right-top-left".

Conversely, suppose that $\mathscr{D} \subset \mathbb{R}^2$ is a Jordan domain and let J' and J'' be two compact disjoint arcs contained in $\partial \mathscr{D}$. Let also L' and L'' be the two compact arcs obtained from the closure of $\partial \mathscr{D} - (J' \cup J'')$. Schoenflies theorem ensures the existence of a homeomorphism $\eta : Q \to \mathscr{D}$ such that

$$\eta(\{0\} \times [0,1]) = J', \quad \eta(\{1\} \times [0,1]) = J''.$$

In this manner, setting

$$\mathscr{D}_l^- := J', \quad \mathscr{D}_r^- := J''$$

and

$$\mathscr{D}^- := J' \cup J'',$$

it follows that the pair $(\mathscr{D}, \mathscr{D}^-) := \widetilde{\mathscr{D}}$ turns out to be an oriented rectangle. In this case we also have that

$$\mathscr{D}^+ = L' \cup L''$$

and the order in which we decide to label the "bottom" and "top" parts is irrelevant.

2.3. Connected sets crossing a rectangle

Definition 9. A space is *connected* if it does not admit a *separation*, that is there do not exist two open nonempty sets with

$$X = U \cup V, \quad U \cap V = \emptyset\,.$$

Notice that the sets U and V are also closed.

Definition 10. Two sets A, B are *connected to each other* in X if there is not a separation U, V in X such that $U \supset A$ and $V \supset B$. On the other hand, we say that A and B are *separated in* X if there exists a separation U, V of X with $U \supset A$ and $V \supset B$.

Remark 2.7. If A and B are separated in X they are not connected to each other in X. Instead the converse assertion is not true.

Definition 11. Consider a pair of points $a, b \in X$. We say that X *is connected from a to b* if the sets $\{a\}$ and $\{b\}$ are connected to each other in X.

The following result, known as Whyburn lemma, plays a crucial role for the proof of the existence of a connected set which crosses a rectangle. It is a key theorem in the theory of connected sets which has found important applications in different areas of nonlinear analysis, mainly in bifurcation theory (see [1, 52]). The proof we give follows closely that contained in the book by Kuratowski [31]. We expose all the details for the readers' convenience (see also [61, p.226]).

Lemma 2.8 (Whyburn Lemma)**.** *Suppose that X is a compact metric space and A, B two nonempty disjoint closed subsets of X. One of these alternatives happens:*

- *there exists a connected set intersecting both A and B;*
- *there exist two compact disjoint sets K_A and K_B such that $K_A \supset A$, $K_B \supset B$ and $X = K_A \cup K_B$.*

Proof. Assume that the second case does not hold, then A and B are not separated in X. Assume also that for every pair of points $(a, b) \in A \times B$ there exists a closed and open set M_{ab} which contains a but such that $b \notin M_{ab}$. For every point $b \in B$ the family $\{M_{ab} : a \in A\}$ is an open covering of the compact set A, so it has a finite subcovering $\{M_{ab} : a \in A'\}$, with A' a finite subset of A. For every point $b \in B$, the set

$$M_b := \bigcup_{a \in A'} M_{ab}$$

is open and closed in X and it contains A. Since every $b \in B$ lies out of everyone of the sets M_{ab},

$$b \in X - M_b \Rightarrow B \subset \{X - M_b : b \in B\}.$$

This means that $\{X - M_b : b \in B\}$ is an open covering of the compact B so there exists a finite subset $B' \subset B$ such that

$$B \subset \bigcup_{b \in B'} X - M_b.$$

Now we consider the open and closed set

$$K_A := \bigcap_{b \in B'} M_b\,.$$

As $M_b \supset A$ for every $b \in B$, it holds that $K_A \supset A$ and $K_A \cap B = \emptyset$, while the set $K_B := X - K_A \supset B$ and $K_B \cap A = \emptyset$, but this is a contradiction with the assumption. Hence we have proved the existence of a pair $(a,b) \in A \times B$ such that the set M_{ab} does not exist. This means that X is connected from a to b.

For such an element $a \in A$, consider now the following set

$$C := \{x \in X : X \text{ is connected from } a \text{ to } x\}\,.$$

If C is not connected then there exists a point $p \in C$ and an open set G such that

$$a \in G\,, \quad p \in C - \overline{G}\,, \quad C \cap \partial G = \emptyset\,.$$

As a consequence X is not connected from a to ∂G, so there exists an open and closed set F such that

$$a \in F, \quad F \cap \partial G = \emptyset \Rightarrow a \in F \cap G\,.$$

Moreover, $F \cap G = F \cap \overline{G}$ and $p \notin \overline{G}$, then $p \in X - (F \cap G)$.

The last thing we need is the fact that $F \cap G$ is both open and closed. F and G are open sets, so also $F \cap G$ is open. Finally

$$\overline{F \cap G} \subset \overline{F} \cap \overline{G} = F \cap \overline{G} = F \cap G \quad \text{and} \quad F \cap G \subset \overline{F \cap G}\,.$$

So $\overline{F \cap G} = F \cap G$, then the set is closed.

We have proved that X is not connected from a to p, but this implies that $p \notin C$ which is a contradiction. Hence we conclude that C is a connected set that intersects both A and B (recall that $b \in B \cap C$). □

Applying this lemma to the case in which the space X is an oriented rectangle, we obtain an important and useful lemma about the paths crossing a rectangle.

Lemma 2.9 (Crossing Lemma). *Let $\widetilde{R}$ be an oriented rectangle and $S \subset R$ a closed subset such that for every path σ in R joining R_l^- and R_r^-,*

$$S \cap \sigma \neq \emptyset\,. \tag{8}$$

Then there exists a compact connected set $C \subset S$ which intersects both R_b^+ and R_t^+.

Proof. Consider the horizontal strip

$$O := \{(x,y) \in \mathbb{R}^2 : -1 \leq y \leq 1\} = \mathbb{R} \times [-1,1]$$

whose boundaries are the horizontal lines

$$O_{-1} := \mathbb{R} \times \{-1\} \quad \text{and} \quad O_1 := \mathbb{R} \times \{1\}. \tag{9}$$

Draw four vertical lines in O

$$V_i := \{i\} \times [-1,1] \quad \text{for} \quad i \in \{-1, -\frac{1}{2}, \frac{1}{2}, 1\} \tag{10}$$

and consider the rectangle $\mathscr{P} := [-\frac{1}{2}, \frac{1}{2}] \times [-1,1] \subset \mathbb{R}^2$ oriented in the standard way. Let $q : \mathscr{P} \to R$ be a homeomorphism which preserves the orientation of the sides and consider the compact set

$$T = q^{-1}(S) \subset \mathscr{P}$$

which intersects every path $\gamma \in O$ going from V_{-1} to V_1.

The set $A := O - T$ is open in O and it is disconnected, because V_{-1} and V_1 belong to different connected components. Hence there exist two open, disjoint sets $A_{-1} \supset V_{-1}$ and $A_1 \supset V_1$ such that $A = A_{-1} \cup A_1$. Consider the function g

$$g(x,y) = w(x,y)\,\mathrm{dist}((x,y);T) \quad \text{with} \quad w(x,y) := \begin{cases} -1 & (x,y) \in A_{-1} \\ 0 & (x,y) \in T \\ 1 & (x,y) \in A_1 \end{cases}$$

which is negative in A_{-1}, vanishes on the set T and is positive in A_1.

Assume for contradiction that a compact set $C \subset S$ which links R_b^+ and R_t^+ does not exists. Applying Whyburn lemma we conclude that there are two closed, disjoint sets $F_{-1} \supset O_{-1} \cap T$ and $F_1 \supset O_1 \cap T$ such that $T = F_{-1} \cup F_1$. Let $Q = [-1,1]^2$ and consider the compact, disjoint sets

$$\hat{F}_{-1} = Q \cap (F_{-1} \cup O_{-1}), \quad \hat{F}_1 = Q \cap (F_1 \cup O_1)$$

and the function

$$f(\cdot) := \mathrm{dist}(\cdot\,; \hat{F}_{-1}) - \mathrm{dist}(\cdot\,; \hat{F}_1).$$

This function is negative on $O_{-1} \cap Q$ and positive on $O_1 \cap Q$ so the vector field $(f,g) : Q \to R$ is continuous and satisfies the hypothesis of Poincaré-Miranda theorem. Hence we conclude that there is a point P^* on which both f and g vanish. A contradiction arises due to the fact that $g(P^*) = 0$ implies $P^* \in T \subset \hat{F}_{-1} \cup \hat{F}_1$, so $f(P^*) > 0$ or $f(P^*) < 0$. $\square$

This proof uses only Poincaré-Miranda theorem and Whyburn lemma; for the sake of completeness we provide also two shorter and simpler proofs of the same result that, however, use more sofisticated topological tools and theorems.

Proof. Let $A = R_b^+ \cap S$ and $B = R_t^+ \cap S$ which are clearly nonempty and disjoint subsets of S. Our aim is to prove that there exists a compact connected set $C \subset S$ such that $C \cap A \neq \emptyset \neq C \cap B$. Assume by contradiction that such a set C does not exist, then the second alternative of Lemma 2.8 holds, which implies that there exists a pair of compact sets K_A and K_B such that

$$S = K_A \cup K_B,\ K_A \cap K_B = \emptyset,\ K_A \supset A,\ K_B \supset B. \tag{11}$$

Define now the compact sets

$$K'_A = K_A \cup R_b^+ \quad \text{and} \quad K'_B = K_B \cup R_t^+ .$$

Since these sets are disjoint, we can introduce the following function

$$f(P) := \frac{\text{dist}(P, K'_A) - \text{dist}(P, K'_B)}{\text{dist}(P, K'_A) + \text{dist}(P, K'_B)} \quad \text{for every} \quad P \in R$$

which assumes the value -1 on K'_A and the value 1 on K'_B. Hence there exists a path (see [49, Lemma 1.1.16]) γ such that $\gamma(0) \in R_l^-$, $\gamma(1) \in R_r^-$ and

$$|f(\gamma(t))| < \frac{1}{2} \quad \forall, t \in [0,1]. \tag{12}$$

The previous assumptions ensure the existence of $t^* \in [0,1]$ such that $\gamma(t^*) \in S = K_A \cup K_B$ which means that $f(\gamma(t^*)) = \pm 1$ in contradiction with inequality (12). $\square$

The third and last proof we present needs also the following theorem (Alexander's lemma) that we borrow from [56] (see also [22]).

Theorem 2.10 (Alexander's Addition Theorem)**.** *The boundary of the oriented rectangle $R = [-1,1]^2$ can be split in two arcs $\rho_1 \subset \{y \geq 0\}$ and $\rho_2 \subset \{y \leq 0\}$ joining the points $(-1,0)$ and $(1,0)$. Consider two closed subsets C_1 and C_2 of a generic metric space X and two paths ω_i with values in $X - C_i$, for $i = 1,2$ whose common endpoints are named P and Q. Let f be a continuous function*

$$f : R \to X - (C_1 \cap C_2) \quad \textit{with} \quad f(\rho_i) = \omega_i \quad \textit{for} \quad i = 1,2.$$

Then there exists a path ω going from P to Q in $X - (C_1 \cup C_2)$.

Proof. Since the function f is continuous, the sets $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are closed; moreover they are disjoint and $f^{-1}(C_1) \cap \rho_2 = \emptyset$. It is possible to split R in a finite number of small squares such that every element of the grid intersects at most one of the sets $f^{-1}(C_1)$ and $f^{-1}(C_2)$.

