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HOPF MODULES IN THE BRAIDED MONOIDAL CATEGORY ${}_L\mathcal{M}$

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Suppose that L is a quasitriangular weak Hopf algebra with a bijective antipode and H is a weak Hopf algebra in the braided nonoidal category ${}_L\mathcal{M}$. We prove that the fundamental theorem for right H-Hopf modules in ${}_L\mathcal{M}$. Our results in this paper generalize previous fundamental theorem for Hopf module on the Hopf algebras and weak Hopf algebras.

1. Introduction

Weak Hopf algebras have been proposed by G. Bohm, F. Nill and K.Szlachanyi as a generalization of ordinary Hopf algebras in the following sense: the defining axioms are the same, but the multiplicativity of the counit and the comultiplicativity of the unit are replaced by weaker axioms. The initial motivation to study weak Hopf algebras is their connection with the theory of algebra extension [1], and another important application of weak Hopf algebras is that they provide a natural framework for the study of dynamical twists in Hopf algebras [2].

In [5] a new theory of weak Hopf algebras has begun to be developed: that of weak Hopf algebras in the monoidal categories. This is the theory with emphasis in ${}^L_L\mathcal{YD}$, the Yetter-Drinfeld category over L, where L is a weak Hopf algebra.

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Our motivation to study quasitriangular weak Hopf algebras is the so-called biproduct construction interpreted in terms of braided categories. More precisely, we are interested in a specific type of braided weak Hopf algebras.

In this paper, we prove the fundamental theorem for right H-Hopf modules in the representations category $Rep(L) = {}_L\mathcal{M}$, where L is a quasitriangular weak Hopf algebra. Since the matrix R gives rise to a natural braiding for ${}_L\mathcal{M}$ and ${}_L^L\mathcal{YD}$, We can show that if H is a weak Hopf algebra in ${}_L\mathcal{M}$, then H is also a weak Hopf algebra in ${}_L^L\mathcal{YD}$.

2. Preliminaries

Throughout this paper we use Sweedler's notation for comultiplication, writing $\Delta(h) = h_1 \otimes h_2$. Let k be a fixed field and all weak Hopf algebras and Hopf algebras are finite dimensional.

Definition 2.1. A weak Hopf algebra is a vector space L with the structure of an associative unital algebra $(L,\ m,\ \mu)$ with multiplication $m : L \otimes L \longrightarrow L$ and unit $1 \in L$ and a coassociative coalgebra $(L,\ \Delta,\ \varepsilon)$ with comultiplication $\Delta : L \longrightarrow L \otimes L$ and counit $\varepsilon : L \longrightarrow k$ such that
(i) The comultiplication Δ is a (not necessarily unit-preserving) homomorphism of algebras such that

$$(\Delta \otimes id)\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1). \tag{2.1}$$

(ii) The counit satisfies the following identity

$$\varepsilon(kgl) = \varepsilon(kg_1)\varepsilon(g_2l) = \varepsilon(kg_2)\varepsilon(g_1l), \quad \forall k,\ g,\ l \in L. \tag{2.2}$$

(iii) There is a linear map $S_L : L \longrightarrow L$ called an antipode, such that, for all $l \in L$

$$m(id \otimes S_L)\Delta(l) \ = \ (\varepsilon \otimes id)(\Delta(1)(l \otimes 1)), \tag{2.3}$$

$$m(S_L \otimes id)\Delta(l) \ = \ (id \otimes \varepsilon)((1 \otimes l)\Delta(1)), \tag{2.4}$$

$$S_L(l) = S_L(l_1)l_2S_L(l_3). \tag{2.5}$$

The linear map defined in (2.3) and (2.4) are called target and source counital maps and denoted by ε_t and ε_s respectively:

$$\varepsilon_t(l) = \varepsilon(1_{(1)}l)1_{(2)} = \varepsilon(S_L(l)1_{(1)})1_{(2)}, \tag{2.6}$$

$$\varepsilon_s(l) = 1_{(1)}\varepsilon(l1_{(2)}) = 1_{(1)}\varepsilon(1_{(2)}S_L(l)). \tag{2.7}$$

For all $l \in L$, we have

$$l_1 \otimes \varepsilon_t(l_2) = 1_{(1)}l \otimes 1_{(2)}, \ \ \varepsilon_s(l_1) \otimes l_2 = 1_{(1)} \otimes l1_{(2)}, \tag{2.8}$$

$$l_1 \otimes \varepsilon_s(l_2) = l1_{(1)} \otimes S_L(1_{(2)}), \ \ \varepsilon_t(l_1) \otimes l_2 = S_L(1_{(1)}) \otimes 1_{(2)}l. \tag{2.9}$$

We will briefly recall the necessary definitions and notions on the weak Hopf algebras.