Let K be the collection of the squares which have nonempty intersection with $f^{-1}(C_1) \cup \rho_2$ and B the set of the edges which belong exactly to one of the squares in K. With these definitions, Sanderson [56] observes that only an even number of the edges in B have an endpoint in common. Since $\rho_2 \subset B$, every node of the graph induced by the set $B - \rho_2$ has even degree, except the nodes $(-1,0)$ and $(1,0)$. It is well known that this property guarantees the existence of an Eulerian path γ from $(-1,0)$ and $(1,0)$ which uses only edges lying in $B - \rho_2$.

Recalling the definitions of B and K, the path γ misses $f^{-1}(C_1)$; our aim is to prove that it misses $f^{-1}(C_2)$ too. For every edge $e \in \gamma$, we consider two possible cases:

$$e \in \partial R \Rightarrow e \in \rho_1 \Rightarrow f(e) \in f(\rho_1) = \omega_1 \subset X - C_2\,.$$

Otherwise, e is a common side of squares in K, so it can not intersect $f^{-1}(C_2)$. Hence $\omega := f(\gamma)$ is the required path. □

Using this theorem, we can modify and significantly shorten the final part of the second proof previously exposed.

Proof. As in the second proof of the crossing lemma, define the nonempty and disjoint sets $A = R_b^+ \cap S$ and $B = R_t^+ \cap S$ and assume that there exists a separation $K_A \cup K_B = S$ with the properties already listed in (11). By this construction it arises that R_t^+ is a path which does not intersect K_A while R_b^+ does not intersect K_B. An easy application of Theorem 2.10 guarantees the existence of a path ω which links R_l^- and R_r^- and contained in $R - (K_A \cup K_B) = R - S$ although the assumption of the lemma requires that $\omega \cap S \neq \emptyset$. □

For other versions of the crossing lemma and some more applications see [53, Appendix].

Now at last we have all the concepts necessary to formalize Poincaré's proof of his theorem (see also [33, 39]).

Lemma 2.11. *If Σ and Ω are two closed connected subsets of $Q = [0,1]^2$ such that*

$$\Sigma \cap Q_b^+ \neq \emptyset\,, \quad \Sigma \cap Q_t^+ \neq \emptyset\,, \quad \Omega \cap Q_l^- \neq \emptyset\,, \quad \Omega \cap Q_r^- \neq \emptyset$$

then $\Sigma \cap \Omega \neq \emptyset$.

Proof. We define four sets

$$W_b = \Sigma \cap Q_b^+, \quad W_t = \Sigma \cap Q_t^+$$
$$W_l = \Omega \cap Q_l^-, \quad W_r = \Omega \cap Q_r^- .$$

Fix an $\varepsilon > 0$ and let $U_\varepsilon^0, U_\varepsilon^1 \subset Q$ be the $\varepsilon-$neighborhood of Σ and Ω respectively. Since Q is connected, there exist two paths $\gamma_\varepsilon{}^0, \gamma_\varepsilon{}^1 : [0,1] \to R$ such that $\gamma_\varepsilon{}^i \subset U_\varepsilon^i$ for $i = 0,1$ and

$$\gamma_\varepsilon{}^0(0) \in W_b, \quad \gamma_\varepsilon{}^0(1) \in W_t, \quad \gamma_\varepsilon{}^1(0) \in W_l, \quad \gamma_\varepsilon{}^1(1) \in W_r .$$

Using the properties of Peano spaces, we can also assume that the paths $\gamma_\varepsilon{}^i$ are (simple) arcs and that they intersect ∂Q only in their endpoints. So we have

$$\gamma_\varepsilon{}^0 \cap \partial R = \{P_b^\varepsilon, P_t^\varepsilon\}, \quad \gamma_\varepsilon{}^1 \cap \partial R = \{P_l^\varepsilon, P_r^\varepsilon\} .$$

Define now a Jordan curve J as the union of the following arcs: $\gamma_\varepsilon{}^1$, the vertical segment from P_r^ε to $(1,0)$, the horizontal segment from $(1,0)$ to $(0,0)$ and the vertical segment from $(0,0)$ to P_l^ε.

Since $P_b^\varepsilon = \gamma_\varepsilon^0(0) \in J$, there exists $t^* \in (0,1)$ such that $\gamma_\varepsilon{}^0(t)$ lies in the internal part A_i of J, for all $t \in (0,t^*)$. On the other hand, $P_t^\varepsilon = \gamma_\varepsilon{}^0(1) \in A_e$ (the external part of J). Hence the arc $\gamma_\varepsilon{}^0$ intersects J and the only possibility is that $\gamma_\varepsilon{}^0$ intersects $\gamma_\varepsilon{}^1$.

Since $\gamma_\varepsilon{}^i \subset U_\varepsilon{}^i$, when $\varepsilon \to 0$ we have that

$$\gamma_\varepsilon{}^0 \cap \gamma_\varepsilon{}^1 \neq \emptyset \Rightarrow \Sigma \cap \Omega \neq \emptyset$$

and this concludes the proof. □

In this last proof we have used a strong property of Peano spaces. By a Peano space we mean a compact, connected and locally connected metric space. Under these assumptions, the pathwise connectivity is equivalent to the arcwise connectivity (see [15, 23, 31]). Recall that by a *path* we mean the image of an interval via a continuous function while an *arc* (compact) is the homeomorphic image of a compact non-degenerate interval.

Theorem 2.12. *Each two points of a compact, connected and locally connected metric space S can be joined by an arc in S.*

We prove this results in the special case of $S = \psi([a,b]) \subset \mathbb{R}^2$, which is the case considered in Lemma 2.11.

Theorem 2.13. *Let $\psi : [a,b] \to \mathbb{R}^2$ be a path (i.e. a continuous function) with $\psi(a) \neq \psi(b)$. Then the set $\psi([a,b])$ contains an arc J whose endpoints are $\psi(a)$ and $\psi(b)$.*

Proof. If ψ is a simple path, it already defines an arc, so simply choose $J = \psi([a,b])$. Otherwise there exist a point $x \in \psi([a,b])$ such that $\#\psi^{-1}(x) \geq 2$, that is, a *multiple point*.

For every multiple point x consider the closed intervals I_x such that ψ assume the value x in both its endpoints and construct sets P, whose elements are pairwise disjoint intervals of the form I_x. For instance, a possible choice of a set P could be $P = \{\mathscr{I}_{\bar{x}}\}$, where $\bar{x}$ is a multiple point and $\mathscr{I}_{\bar{x}}$ is the maximal interval $[t', t'']$ such that $\psi(t') = \psi(t'') = \bar{x}$.

Let $\mathscr{F}$ be the family of all the sets P (defined as above) on which it is possible to define a partial order $\leq$ as follows:

$$P_1 \leq P_2 \Leftrightarrow \forall I_x \in P_1 \; \exists \, I_y \in P_2 : I_x \subset I_y \, .$$

Applying Zorn's lemma to $(\mathscr{F}, \leq)$ we obtain the existence of a maximal element P_M in $\mathscr{F}$ whose elements are closed, pairwise disjoint intervals of $[a,b]$. As a last step, we define a function $h : [a,b] \to [0,1]$ with $h(a) = 0, h(b) = 1$ and such that $h(t) \leq h(s)$ if and only if $t \leq s$, with $h(t) = h(s)$ for $t < s$ if and only if t, s belong to the same interval $I_x \in P_M$. Finally, we define a function $\phi : [0,1] \to \psi([a,b])$ such that $\phi(u) = x$ if $h^{-1}(\{u\}) = I_x$ for some $I_x \in P_M$ and $\phi(u) = \psi(h^{-1}(u))$, otherwise. It is possible to check that ψ is one-to-one and continuous and $\psi([0,1])$ is the desired arc. $\square$

2.3.1. Two different proofs of Poincaré-Miranda Theorem

At the end we will show that it is possible to reprove Poincaré-Miranda theorem simply assuming the crossing lemma. Indeed, let $S = \{(x,y) \in R : f_2(x,y) = 0\}$ and let γ be a path going from R_b^+ to R_t^+. From the hypothesis we know that $f_2(\gamma(0)) \leq 0 \leq f_2(\gamma(1))$, then $\gamma \cap S \neq \emptyset$. Applying now Lemma 2.9 we conclude that there exists a connected set $C \subset S$ which intersects R_l^- and R_r^- on which f_1 changes its sign. Then there exists a point $P \in C$ such that $f_1(P) = 0$, so we have $f(P) = 0$ as required.

More in general we can obtain the following lemma that we state, for simplicity, for the case of the unit square $Q = [0,1]^2$.

Lemma 2.14. *If $f : Q \to \mathbb{R}$ is a continuous function such that*

$$f([0,1] \times \{0\}) \leq 0 \quad \textit{and} \quad f([0,1] \times \{1\}) \geq 0$$

then there is a compact connected set $\Sigma \subset f^{-1}(\{0\})$ which intersects both the right and the left side of Q.

Proof. We define the set $C = f^{-1}(0)$ and consider a path $\gamma : [0,1] \to Q$ such that $\gamma(0) \in Q_b^+$ and $\gamma(1) \in Q_t^+$. If we introduce the composite map computing

f along γ

$$\hat{f} := f \circ \gamma : [0,1] \to \mathbb{R}$$

we observe that

$$\hat{f}(0) \le 0 \quad \text{and} \quad \hat{f}(1) \ge 0 .$$

Hence we can apply the intermediate value theorem to $\hat{f}$ and conclude that there exists $t^* \in [0,1]$ such that

$$\hat{f}(t^*) = f(\gamma(t^*)) = 0 \Rightarrow \gamma(t^*) \in C .$$

Using this argument we have just proved that the set C intersects every path γ which links the top and the bottom sides of Q. Applying Lemma 2.9 we obtain the statement. $\square$

The same result can be read also under the more "modern" point of view of degree theory; indeed it can be proved using the well-known Leray-Schauder continuation theorem. The interested reader can find an exhaustive historic survey on degree theory and its applications in the paper by Mawhin [34].

Consider the auxiliary function $\hat{f} : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}(x,y) = f(x, \max\{0, \min\{y,1\}\}) + \min\{y, \max\{0, y-1\}\} .$$

The map $\hat{f}$ is continuous and such that

$$\hat{f}(x,y) = f(x,y), \; \forall (x,y) \in Q .$$

Moreover it satisfies the following inequalities:

$$\begin{aligned} \hat{f}(x,y) < 0, &\quad \forall (x,y) \in [0,1] \times (-\infty, 0) \\ \hat{f}(x,y) > 0, &\quad \forall (x,y) \in [0,1] \times (1, +\infty) \end{aligned}$$

If we treat the variable $x \in [0,1]$ as a parameter and consider the open set $\Omega := (-1,2)$, we have that $\hat{f}(x,y) \neq 0$ for every $x \in [0,1]$ and $y \in \partial\Omega$. The Brouwer degree $\deg(\hat{f}(0,\cdot), \Omega, 0)$ is well-defined and nontrivial, since by the sign condition $\hat{f}(0,-1) < 0 < \hat{f}(0,2)$ it holds that

$$\deg(\hat{f}(0,\cdot), \Omega, 0) = 1 .$$

The continuation theorem of Leray-Schauder ensures that the set of solution pairs

$$\hat{K} := \{(x,y) \in [0,1] \times \Omega : \hat{f}(x,y) = 0\}$$

contains a continuum along which x assumes all the values in $[0,1]$. By the definition of $\hat{f}$ it is clear that $\hat{K} = K$ and this gives another proof of the existence of S.

Now Poincaré-Miranda theorem can be seen as a corollary of Lemma 2.14, following exactly the original argument given by Poincaré. Indeed, after trivial adjustments we can apply Lemma 2.14 both to f_1 and f_2; hence there exist two sets Ω and Σ with the following properties:

$$\begin{aligned} \Sigma \subset f_2^{-1}(\{0\}) \quad & \text{such that} \quad \Sigma \cap R_r^- \neq \emptyset,\, \Sigma \cap R_l^- \neq \emptyset \\ \Omega \subset f_1^{-1}(\{0\}) \quad & \text{such that} \quad \Omega \cap R_b^+ \neq \emptyset,\, \Omega \cap R_t^+ \neq \emptyset. \end{aligned}$$

Then Ω and Σ satisfy the hypothesis of Lemma 2.11 which implies that $\Omega \cap \Sigma \neq \emptyset$. In this way we conclude that there exists at least one point P such that $f_1(P) = f_2(P) = 0$, that is f vanishes in P.

2.3.2. A classical example

We present now a classical example (borrowed from [18]) of the fact that there exist connected sets joining the pairs of opposite sides of a rectangle which have an empty intersection. This fact seems to go against our intuitions; in order to convince the reader we are going to show a simple example. The key point lies in the fact that the hypothesis of connection is not sufficient, but we have to require that the sets are compact too (namely closed).

Consider this pair of subsets of $Q = [-1,1]^2$

$$\begin{aligned} S_1 = & \{(x,y) : y = \frac{7}{8}x - \frac{1}{8}, -1 \leq x \leq 0\} \cup \\ & \{(x,y) : y = \frac{1}{2}\sin\frac{\pi}{2x} + \frac{1}{4}, 0 < x < 1\} \cup \\ & \{(x,y) : x = 1, \frac{3}{4} \leq y \leq 1\} \end{aligned}$$

and

$$\begin{aligned} S_2 = & \{(x,y) : y = -\frac{7}{8}x + \frac{1}{8}, -1 \leq x \leq 0\} \cup \\ & \{(x,y) : y = \frac{1}{2}\sin\frac{\pi}{2x} - \frac{1}{4}, 0 < x < 1\} \cup \\ & \{(x,y) : x = 1, -1 \leq y \leq \frac{1}{4}\}. \end{aligned}$$

The set S_1 joins the point $(-1,-1)$ with $(1,1)$, while S_2 joins $(-1,1)$ with $(1,-1)$. This means that S_1 crosses the square from the right to the left side and S_2 crosses it from the bottom to the top side. S_1, S_2 are connected sets and their intersection is empty.

2.4. Hex game

There is an extensive literature about the applications of mathematical analysis to game theory, especially to the setting of strategy games; more recent works have also developed the converse approach, with the aim of derive theoretic results using the tools offered by game theory. In this context, the work by David Gale [16] presents an interesting link between planar topology and the combinatorial game of Hex.

Hex is a board game usually played on a rhomboidal grid with 11×11 hexagonal tiles but the model can be generalized to every dimension $n \times n$. A set of colored pieces (usually red and blue) is assigned to each player. The two players move alternately and at every turn they place a colored piece on an empty cell of the board. The goal of each player is to form a connected path of own pieces linking a pair of opposite sides of the board, the top and the bottom for the first player ($\mathscr{R}_t^+$ and $\mathscr{R}_b^+$ in Figure 1), the right and the left ($\mathscr{R}_r^-$ and $\mathscr{R}_l^-$) for the second. John Nash was the first to prove that the first player always has a winning strategy (see [40]), but, since his argument is not constructive, the problem of finding this strategy is a brain teaser. More formally it has been proved that this problem is PSPACE-complete in the dimension of the board. Operative research deals with finding solving algorithms for this problem (see for instance [2]).

John Nash proved that in this game the first player always has a winning strategy. He proceeds by absurd, proving that the existence of a strategy for the player two is not possible. Indeed, if the second player had a winning strategy, then the first could always win, using a "strategy-stealing" argument.

Assume that there exists a winning strategy for the second player. Then player one can steal the strategy following this scheme:

- at the beginning of the game he chooses a square at random
- after any later move, he "forgets" his last move and pretend to change all his remaining tiles with the first player's ones. With this stratagem, he becomes the second player of the match and then he can use the winning strategy for the next move. If the winning strategy prescribes him to mark the same square of his last move, then he can choose any other free square.

With this procedure, the first player wins, in contradiction with the statement that the second player has a winning strategy (see also [4]).

2.4.1. Hex Theorem

An important feature of this game is the so called Hex theorem, which says that Hex can not end in a draw (see [4]). The key observation is that a player can block the other one only creating a winning path.

Theorem 2.15 (Hex Theorem)**.** *If all the cells of the board are covered, then there exists a connected set of blue pieces linking the top and the bottom sides of the board or a set of red pieces linking the right and the left sides.*

The original proof of this theorem was provided by Gale and uses a combinatorial argument dealing with graph theory in order to obtain a winning set for one of the players. We give only a sketch of it, for the sake of completeness. All the details can be found in [16].

Consider the Hex board as a graph G and introduce four auxiliary nodes u, u', v, v' as in Figure 1; split the region outside the board in four zones: R and R' are red, while B and B' are blue. An algorithm for finding a winning path proceeds as follows. Start from the node u and visit the graph following this rule: at every step move along an edge which separates a red cell from a blue cell or a pair of regions in $\mathbb{R}^2 - \mathscr{R}$ of different color. This rule univocally specifies a path in G which visits every node at most once so that the construction must end and the only possibility is that the path terminates in one of the auxiliary nodes u', v, v'. If the last node of this path is v then the path is the boundary of a winning red set for the red player. In the other case, the first player has a winning set.

We propose now an alternative proof of Theorem 2.15, which uses topological tools instead of the combinatorial ones. Indeed Hex theorem turns out to be a corollary of the Crossing Lemma 2.9.

Proof. The Hex board can be thought as a generalized rectangle oriented in a way such that the first player has to connect $\mathscr{R}_t^+$ and $\mathscr{R}_b^+$ while the second one has to cross the board in the horizontal direction. Assume that all the tiles are covered and let S be the set of the blue pieces (which own to the first player). Clearly the complement of S in the board is the set of the red pieces. Two alternatives can occur: if there exists a winning path σ in S^c, then the second player has won. If such a path σ does not exist, then $\sigma \cap S \neq \emptyset$ for every path that links the right and the left sides of the board. In this case we are in the hypothesis of Lemma 2.9 that guarantees the existence of a subset $C \subset S$ (i.e. made only by blue pieces) which connects the top and the bottom side of the board. In this case the first player has won. □

2.4.2. Equivalence with Brouwer fixed point theorem

Hex theorem is essentially a planar topological theorem. The topic of this section concerns the equivalence between Hex theorem 2.15 and the version of

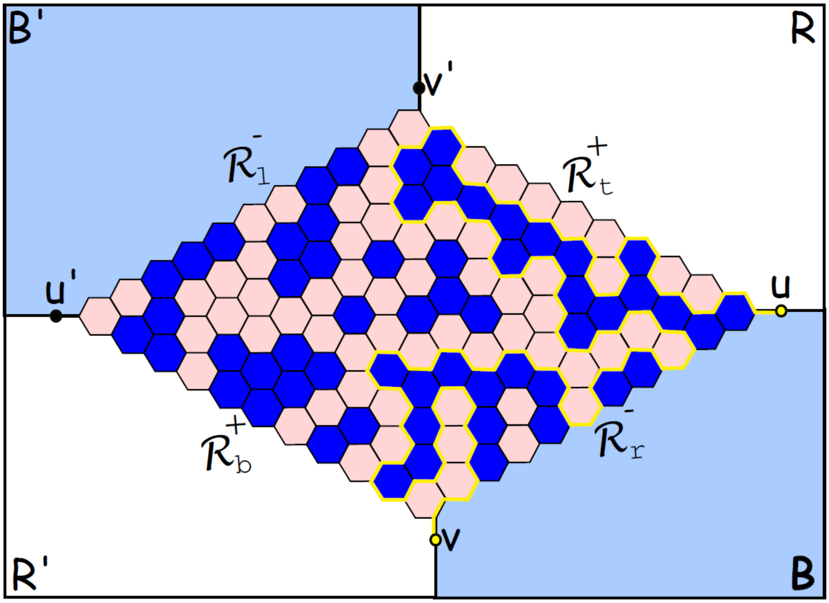

Figure 1: A possible outcome of the game and a graphical sketch of the proof of Theorem 2.15.

Brouwer theorem for the square $Q = [0,1]^2$, following Gale [16]. In order to prove this result it is more convenient to consider the model of the hex board originally proposed by Nash. The board is a $n \times n$ square grid and every cell is adjacent also to the two cells that lies on the positive diagonal of the square.

In order to formalize the setting, we introduce some definitions. For every point $x \in \mathbb{R}^n$ we consider the norm $|x| = ||x||_\infty = \max_{i=1,\dots,n} x_i$. For every pair of points $x, y \in \mathbb{R}^n$ we say that $x < y$ if and only if $x_i \leq y_i$ for all the components $i = 1, \dots, n$. Two points x, y are comparable if $x < y$ or $y < x$.

Let B be the $(k-1) \times (k-1)$ board, which have k vertex on each side; we provide it with the structure of an oriented rectangle with the standard orientation of its sides. The board B is the set of the points (x,y) of $\mathbb{Z}^2$ such that $(0,0) \leq (x,y) \leq (k,k)$. In this setting two vertex z, z' are adjacent if are comparable and $|z - z'| = 1$.

Hex $\Rightarrow$ Brouwer

Let $f = (f_1, f_2) : Q \to Q$ be a continuous function. Since Q is a compact metric space, f is also uniformly continuous, that is

$$\forall \varepsilon > 0 \, \exists \delta > 0 : |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon.$$

Fix $\varepsilon > 0$, take δ as above with $\delta < \varepsilon$ and choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$. Consider the board B in which we define four subsets H^+, H^-, V^+, V^- as follows

$$H^+ = \{z \in B : f_1(\frac{z}{k}) - \frac{z_1}{k} > \varepsilon\}$$
$$H^- = \{z \in B : \frac{z_1}{k} - f_1(\frac{z}{k}) > \varepsilon\}$$
$$V^+ = \{z \in B : f_2(\frac{z}{k}) - \frac{z_2}{k} > \varepsilon\}$$
$$V^- = \{z \in B : \frac{z_2}{k} - f_2(\frac{z}{k}) > \varepsilon\}$$

In plain words H^+, H^-, V^+, V^- are the sets of the vertices z such that the displacement of $\frac{z}{k}$ under the action of f is at least ε in the right, left, up, down directions respectively. We want to prove that these sets do not cover the board, that is there exists a point whose displacement is less than ε.

Consider a pair of vertices $z \in H^+$ and $z' \in H^-$ so that

$$f_1(\frac{z}{k}) - f_1(\frac{z'}{k}) - \frac{z_1}{k} + \frac{z'_1}{k} > 2\varepsilon$$

If z, z' were adjacent, then $|z' - z| = 1$ and we would obtain the following inequalities

$$\frac{z'_1}{k} - \frac{z_1}{k} \leq \frac{1}{k} < \delta < \varepsilon \Rightarrow \frac{z_1}{k} - \frac{z'_1}{k} > -\varepsilon$$

and conclude that $f_1(\frac{z}{k}) - f_1(\frac{z'}{k}) > \varepsilon$ against the hypothesis. With this argument we have proved that H_+ and H^- are not adjacent as well V^+ and V^-.