Definition 2.2. An algebra H is a left L-module algebra if H is a left L-module via $l\otimes x \longmapsto l \longrightarrow x$ such that

(1) $l \longrightarrow xy = (l_1 \longrightarrow x)(l_2 \longrightarrow y)$,

(2) $l \longrightarrow 1 = \varepsilon_t(l) \longrightarrow 1, \quad \forall x,\ y \in H,\ l \in L.$

The second equation is equivalent to $\varepsilon_t(l) \longrightarrow x = (l \longrightarrow 1)x$.

Definition 2.3. An algebra H is a left L-module coalgebra if H is a left L-module via $l\otimes x \longmapsto l \longrightarrow x$ such that

(1) $\Delta(l \longrightarrow x) = (l \longrightarrow x)_1 \otimes (l \longrightarrow x)_2 = (l_1 \longrightarrow x_1)\otimes(l_2 \longrightarrow x_2)$,

(2) $\varepsilon_s(l) \longrightarrow x = x_1\varepsilon(l \longrightarrow x_2), \quad \forall l \in L,\ x \in H.$

the second equation is equivalent to

$$\varepsilon(lk \longrightarrow h) = \varepsilon(lk_2)\varepsilon(k_1 \longrightarrow h), \quad \varepsilon(\varepsilon_s(l) \longrightarrow h) = \varepsilon(l \longrightarrow h),\ l,\ k \in L,\ h \in H.$$

Definition 2.4. A quasitriangular weak Hopf algebra is a pair $(L,\ R)$ where L is a weak Hopf algebra and $R \in \Delta^{op}(1)(L\otimes L)\Delta(1)$ (called the R-matrix) satisfying the following conditions:

$$\Delta^{op}(l)R = R\Delta(1) \tag{2.10}$$

for all $l \in L$, where Δ^{op} denotes the conditions apposite to Δ,

$$(id\otimes\Delta)(R) = R_{13}R_{12}, \tag{2.11}$$

$$(\Delta\otimes id)(R) = R_{13}R_{23}. \tag{2.12}$$

where $R_{12} = R\otimes 1,\ R_{23} = 1\otimes R$, etc. as usual, and such that there exits $\overline{R} \in \Delta(1)(L\otimes L)\Delta^{op}(1)$ with

$$R\overline{R} = \Delta^{op}(1), \quad \overline{R}R = \Delta(1)$$

Furthermore, $(L,\ R)$ is called triangular if $\overline{R} = R_{21}$, where we write $R = R^1\otimes R^2$, then $R_{21} = R^2\otimes R^1$.

Note that $\overline{R}$ is uniquely determined by R. R satisfies the quantum Yang-Baxter equation. By [3], we can obtain that

Proposition 2.5. *For any quasitriangular weak Hopf algebra* $(L,\ R)$*, we have*

$$(\varepsilon_s\otimes id)(R) = \Delta(1), \qquad (id\otimes\varepsilon_s)(R) = (S_L^{-1}\otimes id)\Delta^{op}(1),$$
$$(\varepsilon_t\otimes id)(R) = \Delta^{op}(1), \qquad (id\otimes\varepsilon_t(R) = (S_L^{-1}\otimes id)\Delta(1),$$
$$(S_L\otimes id)(R) = (id\otimes S_L^{-1})(R) = \overline{R}, \qquad (S_L\otimes S_L)(R) = R,$$
$$(\varepsilon\otimes id)(R) = (id\otimes R) = 1.$$

3. Weak Hopf Algebras in the Braided Monoidal Category

Let L be a quasitriangular weak Hopf algebra with a bijective antipode S_L. We recall that the category ${}_L\mathcal{M}$ is the braided monoidal categories whose objects V are left L-modules and satisfy the following conditions:

Proposition 3.1. *The category $Rep(L) = {}_L\mathcal{M}$ is a braided monoidal category. The braiding $\tau_{V,W}: V \otimes W \longrightarrow W \otimes V$ is defined by*

$$\tau_{V,\, W}(v \otimes w) = (R^2 \longrightarrow w) \otimes (R^1 \longrightarrow v), \ \ \forall v \in V, \ w \in W$$

and the inverse of $\tau_{V,W}$ is given by

$$\tau_{V,W}^{-1}(w \otimes v) = (\overline{R}^1 \longrightarrow v) \otimes (\overline{R}^2 \longrightarrow w).$$

In [5] Bing-liang et al introduces the definition of Weak Hopf algebra in the braided monoidal category ${}_L\mathcal{M}$. Moreover they have showed that if H is a finite-dimensional weak Hopf algebra in ${}_L\mathcal{M}$, then its dual H^* is a weak Hopf algebra in ${}_L\mathcal{M}$.

Definition 3.2. Let (L,R) be a quasitriangular weak Hopf algebra. An object $H \in {}_L\mathcal{M}$ is called a weak bialgebra in this category if it is both an algebra and a coalgebra satisfying the following conditions:

(1) Δ and ε are not necessarily unit-preserving, such that

$$\begin{aligned}
\Delta(xy) =& x_1(R^2 \longrightarrow y_1) \otimes (R^1 \longrightarrow x_2)y_2,\\
\varepsilon(xyz) =& \varepsilon(xy_1)\varepsilon(y_2z),\\
\varepsilon(xyz) =& \varepsilon(x(R^2 \longrightarrow y_2))\varepsilon((R^1 \longrightarrow y_1)z),\\
\Delta^2(1) =& 1_1 \otimes 1_2 1'_1 \otimes 1'_2,\\
\Delta^2(1) =& 1_1 \otimes (R^2 \longrightarrow 1'_1)(R^1 \longrightarrow 1_2) \otimes 1'_2.
\end{aligned}$$

(2) H is both a left L-module algebra and L-module coalgebra.

(3) Furthermore, H is called a weak Hopf algebra in ${}_L\mathcal{M}$ if there exists an antipode $S: H \longrightarrow H$ (here S is left L-linear i.e., S is a morphism in the category of ${}_L\mathcal{M}$) satisfying

$$\begin{aligned}
x_1S(x_2) &= \varepsilon((R^2 \longrightarrow 1_1)(R^1 \longrightarrow x))1_2,\\
S(x_1)x_2 &= 1_1\varepsilon((R^2 \longrightarrow x)(R^1 \longrightarrow 1_2)),\\
S(x_1)x_2S(x_3) &= S(x), \ \ \forall x \in H.
\end{aligned}$$

Similar to the notation of weak Hopf algebra, we denote

$$\varepsilon_t(x) = \varepsilon((R^2 \longrightarrow 1_1)(R^1 \longrightarrow x))1_2, \tag{2.13}$$

$$\varepsilon_s(x) = 1_1\varepsilon((R^2 \longrightarrow x)(R^1 \longrightarrow 1_2)). \tag{2.14}$$

If $x = 1$ one can obtain $\varepsilon_t(1) = \varepsilon_s(1) = 1$. According to the definitions of $\varepsilon_t,\ \varepsilon_s$ one obtains explicit expressions for these coproducts

$$\Delta(\varepsilon_t(x)) = \varepsilon_t(x)1_1 \otimes 1_2, \quad \Delta(\varepsilon_(x)) = 1_1 \otimes 1_2\varepsilon_s(x)$$

Furthermore, for $x \in H$

$$\begin{aligned}
\varepsilon(\varepsilon_t(x)) =&\varepsilon((R^2 \longrightarrow 1)(R^1 \longrightarrow x)),\\
=&\varepsilon((\varepsilon_t(R^2) \longrightarrow 1)(R^1 \longrightarrow x)),\\
=&\varepsilon((1_{(2)} \longrightarrow 1)(S_L^{-1}(1_{(1)}) \longrightarrow x)),\\
=&\varepsilon((S_L(1_{(1)}) \longrightarrow 1)(1_{(2)} \longrightarrow x)),\\
=&\varepsilon(\varepsilon_t(S_L(1_{(1)}))1_{(2)} \longrightarrow x),\\
=&\varepsilon(S_L(\varepsilon_s(1_{(1)}))1_{(2)} \longrightarrow x),\\
=&\varepsilon(\varepsilon_s(1) \longrightarrow x),\\
=&\varepsilon(1).
\end{aligned}$$