Define now the sets $H = H^+ \cup H^-$ and $V = V^+ \cup V^-$ and let S be a connected set of $H \subset B$. For the connection of S we have that $S \subset H^+$ or $S \subset H^-$. Moreover since $f : Q \to Q$ we have that $H^+ \cap B_r^- = \emptyset$ and for the same argument also $H^- \cap B_l^- = \emptyset$. So we conclude that S does not intersect both the right and the left sides of the board and also V does not contain connected subsets which intersect both the top and the bottom sides of B.

Recalling Hex theorem, we know that there exists a connected set that intersects a pair of opposite sides of the board, hence the sets H and V do not cover all the board. We have proved that for every $\varepsilon > 0$ there exists a point $x \in B$ such that $|f(x) - x| < \varepsilon$ and then we obtain a fixed point for f in Q with a standard compactness argument.

Brouwer $\Rightarrow$ Hex

The structure of the board B can be modelled by a triangulation of the square $Q = [0,k]^2$ as shown in Figure 2. Every function defined on the set of the vertices of the board can be extended to a function $\hat{f}$ defined on the all square Q. Indeed every point $x \in Q$ can be expressed as a convex combination of three vertices

$$x = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 \quad \text{with} \quad \lambda_1 + \lambda_2 + \lambda_3 = 1 \text{ and } \lambda_i \geq 0 \; \forall i = 1,2,3$$

hence by definition $\hat{f}(x) = \lambda_1 f(z_1) + \lambda_2 f(z_2) + \lambda_3 f(z_3)$. Assume that the board is covered by two sets R and B (red and blue) and that the game has finished in a draw. Define the following subsets: Q_L is the set of the points of the board that are connected to Q_l^- by a path in R; $Q_R := R - Q_L$; Q_T is the set of the points of the board that are connected to Q_t^+ by a blue path; $Q_B := B - Q_T$. With this definitions it is clear that Q_L and Q_R are not contiguous as well Q_T and Q_B.

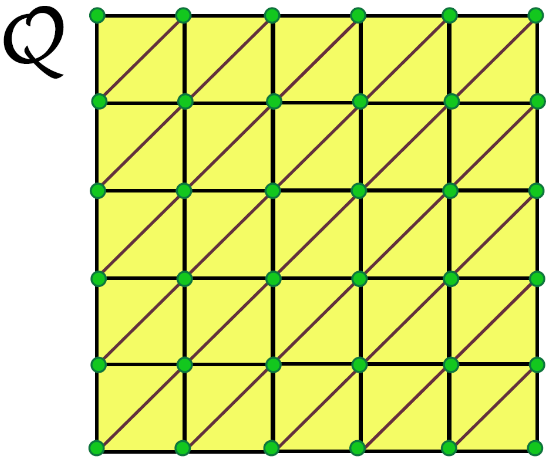

Figure 2: The Hex board modelled as a triangulation of the square

Assume now, by contradiction, that the game finishes in a draw, that is neither a connected subset of R intersecting Q_l^- and Q_r^- nor a connected subset of B going from Q_t^+ to Q_b^+ exists. Let e_1, e_2 be the vectors of the canonic base of $\mathbb{R}^2$ and define a function f of the board B

$$f(z) = \begin{cases} z + e_1 & z \in Q_L \\ z - e_1 & z \in Q_R \\ z + e_2 & z \in Q_B \\ z - e_2 & z \in Q_T \end{cases}$$

Extend f to the square, such that $\hat{f} : Q \to Q$. Applying Brouwer theorem the existence of a fixed point z^* holds. Since the map moves every vertex of the board, the point z^* can not be a vertex and it can be expressed as the convex combination of three points of the board, so that $z^* = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3$. Since the non-contiguousness of the sets defined above, only two cases are possible. If the vertices z_i belong to the same set, say Q_L without loss of generality, then

$$\hat{f}(z^*) = \lambda_1(z_1 + e_1) + \lambda_2(z_2 + e_1) + \lambda_3(z_3 + e_1) = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 = z^*$$

which is satisfied if and only if $e_1 = 0$. Absurd. The other possible situation can be exemplified by the case in which $z_1, z_2 \in Q_L$ and $z_3 \in Q_B$. Then

$$\hat{f}(z^*) = \lambda_1(z_1 + e_1) + \lambda_2(z_2 + e_1) + \lambda_3(z_3 + e_2) = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 = z^*$$

which leads to a contradiction too. Then we conclude that f can not have a fixed point, so the game can not end in a draw.

2.5. Other related problems

The aim of this last section is to present some other combinatorial settings related with the problem of "crossing" a given set.

First of all we expose the so called *Steinhaus chessboard theorem* which is an extension of the results about the Hex presented in [16] to the case of a board whose tiles are generic polygons, as exposed in [30]. The model of the board consists of the square $Q = [0,1]^2$ on which a tiling $\mathscr{P} = \{P_i : i = 1, \dots, n\}$ is defined. We assume that every P_i is a polygon and

$$Q = \bigcup_{i=1}^{n} P_i \quad \text{with} \quad \mathring{P}_i \cap \mathring{P}_j = \emptyset \quad \forall i \neq j\,.$$

A 2-coloring function $f : \mathscr{P} \to \{w, b\}$ is defined on the board. If $P_i \cap P_j \neq \emptyset$ and $f(P_i) \neq f(P_j)$ then we say that $P_i \cap P_j$ is a *black-white side*.

We place on the board a king and a tower which move according to the following rule: the king moves only on the black tiles, while the tower moves only on the white ones. Then an admissible path for the tower (say a tower-path) is a sequence $P_0, \dots, P_m$ of white tiles such that $P_i \cap P_{i+1} \neq \emptyset$ for every $i = 0, \dots, m-1$. In a similar way an admissible path for the king can be defined. In this scenery the following statement holds:

Theorem 2.16. *There exists a tower-path going from Q_l^- to Q_r^- or a king-path from Q_t^+ to Q_b^+.*

It is easy to see that this theorem can be proved with the same argument used in the proof of Theorem 2.15 due to the fact that in our proof we do not assume any hypothesis on the shape of the tiles.

A second scenery, presented in [29], models the problem of *navigating a fjord*, that is the problem of finding an optimal path between the top and the bottom sides of a square in the so called "fjord scenery". As in the previous case, we consider a polygonal tessellation of the square and a coloring function of the vertices of the tiles which defines *water points* and *rock points*. We also assume that every vertex on the left side of Q is a water point while the vertices on the right side are rocks. The main theorem proved in [29] claims that it is possible to find a path in the water region going from the top to the bottom of Q and avoiding the rocks.

More precisely, consider the unit square $Q = [0,1]^2 \subset \mathbb{R}^2$ with the standard structure of oriented rectangle and let $\mathscr{P}$ be a polygonal partition of Q such that the family $\mathscr{P}$ is closed respect to intersection. Call $\hat{\mathscr{P}}$ the set whose elements are the empty set and all the sides of the polygons in $\mathscr{P}$ and let V be the set of the vertices of the polygons of the partition. Assume that a $2-$coloring function $f : V \to [0,1]$ is defined on the set of vertices and satisfies the boundary conditions

$$f(p) = \begin{cases} 0 & \text{if } p \in Q_l^- \\ 1 & \text{if } p \in Q_r^- \end{cases}$$

We say that a side of a polygon $\ell = \{p,q\}$ is a *gate* if $f(\{p,q\}) = \{0,1\}$. Given the definitions above, we can expose the main result.

Theorem 2.17. *For every polygonal partition $\mathscr{P}$ of the square $Q \subset \mathbb{R}^2$ and for every coloring function $f : V \to \{0,1\}$, there exists a set $Z = \{P_1,\dots,P_n\}$ such that $P_i \cap P_{i+1}$ is a gate for every $i = 1,\dots,n-1$. Moreover Z crosses the square, that is*

$$P_1 \cap Q_b^+ \neq \emptyset \neq P_n \cap Q_t^+$$

This proves the possibility of crossing the square in the vertical direction "navigating in the water region".

The ones we have just exposed belong to a class of combinatorial problems dealing with computational geometry and having many applications in the field of transportation, shipping and distribution problems. Nevertheless we notice that completely similar theorems arises also in more theoretic settings, as the research of periodic solutions for second order scalar ODEs. In a paper by Butler the following lemma is proved ([8, Lemma 4]).

Lemma 2.18. *Let Ω be an open, connected subset of $\mathbb{R}^2$ with the property that for each vertical line L lying between (and including) two fixed vertical lines L_1, L_2, $L \cap \Omega$ is a nonempty bounded set. Let $\{\Gamma_i\}_{i=0}^m$ be a collection of continua contained in Ω such that for $i = 1,2,\dots,m$ the Γ_i are mutually disjoint, and $L_1 \cap \Gamma_0 \neq \emptyset \neq L_2 \cap \Gamma_0$, while for $i = 1,\dots,m$, at least one of $L_1 \cap \Gamma_i$ and $L_2 \cap \Gamma_i$*

is empty. Then there exist $p \in \Gamma_0, q \in \partial\Omega$ *and an arc* γ *from* p *to* q *with* $\gamma \subset \overline{\Omega} \cap S(L_1, L_2)$*, where* $S(L_1, L_2)$ *is the infinite strip of* $\mathbb{R}^2$ *contained between* L_1 *and* L_2*, such that* γ *is disjoint from* $\bigcup_{i=1}^{m} \Gamma_i$*.*

With minor variants, this theorem has the same meaning of Theorem 2.17. See also [3, p.330 and following] for a similar result applied in the proof of Hex theorem.

3. A fixed point theorem in oriented rectangles

In this section we prove a fixed point theorem for continuous maps which are expansive along some direction. It is based on the Crossing Lemma (Lemma 2.9) and depends also on the concept of "Stretching Along the Paths" (SAP) that we define below. We also need here to recall the definition of *oriented rectangle* (see Definition 8).

Definition 12. Let $\widetilde{\mathscr{A}} := (\mathscr{A}, \mathscr{A}^-)$ and $\widetilde{\mathscr{B}} := (\mathscr{B}, \mathscr{B}^-)$ be two oriented rectangles and let $K \subset \mathscr{A}$ be a (nonempty) compact set. Suppose also that $f : K \to \mathbb{R}^2$ is a continuous map. We say that (K, f) *stretches* $\mathscr{A}$ *to* $\mathscr{B}$ *along the paths* and write

$$(K, f) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}},$$

if, for every (continuous) path $\gamma : [t_0, t_1] \to \mathscr{A}$, with $\gamma(t_0)$ and $\gamma(t_1)$ belonging to different components of $\mathscr{A}^-$, there exists $[s_0, s_1] \subset [t_0, t_1]$ such that

$$\gamma(t) \in K, \quad f(\gamma(t)) \in \mathscr{B}, \quad \forall t \in [s_0, s_1]$$

with $f(\gamma(s_0))$ and $f(\gamma(s_1))$ belonging to different components of $\mathscr{B}^-$.
If $K = \mathscr{A}$, we simply write

$$f : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}},$$

for $(\mathscr{A}, f) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$.

From the definition, it is clear that, without loss of generality, we can also assume $f(K) \subset \mathscr{B}$. Indeed, $(K, f) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$, if and only if $(K', f) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$, with $K' := K \cap f^{-1}(\mathscr{B})$. Thus, we can replace K with K' in Definition 12 and hence suppose that $f(K) \subset \mathscr{B}$, when this may be convenient. We stress, however, that $f : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$, *does not imply* that $f(\mathscr{A}) \subset \mathscr{B}$. If we know that $f : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$, we can only infer that $(H, f) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$, for $H := f^{-1}(\mathscr{B})$. In general, for a continuous map $f : \mathscr{A} \to \mathbb{R}^2$, it holds that if $(K, f) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$, for a suitable set $K \subset \mathscr{A}$, then $(H, f) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$, for any compact set H such that $K \cap f^{-1}(\mathscr{B}) \subset H \subset \mathscr{A}$.

Now we are in position to present our fixed point theorem. Its proof was already given in some preceding papers (see [44, 47]). We repeat the proof here in order to make our presentation self-contained and also because in this manner we can show the role of the Crossing Lemma in our result.

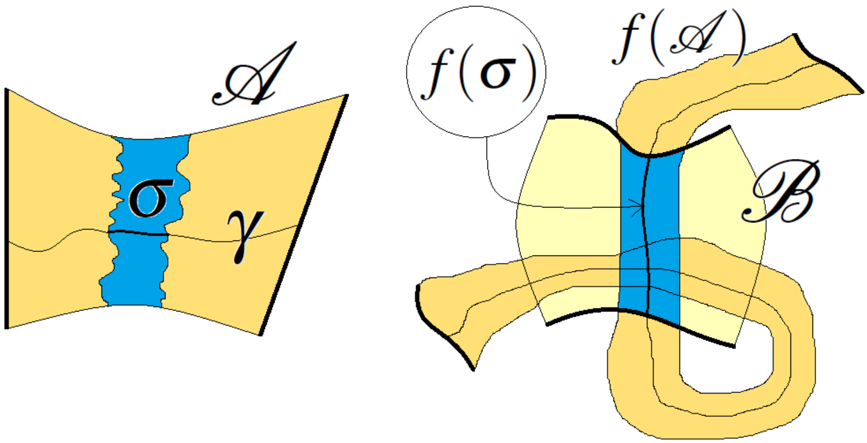

Figure 3: A pictorial description of the SAP property: A map f transforms a generalized rectangle $\mathscr{A}$ to a snake-like set $f(\mathscr{A})$ which crosses the generalized rectangle $\mathscr{B}$. Both $\mathscr{A}$ and $\mathscr{B}$ are oriented by putting in evidence with bold lines their $[\cdot]^-$-sets. A path γ in $\mathscr{A}$ connecting the two components of $\mathscr{A}^-$ contains a sub-path σ such that $f(\sigma)$ is contained in $\mathscr{B}$ and connects the two components of $\mathscr{B}^-$. For a suitable compact set $K \subset \mathscr{A}$ (for instance, the part of $\mathscr{A}$ indicated in figure with a darker color), we have that $(K, f) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$.

Theorem 3.1. *Let $\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$ be an oriented rectangle and let $f : H \to \mathbb{R}^2$ be a continuous map defined on a compact set $H \subset \mathscr{R}$. Assume that*

$$(H, f) : \widetilde{\mathscr{R}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{R}}.$$

Then there exists $w \in H$ such that $f(w) = w$.

Proof. Without loss of generality, we suppose that $f(H) \subset \mathscr{R}$. Let $\eta : Q \to \eta(Q) = \mathscr{R}$ be a homeomorphism, with $Q := [0,1]^2$, such that

$$\mathscr{R}_l^- := \eta(\{0\} \times [0,1]), \quad \mathscr{R}_r^- := \eta(\{1\} \times [0,1])$$

and let $K := \eta^{-1}(H)$, $\phi := \eta^{-1} \circ f \circ \eta$. By the assumptions, $\phi : K \to Q$ and $(K, \phi) : \widetilde{Q} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{Q}$, where $\widetilde{Q} = (Q, Q^-)$ with Q^- the union of the left and right sides of the unit square. For $\phi = (\phi_1, \phi_2)$, observe that $0 \leq \phi_2(x,y) \leq 1$, for all $(x,y) \in K$ and, moreover, let $S \subset Q$, be the compact set defined by

$$S := \{(x,y) \in K : x - \phi_1(x,y) = 0\}.$$

Let $\gamma = (\gamma_1, \gamma_2) : [0,1] \to Q$ be a continuous map such that $\gamma_1(0) = 0$ and $\gamma_1(1) = 1$. By the SAP property, a sub-path of γ takes its values in K and is stretched by

ϕ across Q. Hence, there exists $[s_0,s_1] \subset [0,1]$ such that $\gamma(t) \in K$ and $\phi(\gamma(t)) \in Q$ for every $t \in [s_0,s_1]$ and, moreover, $\phi_1(\gamma(s_0)) = 0$, $\phi_1(\gamma(s_1)) = 1$ (respectively, $\phi_1(\gamma(s_0)) = 1$, $\phi_1(\gamma(s_1)) = 0$). Then, by the Bolzano's theorem, the map $[s_0,s_1] \ni t \mapsto \gamma_1(t) - \phi_1(\gamma_1(t),\gamma_2(t))$ vanishes at some point $t^* \in [s_0,s_1]$, with $\gamma(t^*) \in K$. We have thus proved that $S \cap \bar{\gamma} \neq \emptyset$ for each path γ with values in Q and joining Q_l^- with Q_r^-. The Crossing Lemma (see Lemma 2.9) guarantees that S contains a continuum C which intersects $Q_b^+ := [0,1] \times \{0\}$ and $Q_t^+ := [0,1] \times \{1\}$ at some points, say $p = (p_1,0)$ and $q = (q_1,1)$, respectively. Evaluating $\psi(x,y) := y - \phi_2(x,y)$ along C, we have that $\psi(p) = -\phi_2(p) \leq 0$ and $\psi(q) = 1 - \phi_2(q) \geq 0$. Therefore, there exists $z \in C$ such that $\psi(w) = 0$. By the definition of S and the inclusions $C \subset S \subset K$, we conclude that $\phi(z) = z$ and, finally, $f(w) = w$, for $w := \eta(z) \in H$. $\square$

In Theorem 3.1 we have required f to be defined and continuous only on $H \subset \mathscr{R}$. If we like (using Tietze theorem) we could assume $f : \mathscr{R} \to \mathbb{R}^2$ continuous. In any case, the behavior of f outside H, as well as possible discontinuities of f in $\mathscr{R} - H$, do not effect our result.

An immediate consequence of Definition 12 is the fact that the SAP property is preserved by the composition of maps, in fact, we have:

Lemma 3.2. *Let $\widetilde{\mathscr{A}_i} := (\mathscr{A}_i, \mathscr{A}_i^-)$, for $i = 1,\ldots,\ell$ and $\ell \geq 3$, be oriented rectangles and let, for $i = 1,\ldots,\ell-1$, be $f_i : K_i \to \mathbb{R}^2$ be continuous maps defined on the compact sets $K_i \subset \mathscr{A}_i$. Let K be the (compact) subset of K_1 where the map $f := f_{\ell-1} \circ \ldots f_2 \circ f_1$ is defined and such that $f_1(x) \in K_2, (f_2 \circ f_1)(x) \in K_3, \ldots, (f_{j-1} \circ \ldots f_2 \circ f_1)(x) \in K_j, \ldots, (f_{\ell-2} \circ \ldots f_2 \circ f_1)(x) \in K_{\ell-1}$, $\forall x \in K$. If*

$$(K_i, f_i) : \widetilde{\mathscr{A}_i} \Leftrightarrow\!\!\!\!\rightarrow \widetilde{\mathscr{A}_{i+1}}, \quad \forall i = 1,\ldots,\ell-1,$$

then

$$(K, f) : \widetilde{\mathscr{A}_1} \Leftrightarrow\!\!\!\!\rightarrow \widetilde{\mathscr{A}_\ell}.$$

Then, from Theorem 3.1 and Lemma 3.2, the following result holds:

Theorem 3.3. *Let $\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$ be an oriented rectangle, let $H_0, H_1, \ldots, H_{m-1}$ be $m \geq 2$ nonempty compact and pairwise disjoint subsets of $\mathscr{R}$, and let $f : \mathscr{H} := \bigcup_{j=0}^{m-1} H_j \to \mathbb{R}^2$ be a continuous map. Assume that*

$$(H_j, f) : \widetilde{\mathscr{R}} \Leftrightarrow\!\!\!\!\rightarrow \widetilde{\mathscr{R}}, \quad \forall j = 0,1,\ldots,m-1 \tag{13}$$

holds. Then, for any k-periodic sequence $(s_i)_i \in \{0,1,\ldots,m-1\}^{\mathbb{Z}}$, with $k \geq 1$, there exists at least one two-sided sequence $(w_i)_{i\in\mathbb{Z}}$ such that

$$w_i \in H_{s_i} \quad \text{and} \quad w_{i+1} = f(w_i), \quad \forall i \in \mathbb{Z},$$

with $w_{i+k} = w_i, \forall i \in \mathbb{Z}$.

Proof. Let $(s_i)_i \in \{0,1,\ldots,m-1\}^{\mathbb{Z}}$, with $k \geq 1$, be a given k-periodic sequence. Let K be the (compact) subset of H_{s_0} such that

$$f(x) \in K_{s_1}, f^2(x) \in K_{s_2}, \ldots, f^i(x) \in K_{s_i}, \ldots, f^{k-1}(x) \in K_{s_{k-1}}, \ \forall x \in K.$$

Then Lemma 3.2 implies that

$$(K, f^k) : \widetilde{\mathscr{R}} \Longleftrightarrow\!\!\!\!\!\!\rightarrow \widetilde{\mathscr{R}}.$$

A fixed point $z^* \in K$ for f^k (guaranteed by Theorem 3.1) is such that

$$z^* \in H_{s_0}, f(z^*) \in H_{s_1}, \ldots, f^i(z^*) \in H_{s_i}, \ldots, f^k(z^*) = z^*.$$

The two-sided k-periodic sequence $(w_i)_i$ with $w_0 = z^*$ and $w_i = f^i(z^*)$ for each $i = 1, \ldots, k$, allows to conclude the proof. □

On the other hand, Lemma 3.2 and Lemma 2.9 give the following.

Theorem 3.4. *Let $\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$ be an oriented rectangle, let $H_0, H_1, \ldots, H_{m-1}$ be $m \geq 2$ nonempty compact and pairwise disjoint subsets of $\mathscr{R}$, and let $f : \mathscr{H} := \bigcup_{j=0}^{m-1} H_j \to \mathbb{R}^2$ be a continuous map. Assume that* (13) *holds. Then, for any sequence $\xi := (s_i)_{i \in \mathbb{N}} \in \{0,1,\ldots,m-1\}^{\mathbb{N}}$, there exists a continuum*

$$\mathscr{C}_\xi \in H_{s_0}, \quad \textit{with } \mathscr{C}_\xi \cap \mathscr{R}^+_b \neq \emptyset, \ \mathscr{C}_\xi \cap \mathscr{R}^+_t \neq \emptyset,$$

such that, for each $w \in \mathscr{C}_\xi$, the sequence

$$w_0 = w, \quad w_{i+1} = f(w_i), \ \forall i \in \mathbb{N},$$

is such that

$$w_i \in H_{s_i}, \quad \forall i \in \mathbb{N}.$$

Proof. Let $(s_i)_{i \in N} \in \{0,1,\ldots,m-1\}^{\mathbb{N}}$ be a given sequence. Let S be the (compact) subset of H_{s_0} defined by

$$S := \bigcap_{k=0}^{\infty} S_k,$$

with $S_0 := K_{s_0}$ and

$$S_k := \bigcap_{i=1}^{k} f^{-i}(K_{s_i}).$$

By Lemma 3.2 we have that

$$(S_k, f^{k+1}) : \widetilde{\mathscr{R}} \Longleftrightarrow\!\!\!\!\!\!\rightarrow \widetilde{\mathscr{R}}, \quad \forall k \in \mathbb{N}.$$

Let $\gamma : [0,1] \to \mathscr{R}$ be a continuous map such that $\gamma(0) \in \mathscr{R}_l^-$ and $\gamma(1) \in \mathscr{R}_r^-$. Since the SAP property is satisfied for all the (S_k, f^{k+1})'s, we obtain a decreasing sequence of compact (nonempty) sets

$$\bar{\gamma} \supset C_0 \supset C_1 \supset \cdots \supset C_k \supset C_{k+1} \supset \dots$$

with

$$C_k \subset S_k, \quad \forall k \in N,$$

each of the C_k being the image of γ restricted to a suitable compact subinterval I_k of $[0,1]$, that is $C_k = \gamma(I_k)$, with $I_k \supset I_{k+1}$ for all $k \geq 0$. By Cantor's lemma we find that $\bigcap_{k\geq 0} C_k \neq \emptyset$ and thus we have proved that $\bar{\gamma} \cap S \neq \emptyset$. The proof ends just by applying the Crossing Lemma. □

As remarked in [25, Proposition 5], from one-sided sequences one gets two-sided sequences via a diagonal argument (see also [44, Theorem 2.2]). Thus, as a corollary of Theorem 3.3 and Theorem 3.4 one easily obtains the following.