In a similar way we can compute $\varepsilon(\varepsilon_s(x)) = \varepsilon(x)$. Applying the 3.2 one obtains immediately the following identities

$$\begin{aligned}
\varepsilon(x\varepsilon_t(y)) =&\varepsilon(xy_1S(y_2)) = \varepsilon(xy_1)\varepsilon(y_2S(y_3)) = \varepsilon(xy),\\
\varepsilon(\varepsilon_s(x)y) =&\varepsilon(S(x_1)x_2y) = \varepsilon(S(x_1)x_2)\varepsilon(x_3y) = \varepsilon(xy).
\end{aligned}$$

As S is left L-linear, we can easily check that ε_t and ε_s are also left L-linear. Moreover it is both an anti-algebra map and an anti-coalgebra map, that is

$$\begin{aligned}
Sm =&m\tau_{H,H}(S\otimes S),\ i.e., S(xy) = (R^2 \longrightarrow S(y))(R^1 \longrightarrow S(x)),\ \ x,\ y \in H,\\
\Delta S =&(S\otimes S)\tau_{H,H}\Delta,\ i.e., \Delta(S(x)) = R^2 \longrightarrow S(x_2) \otimes R^1 \longrightarrow S(x_1).
\end{aligned}$$

In this paper, we will always assume that the antipode S is bijective. The composite-inverse S^{-1} satisfies

$$\begin{aligned}
S^{-1}m =&m(S^{-1}\otimes S^{-1})\tau^{-1}, \quad i.e.,\\
S^{-1}(xy) =&\overline{R}^1 \longrightarrow S^{-1}(y) \otimes \overline{R}^2 \longrightarrow S^{-1}(x)),\\
\Delta S^{-1} =&(S^{-1}\otimes S^{-1})\tau^{-1}\Delta, \quad i.e.,\\
\Delta(S^{-1}(x)) =&\overline{R}^1 \longrightarrow S^{-1}(x_2) \otimes \overline{R}^2 \longrightarrow S^{-1}(x_1).
\end{aligned}$$

Proposition 3.3. *Suppose H is a weak Hopf algebra in $_L\mathcal{M}$, the following identities hold*

$$\varepsilon_t \circ S = S\circ \varepsilon_S,\ \ \varepsilon_s \circ S = S\circ \varepsilon_t.$$

Proof. For $x \in H$ we have

$$\begin{aligned}\varepsilon_t \circ S(x) =& [S(x)]_1 S([S(x)]_2) = (x_1^{-1} \longrightarrow S(x_2))S(S(x_1^0)),\\ =& S(S(x_1)x_2) = S \circ \varepsilon_S(x).\end{aligned}$$

In a similar way one can verify $\varepsilon_s \circ S = S \circ \varepsilon_t$. □

As a preparation for the proposition below we notice that the definitions (2.13)(2.14) have counterparts involving the antipode,

$$\varepsilon_t(x) = \varepsilon(S(x)1_1)1_2, \quad \varepsilon_s(x) = 1_1\varepsilon(1_2S(x)).$$

As a matter of fact

$$\varepsilon_t(x) = \varepsilon(\varepsilon_t(x)1_1)1_2 = \varepsilon(x_1S(x_2)1_1)1_2 = \varepsilon(\varepsilon_s(x_1)S(x_2)1_1)1_2 = \varepsilon(S(x)1_1)1_2.$$

The second equation can be proven analogously. Applying Proposition 3.2 one can verify

$$\begin{aligned}\varepsilon_s(x) =& (S \circ \varepsilon_t \circ S^{-1})(x) = S(\varepsilon(x1_1)1_2) = \varepsilon(x1_1)S(1_2),\\ \varepsilon_t(x) =& (S \circ \varepsilon_s \circ S^{-1})(x) = S(\varepsilon(1_2x)1_1) = \varepsilon(1_2x)S(1_1).\end{aligned}$$

Proposition 3.4. *Suppose H is a weak Hopf algebra in ${}_L\mathcal{M}$. For all $x \in H$ we have the identities*

$$x_1 \otimes \varepsilon_s(x_2) = x1_1 \otimes S(1_2), \quad \varepsilon_t(x_1) \otimes x_2 = S(1_1) \otimes 1_2x.$$