Theorem 3.5. *Let $\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$ be an oriented rectangle, let $H_0, H_1, \dots, H_{m-1}$ be $m \geq 2$ nonempty compact and pairwise disjoint subsets of $\mathscr{R}$, and let $f : \mathscr{H} := \bigcup_{j=0}^{m-1} H_j \to \mathbb{R}^2$ be a continuous map. Assume that* (13) *holds. Then f induces chaotic dynamics on m symbols in $\mathscr{H}$.*

Theorem 3.5 is strongly related to a result by Kennedy and Yorke [25, Theorem 1]. Indeed, in the setting of [25], and calling end_0 and end_1 the two components of $\mathscr{R}^-$, we have that the assumption $(H_j, f) : \widetilde{\mathscr{R}} \Longleftrightarrow\!\!\!\!\rightarrow \widetilde{\mathscr{R}}$, for $j = 0, 1, \dots, m-1$ implies the *horseshoe hypothesis* Ω with a *crossing number* $M \geq m$ of Kennedy and Yorke. Then, [25, Theorem 1] ensures the existence of a closed invariant set $\mathscr{R}_I \subset \mathscr{R}$ for f such that $f|_{\mathscr{R}_I}$ is semiconjugate to a one-sided M-shift (respectively, semiconjugate to a two-sided M-shift if f is one-to-one). The results in [25] hold in the more general setting of mappings defined on locally connected compact sets of a separable metric space; however, the special geometry for our simplified setting allows us to draw as a further conclusion with respect to [25, Theorem 1] also the information about periodic points for f which is required in our definition of chaotic dynamics (see Definition 1).

In some applications of Theorem 3.5 there is a natural splitting of the map f as

$$f := \psi \circ \phi. \tag{14}$$

For instance, if f is the Poincaré map associated to some planar differential system, it may be natural to decompose f as two (or more than two) Poincaré maps which take into account of some peculiar behaviors of the system in different time intervals. In [46, 47] we have introduced a special corollary of Theorem 3.5 to deal with this situation. Namely, we have the following result where, for sake of simplicity, we have confined ourselves to the case $m = 2$.

Corollary 3.6. *Let $\widetilde{\mathscr{M}} := (\mathscr{M}, \mathscr{M}^-)$ and $\widetilde{\mathscr{N}} := (\mathscr{N}, \mathscr{N}^-)$ be oriented rectangles, and let H_0 and H_1 be two nonempty compact and disjoint subsets of $\mathscr{M}$. Let also $\phi : \mathscr{H} := H_0 \cup H_1 \to \mathbb{R}^2$ and $\psi : \mathscr{N} \to \mathbb{R}^2$ be continuous maps. Assume*

(i) $(H_j, \phi) : \widetilde{\mathscr{M}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{N}}$, *for* $j = 0, 1$,

(ii) $\psi : \widetilde{\mathscr{N}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{M}}$.

Then $f = \psi \circ \phi$ induces chaotic dynamics on two symbols in $\mathscr{H}$.

Proof. Setting $H_j' := H_j \cap \phi^{-1}(\mathscr{N})$ for $j = 0, 1$, we easily see from $(i) - (ii)$ and Lemma 3.2 that

$$(H_j', f) : \widetilde{\mathscr{M}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{M}}, \quad \text{for } j = 0, 1.$$

Then, Theorem 3.5 implies that f induces chaotic dynamics on two symbols in $\mathscr{H}' := H_0' \cup H_1' \subset \mathscr{H}$. $\square$

Corollary 3.6 has been applied in [48] in connection to the theory of the linked twist maps.

For extensions of this theory to higher dimensions, see [45], [49, 50] and [55].

4. Applications

In this last part of our article, we outline a possible application of our topological theorems to the search of periodic solutions and chaotic like dynamics associated to some second order scalar ODEs. The results are obtained by means of the study of the associated Poincaré map. As a model example, we focus our attention to the pendulum equation with a moving support. As explained in [21, Ch.8], a pendulum equation with a harmonically moving support is equivalent to a pendulum with a stationary support in a space with a periodically varying constant of gravity. Accordingly, mechanical systems of this kind are modelled by a second order equation of the form

$$u'' + w(t) \sin u = 0, \tag{15}$$

or, equivalently, by the first order system in the phase plane $(x, y) = (u, u')$

$$\begin{cases} x' = y \\ y' = -w(t) \sin x, \end{cases} \tag{16}$$

where the weight $w(t)$ is a periodic function of period $T > 0$. Following a classical approach (see [21]) one is usually led to study the linearized equation

$$\begin{cases} x' = y \\ y' = -w(t)x, \end{cases} \tag{17}$$

which represents a reasonable approximation of (16) in the case of *small solutions*. In this special case we have to study a Hill equation, which, for a general $w(t)$, represents still a nontrivial task. In [21, p. 344] Den Hartog suggests to consider a simplified form of (17) by assuming a squarewave weight function. For recent results about the Hill equation with stepwise coefficients we refer also to [17, 41, 42] and the references therein. Following the same suggestion, we propose to analyze the global dynamics for system (16) in the simplified case in which $w(t)$ is a stepwise function. In [6] and in [7] the case in which $w(t) > 0$ for all $t \in \mathbb{R}$ and the case in which $w(t)$ changes its sign were discussed, respectively. Therefore, here we consider a situation not previously investigated in [6, 7], namely the case in which the weight may vanish during some time interval. Physically, for $u(t) = \theta(t)$, which is the angle between the rod and the vertical line pointing downward, this corresponds to a model in which the pendulum winds around its pivot with constant angular speed $\theta'(t) = \text{constant} = \theta_0$ and without the effect of a gravity field for some time interval, coming back to the usual oscillation mode under the effect of a constant gravity field for a subsequent time interval. We also assume that the switching between these two oscillatory modes occurs in a T-periodic fashion. In conclusion, we suppose that the weight $w : \mathbb{R} \to \mathbb{R}$ is a T-periodic function and there are $T_0, T_1 \in]0, T[$ with

$$T_0 + T_1 = T,$$

such that

$$w(t) := \begin{cases} K, & \text{for } 0 \le t < T_0 \\ 0, & \text{for } T_0 \le t < T. \end{cases} \tag{18}$$

We assume $K > 0$ (it is easy to check that the case $K < 0$, which corresponds to the so-called inverted pendulum, can be treated with minor modifications in our forthcoming analysis).

Equation (16) with a weight function as in (18) can be viewed as a superposition of the equations

$$\begin{cases} x' = y \\ y' = -K \sin x, \end{cases} \tag{19}$$

and

$$\begin{cases} x' = y \\ y' = 0, \end{cases} \tag{20}$$

the first acting on an interval of length T_0 and the second one on an interval of length T_1. The choice of which between the two equations has to be considered as a first equation and which as a second equation is merely conventional, because all the process repeats in a periodic manner. As a last but a crucial remark, we have to notice that system (16), as well as (19) and (20) is studied in the *cylindrical phase space*, namely, we assume

$$(x_1, y) \equiv (x_2, y) \quad \text{for } \frac{x_2 - x_1}{2\pi} \in \mathbb{Z}.$$

This last remark, however, will be not used in the proof of Theorem 4.1 below; however, it turns out to be useful in view of extending our theorem to more general situations (see Remark 4.3).

The Poincaré map $\Phi : z \mapsto \zeta(T, z)$ associated to system (16) splits as

$$\Phi := \Phi_1 \circ \Phi_0 ,$$

where Φ_0 is the Poincaré map associated to the classical pendulum equation (19) for the time interval $[0, T_0]$, while Φ_1 is the Poincaré map associated to Eq. (19) for the time interval $[0, T_1]$. By a direct integration of the equation, Φ_1 can be easily described as the shift

$$\Phi_1(x, y) = (x + T_1 y, y).$$

We describe now the main steps in the proof of the presence of chaotic dynamics for equation (15), using Corollary 3.6. To this aim, first of all, we recall some basic facts about the phase plane analysis of (19), which corresponds to the nonlinear simple pendulum equation

$$x'' + K \sin x = 0, \quad K > 0.$$

Equation (19) is a simple example of a first order planar Hamiltonian system

$$\begin{cases} x' = y \\ y' = -g(x), \end{cases} \tag{21}$$

with $g : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz continuous function. The orbits associated to (21) lie on the level lines of the energy function

$$E(x, y) := \frac{1}{2} y^2 + G(x), \quad \text{with } G(x) := \int_0^x g(s)\, ds.$$

For the pendulum equation we have

$$G(x) = K(1 - \cos x)$$

and the (well known) phase portrait is that of Figure 4 below. We set

$$d := 2\sqrt{K}$$

and consider, for each $\mu \in]0,d]$, the set

$$\Gamma^{\mu} := \{(x,y) \in [-\pi,\pi] \times \mathbb{R} : E(x,y) = \frac{1}{2}\mu^2\}.$$

For each $0 < \mu < d$, the set Γ^{μ} is a closed curve surrounding the origin and intersecting the x-axis at the points $(\pm \arccos(1-\mu^2/2K), 0)$ and the y-axis at the points $(0, \pm\mu)$. Actually, Γ^{μ} is a periodic orbit which is run in the clockwise sense and its period, that we denote by τ_{μ}, can be expressed by means of an elliptic integral (see [20, pp. 180-181]). The time-mapping $e \mapsto \tau_{\mu}$ is a strictly increasing function with

$$\lim_{\mu \to 0^+} \tau_{\mu} = \frac{2\pi}{\sqrt{K}} \quad \text{and} \quad \lim_{\mu \to d^-} \tau_{\mu} = +\infty$$

(see [60, Figure 14]). On the other hand, for $\mu = d$, the level set Γ^d is the union of four orbits which are the two equilibrium points $(-\pi,0)$ and $(\pi,0)$ (which coincide each other in the cylindrical phase space and correspond to the unstable equilibrium position of the pendulum) and the two connecting orbits

$$L^+ := \{(x,y) \in]-\pi,\pi[\times \mathbb{R} : E(x,y) = 2K,\, y > 0\},$$

$$L^- := \{(x,y) \in]-\pi,\pi[\times \mathbb{R} : E(x,y) = 2K,\, y < 0\}.$$

The line L^+ is the orbit through $(0,d)$ which connects $(-\pi,0)$ (for $t \to -\infty$) to $(\pi,0)$ (for $t \to +\infty$) in the upper half-plane, while L^- is the orbit through $(0,-d)$ which connects $(\pi,0)$ (for $t \to -\infty$) to $(-\pi,0)$ (for $t \to +\infty$), in the lower half-plane.

We are ready now to define two generalized rectangles $\mathscr{M}$ and $\mathscr{N}$ and choose a suitable orientation for each of them (see Figure 5). To this end, we fix two numbers b,c with

$$0 < b < c < d$$

and consider (in the upper half plane) the intersection of the region

$$\mathscr{W} := \{(x,y) \in [-\pi,\pi] \times \mathbb{R} : \frac{1}{2}c^2 \leq E(x,y) \leq \frac{1}{2}d^2\}$$

with the strip

$$\mathscr{S} := \mathbb{R} \times [0,b]$$

Such intersection is made by two disjoint sets which are topological rectangles. We call $\mathscr{M}$ the component of $\mathscr{W} \cap \mathscr{S}$ contained in the right half-plane and we

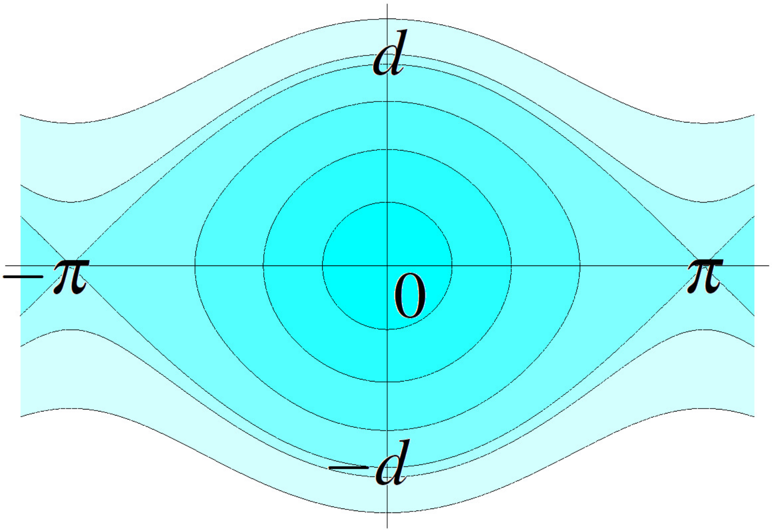

Figure 4: Energy level lines for Eq. (19) in the phase plane. The two separatrices (heteroclinic connections) connecting the unstable equilibria (saddle points) $(-\pi, 0)$ and $(\pi, 0)$ intersect the vertical axis at $(0, d)$ (in the upper half-plane) and $(0, -d)$ (in the lower half-plane), respectively, for $d = 2\sqrt{K}$.

call $\mathscr{N}$ the symmetric one with respect to the y-axis. One can easily find a homeomorphism mapping the unit square onto $\mathscr{M}$. Indeed, the function

$$h : (\mu, y) \mapsto (\arccos(1 - (2K)^{-1}(\mu^2 - y^2), y)$$

maps the rectangle $[c, d] \times [0, b]$ homeomorphically onto $\mathscr{M}$ and from this it is a simple task to obtain the desired homeomorphism defined on $[0, 1]^2$ onto $\mathscr{M}$. Having checked that $\mathscr{M}$ is a topological rectangle, we have that also $\mathscr{N}$ is a topological rectangle, using the symmetry $(x, y) \mapsto (-x, y)$ transforming $\mathscr{M}$ into $\mathscr{N}$.

Observe that $\mathscr{W}$ is an invariant set for system (19). Indeed, each point $z_0 = (x_0, y_0) \in \mathscr{W}$ belongs to the energy level line Γ^{μ_0} with

$$\mu_0 = 2\sqrt{E(x_0, y_0)} \in [c, d]$$

and the solution of (19) with initial point z_0 lies on Γ^{μ_0}. In particular, for each $z_0 \in \mathscr{M}$, we can represent the solution $(x(t), y(t))$ of (19) with $(x(0), y(0)) = z_0$

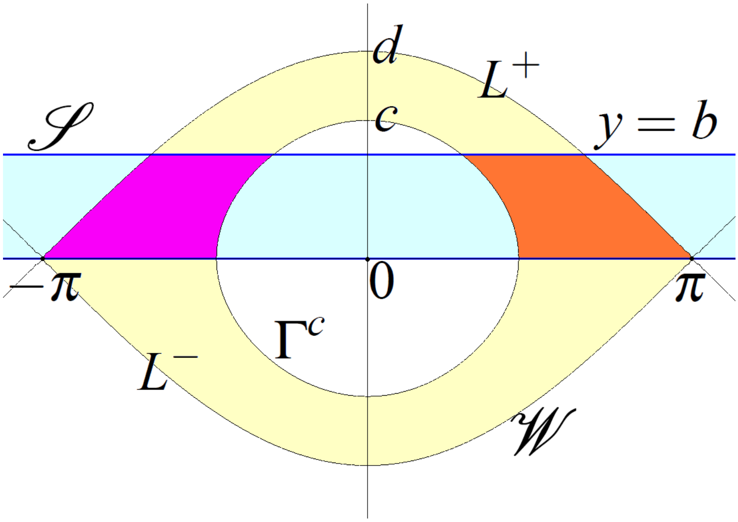

Figure 5: The set $\mathscr{W}$ is the part of the strip $[-\pi, \pi] \times \mathbb{R}$ between the energy level lines Γ^c and Γ^d. The intersection of $\mathscr{W}$ with the strip $\mathscr{S}$ produces two rectangular regions (generalized rectangles), painted with a darker color. The set $\mathscr{N}$ is the component of the intersection with $x < 0$, while $\mathscr{M}$ is the component of the intersection with $x > 0$. The sets $\mathscr{M}$ and $\mathscr{N}$ are symmetric with respect to the y-axis.

in polar coordinates, so that

$$x(t) = \rho(t, z_0) \cos \vartheta(t, z_0), \quad y(t) = \rho(t, z_0) \sin \vartheta(t, z_0),$$

The angular function $\vartheta(t, z_0)$ is well defined, continuous with respect to $(t, z_0) \in \mathbb{R} \times \mathscr{M}$ and satisfies $\vartheta(0, z_0) \in [0, \pi/2]$ (in fact, $\mathscr{M}$ is contained in the first quadrant). It is easy to check that $t \mapsto \vartheta(t, z_0)$ is a strictly decreasing function provided that $z_0 \neq (\pi, 0)$. As we have already observed, for $\mu_0 \in [c, d[$, we know that Γ^{μ_0} is a periodic orbit of period τ_{μ_0} which is run in the clockwise sense. Hence, if we take any initial point z_0 with $z_0 \in \mathscr{M} - \Gamma^d$, we conclude that z_0 is a periodic point of system (19) of period τ_{μ_0} and therefore, we have that for j a nonnegative integer,

$$\vartheta(t, z_0) - \vartheta(0, z_0) \lesseqqgtr -2j\pi \quad \text{if and only if} \quad t \gtreqqless j\,\tau_{\mu_0}$$

(remember that the motion associated to (19) occurs in the clockwise sense and therefore the angle decreases when the time increases).

As $\mathscr{W}$ is invariant for Φ_0, similarly, the strip $\mathscr{S}$ is invariant for Φ_1. In this case, under the effect of (20), all the points of $\mathscr{N}$ which belong also to the x-axis are rest points, while all the other points in $\mathcal{N}$ (with $y > 0$) move from the left to the right along the lines $y = \text{constant} = y_0$ with constant speed $x'(t) = y_0 > 0$.

For $\mathcal{M}$ and $\mathcal{N}$ we consider now the following orientations:

$$\mathcal{M}_l^- := \mathcal{M} \cap \Gamma^c, \quad \mathcal{M}_r^- := \mathcal{M} \cap \Gamma^d = \mathcal{M} \cap L^+ \cup \{(\pi, 0)\}.$$

$$\mathcal{N}_l^- := \mathcal{N} \cap [-\pi, 0] \times \{0\}, \quad \mathcal{N}_r^- := \mathcal{N} \cap [-\pi, 0] \times \{b\}.$$

Step 1. Stretching the paths from $\mathcal{M}$ to $\mathcal{N}$ by Φ_0. Assume that T_0 is fixed with

$$T_0 \geq 2\tau_c\,.$$

Consider any initial point $z_0 \in \mathcal{M}_r^-$. Since $\mathcal{M}_r^- \subset L^+ \cup \{(\pi, 0)\}$, which is an invariant set, we have that, for every $t \in [0, T_0]$, the solution of (19) with $(x(0), y(0)) = z_0$ belongs to $\mathcal{M}_r^-$. Hence

$$\vartheta(T_0, z_0) \geq 0, \quad \forall z_0 \in \mathcal{M}_r^-\,.$$

On the other hand, if $z_0 \in \mathcal{M}_l^-$, then

$$\vartheta(T_0, z_0) \leq \vartheta(0, z_0) - 4\pi < \frac{\pi}{2} - 4\pi = -3\pi - \frac{\pi}{2}\,.$$

Now we define the compact sets

$$H_0 := \{z \in \mathcal{M} : \Phi_0(z) \in \mathcal{N} \text{ and } \vartheta(T_1, z) \in [-3\pi/2, -\pi]\}$$

and

$$H_1 := \{z \in \mathcal{M} : \Phi_0(z) \in \mathcal{N} \text{ and } \vartheta(T_1, z) \in [-7\pi/2, -3\pi]\}.$$

Using the fact that the angular coordinates of the points of $\mathcal{N}$ belong to the intervals $]\pi/2 + 2k\pi, \pi + 2k\pi]$ (for $k \in \mathbb{Z}$) and using the above angular estimates, we conclude that H_0 and H_1 are both nonempty and, moreover, $H_0 \cap H_1 = \emptyset$.

Note that a point $z \in \mathcal{M}$ belongs to H_0 $(j = 0, 1)$ if and only if the solution $(x(t), y(t))$ of (19) with $(x(0), y(0)) = z$ is such that $(x(T_0), y(T_0)) \in \mathcal{N}$ and, moreover, $x(0) > x(T_0)$ with $x(t)$ having exactly $2j + 1$ simple zeros in $]0, T_0[$ where it changes its sign with $x' \neq 0$.

Let $\gamma : [0, 1] \ni s \mapsto \gamma(s) \in \mathcal{M}$ be a continuous curve with $\gamma(0) \in \mathcal{M}_l^-$ and $\gamma(1) \in \mathcal{M}_r^-$. By the previous estimates, we know that

$$\vartheta(T_0, \gamma(0)) < -3\pi - \frac{\pi}{2} \quad \text{and} \quad \vartheta(T_0, \gamma(1)) \geq 0.$$

By a continuity argument, we can find two subintervals $[s_0', s_0'']$ and $[s_1', s_1'']$ of $[0, 1]$, with

$$0 < s_1' < s_1'' < s_0' < s_0'' < 1,$$

such that

$$\gamma(s) \in H_j, \quad \forall s \in [s'_j, s''_j], \; j = 0, 1,$$

$$\Phi_0(\gamma(s)) \in \mathscr{N}, \quad \forall s \in [s'_1, s''_1] \cup [s'_0, s''_0],$$

$$\Phi_0(\gamma(s'_j)) \in \mathscr{N}_r^-, \quad \Phi_0(\gamma(s''_j)) \in \mathscr{N}_l^-, \quad \text{for } j = 0, 1.$$

We have thus proved that condition (i) of Corollary 3.6 holds for $\phi = \Phi_0$. $\square$

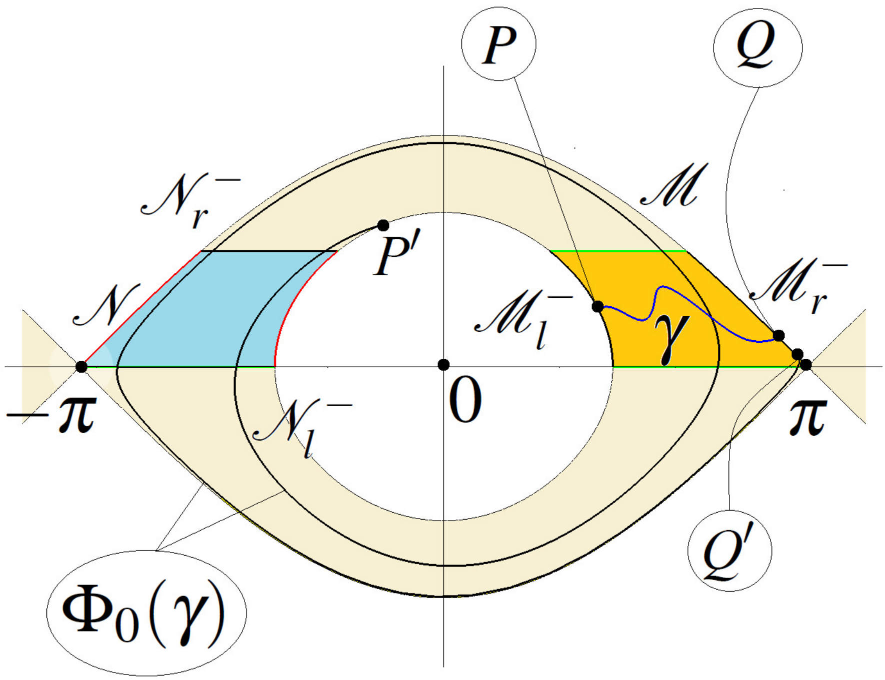

Figure 6: A graphical illustration of the property $\Phi_0 : \widetilde{\mathscr{M}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{N}}$ with crossing number larger than or equal to two. The path γ in $\mathscr{M}$ joining a point $P \in \mathscr{M}_l^-$ to a point $Q \in \mathscr{M}_r^-$ is transformed by Φ_0 into a path $\Phi_0(\gamma)$ joining P' to Q'. If the time T_0 is sufficiently large, the path $\Phi_0(\gamma)$ will make a certain number of windings around the origin and will cross the set $\mathscr{N}$ at least twice.

Step 2. Stretching the paths from $\mathscr{N}$ to $\mathscr{M}$ by Φ_1. Assume that

$$T_1 \geq \frac{2\pi}{b}.$$

By the simple form of Φ_1 we immediately see that

$$\Phi_1(z) = z, \quad \forall z \in \mathcal{N}_l^-$$

and,

$$x_1 > \pi, \text{ for } (x_1, b) = \Phi_1(z) \text{ with } z = (x, b) \in \mathcal{N}_r^- .$$

Let $\gamma\colon [0,1] \ni s \mapsto \gamma(s) \in \mathcal{N}$ be a continuous curve with $\gamma(0) \in \mathcal{N}_l^-$ and $\gamma(1) \in \mathcal{N}_r^-$. Hence, for $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, and $\Phi_1(\gamma(s)) = (\sigma_1(s), \sigma_2(s))$, we have that

$$\sigma_1(0) = \gamma_1(0) < 0, \quad \sigma_1(1) > \pi/2$$

and

$$\sigma_2(s) = \gamma_2(s) \in [0, b], \ \forall s \in [0,1].$$

By a continuity argument, we can find a subinterval $[s', s'']$ of $[0,1]$ such that $\Phi_1(\gamma(s)) \in \mathcal{M}$ for all $s \in [s', s'']$ and

$$\Phi_1(\gamma(s')) \in \mathcal{M}_l^- , \quad \Phi_1(\gamma(s'')) \in \mathcal{M}_r^- .$$

We have thus proved that condition (ii) of Corollary 3.6 holds for $\psi = \Phi_1$. □

In conclusion, using Corollary 3.6 we have proved the following.

Theorem 4.1. *Let $w(t)$ be a T-periodic stepwise function defined as in* (18). *Let us fix two constants b, c with*

$$0 < b < c < d := 2\sqrt{K}$$

and let τ_c be the fundamental period of the periodic orbit of the pendulum equation $x'' + K \sin x = 0$ with $x(0) = 0$ and $x'(0) = c$. Then, for

$$T_0 \geq 2\tau_c \quad \text{and} \ T_1 \geq \frac{2\pi}{b}, \quad T = T_0 + T_1,$$

equation (15) *exhibits chaotic dynamics on two symbols. The precise behavior of the chaotic like solutions can be described as follows.*

There exists a (nonempty) compact set Λ which is contained in the first quadrant of the phase plane which is invariant for the Poincaré map Φ associated to (16) *and such that $\Phi|_\Lambda$ is semiconjugate (via a continuous map g) to the two-sided Bernoulli shift on two symbols. In particular, for any sequence $\xi := (s_i)_i \in \{0,1\}^{\mathbb{Z}}$ there exists a point $z \in g^{-1}(\xi) \in \Lambda$ such that the solution $x(t)$ of* (15) *with $(x(0), x'(0)) = z$ has precisely $2s_i + 1$ simple zeros in $]iT, T_0 + iT[$ and exactly one zero in $]T_0 + iT, (i+1)T[$. Moreover, if the sequence of symbols $\xi = (s_i)_i$ is k-periodic, then there exists a $z \in g^{-1}(\xi)$ which is a k-periodic point for Φ in Λ and, consequently, the solution $x(t)$ is kT-periodic.*

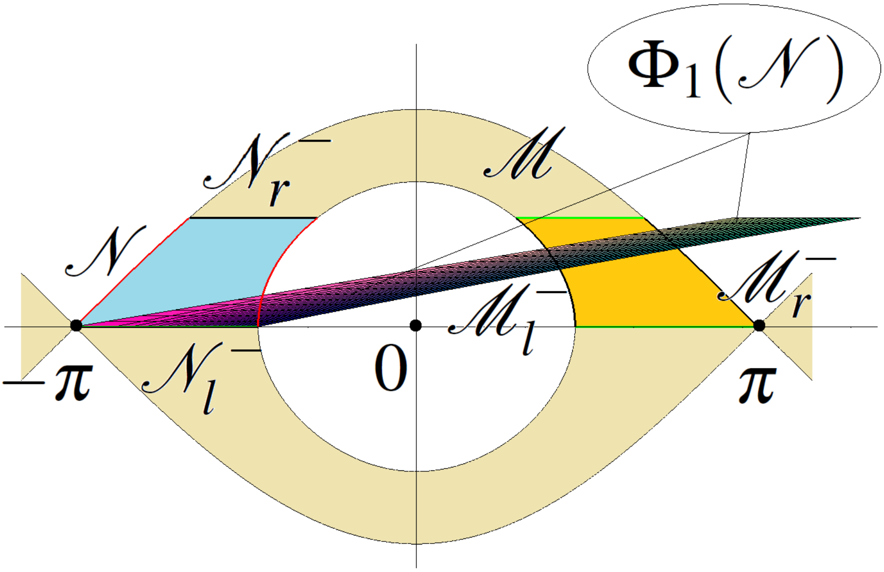

Figure 7: A graphical illustration of the property $\Phi_1 : \widetilde{\mathscr{N}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{M}}$. In this case, the SAP property is evident from the way in which $\Phi_1(\mathscr{N})$ goes across $\mathscr{M}$.

By the oddness of the sin function, it is clear that there is another family of chaotic solutions with initial points belonging to an invariant set (for the Poincaré map) contained in the third quadrant of the phase plane.

A careful checking of the proof will convince the reader that our argument is stable with respect to small perturbations on all the coefficients of the equation. The same observation was already employed in the previous papers [6, 7] where Corollary 3.6 was applied to the pendulum equation (under different conditions on the weight coefficient). The mathematical details which justify this assertion about the robustness of our result are fully developed in [48] and we refer to [48, pp. 900-902] for a description how to modify slightly the sets $\mathscr{M}$ and $\mathscr{N}$ in order to make the proof valid also in presence of small perturbations. Thus the following result holds true as well.

Theorem 4.2. *Let $w(t)$ be a T-periodic stepwise function defined as in* (18). *Let* *us fix two constants* b,c *with* $0 < b < c < d := 2\sqrt{K}$ *and let* τ_c *be the fundamental period of the periodic orbit of the pendulum equation* $x'' + K\sin x = 0$ *with* $x(0) = 0$ *and* $x'(0) = c$. *Then, for* T_0 *and* T_1 *fixed and satisfying*

$$T_0 > 2\tau_c \quad \textit{and} \;\; T_1 > \frac{2\pi}{b}, \quad T = T_0 + T_1,$$

there exists $\varepsilon > 0$ *such that for every* T*-periodic* L^1_{loc} *functions* $q(t)$ *and* $p(t)$ *satisfying*

$$\int_0^T |q(t) - w(t)|\,dt < \varepsilon, \quad \int_0^T |p(t)|\,dt < \varepsilon$$

and for every κ *with* $|\kappa| < \varepsilon$, *the pendulum equation*

$$u'' + \kappa u' + q(t)\sin u = p(t)$$

exhibits chaotic dynamics on two symbols.

Remark 4.3. Here is a list of possible directions toward which our results (Theorem 4.1 and Theorem 4.2) can be easily extended.

- If we work in the cylindrical phase plane and take T_1 sufficiently large we can easily find conditions in order to have that $\Phi_1(\mathscr{N})$ crosses multiple times the set $\mathscr{M}$ (mod 2π). In this manner, and with a minimal expense in the computations needed for the proof, we can prove the presence of even more complicated dynamics (namely on a larger set of symbols) in which the classical oscillations of the pendulum around the equilibrium position $u(t) = \theta(t) = 0$ alternate with a certain number of full revolutions.

- In Theorem 4.2, due to the particular form of $w(t)$ in (18), we assume that $\int_0^{T_1} |q(t)|\,dt < \varepsilon$. Actually, it is not difficult to get a more precise and better upper bound in terms of the L^1-norm of $q(\cdot)$ in $[0,T_1]$ so that our perturbative argument works.

- We have confined ourselves to the study of the nonlinear equation

  $$u'' + w(t)g(u) = 0,$$

  for $g(x) = \sin x$, motivated by the study of a pendulum type equation with moving support. It is possible to adapt our argument to some more general functions $g(x)$ (see [6, 7] for a similar treatment in the cases when $w(t)$ is of constant sign or $w(t)$ is a sign-changing weight).

The geometry for the application of Corollary 3.6 is that of the composition of a twist map acting on a topological annulus (the set $\mathscr{W}$ in our proof) with a squeezing and stretching map on a strip (the set $\mathscr{S}$ in our proof). Such kind of geometry, unlike the case of the *linked twist maps* [14, 59, 64], requires only one of the two mappings twisting the boundaries of an annulus. The same kind of geometrical configuration as the one considered in our example was proposed in an abstract setting in [46]. Concrete examples of ODEs presenting such kind of geometry have been obtained by Ruiz-Herrera in [54] dealing with population dynamics models. Some geometric configurations which are topologically equivalent (in the sense that an annulus is crossed by a topological strip) have been considered in [43, 47, 65, 66] and in 1997 by Kennedy and Yorke [26] dealing with a problem of turbulent fluid dynamics.

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