Proof. Using $\varepsilon_s(x) = \varepsilon(x1_1)S(1_2)$, $\varepsilon_t(x) = \varepsilon(1_2x)S(1_1)$, one obtains

$$\begin{aligned}x_1 \otimes \varepsilon_s(x_2) =& x_1 \otimes S(1_2)\varepsilon(x_21_1),\\ =& x_1(R^2 \longrightarrow 1_{1'})\varepsilon((R^1 \longrightarrow x_2)1_{2'}1_1) \otimes S(1_2),\\ =& x_1(R^2 \longrightarrow 1_1)\varepsilon((R^1 \longrightarrow x_2)1_2) \otimes S(1_3),\\ =& (x1_1)_1\varepsilon((x1_1)_2) \otimes S(1_2),\\ =& x1_1 \otimes S(1_2).\end{aligned}$$

$$\begin{aligned}\varepsilon_t(x_1) \otimes x_2 =& S(1_1)\varepsilon(1_2x_1) \otimes x_2,\\ =& S(1_1)\varepsilon(1_21_{1'}(R^2 \longrightarrow x_1)) \otimes (R^1 \longrightarrow 1_{2'})x_2,\\ =& S(1_1)\varepsilon(1_2(R^2 \longrightarrow x_1)) \otimes (R^2 \longrightarrow 1_3)x_2,\\ =& S(1_1) \otimes (1_2x)_1\varepsilon((1_2x)_2),\\ =& S(1_1) \otimes 1_2x,\end{aligned}$$

□

Applying the above proposition we obtain $x_1S(x_2)x_3 = x_1\varepsilon_s(x_2) = x$.

4. Hopf Modules in the Yetter-Drinfeld Categories

Since a weak Hopf algebra H in the weak Yetter-Drinfeld categories $_L\mathcal{M}$ is both algebra and coalgebra, one can consider modules and comodules over H. As in the theory of Hopf algebras, an H-Hopf module is an H-module which is also an H-comodule such that these two structures are compatible (the action "commutes" with coaction):

Definition 4.1. Let H be a weak Hopf algebra in $_L\mathcal{M}$. A right H-Hopf module M in $_L\mathcal{M}$ is an object $M \in {_L\mathcal{M}}$ such that it is both a right H-module and a right H-comodule via $\rho_M: M \longrightarrow M \otimes H$, $\rho_M(m) = m_0 \otimes m_1$ and the following equations hold:

$$
\begin{aligned}
&(1)\ \rho_M(mh) = m_0(R^2 \longrightarrow h_1) \otimes (R^1 \longrightarrow m_1)h_2, m \in M,\ h \in H,\\
&(2)\ l \longrightarrow (mh) = (l_1 \longrightarrow m)(l_2 \longrightarrow h), l \in L,\ m \in M,\ h \in H,\\
&(3)\ \rho_M(l \longrightarrow m) = (l_1 \longrightarrow m_0)(l_2 \longrightarrow m_1), l \in L,\ m \in M.
\end{aligned}
$$

We remark that $M \otimes_t H$ is a right H-module by $(m \otimes h)x = m(R^2 \longrightarrow x_1) \otimes (R^1 \longrightarrow h)x_2$ and a right H-comodule $\rho_{M\otimes H}(m \otimes h) = m_0 \otimes (R^2 \longrightarrow h_1) \otimes (R^1 \longrightarrow m_1)h_2$. The condition (1) means that the H-comodule structure $\rho_M: M \longrightarrow M \otimes H$ is H-linear, or equivalently the H-module structure map $\varphi_M: M \otimes H \longrightarrow M$ is H-colinear. Also, (2)$\Longleftrightarrow$ φ_M is L-linear; (3)$\Longleftrightarrow$ ρ_M is L-linear.

Example 4.2. H itself is a right H-Hopf module (in $_L\mathcal{M}$) in the natural way. If V is an object in $_LM$, then so is $V \otimes_t H$ by $l \longrightarrow (v \otimes h) = (l_1 \longrightarrow v) \otimes (l_2 \longrightarrow h)$ It is also both a right H-module and a right H-comodule by $(v \otimes h)x = v \otimes hx$ and $\rho_{V\otimes H}(v \otimes h) = v \otimes h_1 \otimes h_2$. One easily checks that $V \otimes_t H$ is an H-Hopf module.

Theorem 4.3. *If H is a weak Hopf algebra in $_L\mathcal{M}$ and M a right H-Hopf module in $_L\mathcal{M}$, then*

(1) $M^{coH} = \{m \in M | \rho_M(m) = m1_1 \otimes 1_2\}$ is a L-submodule. So $M^{coH} \in {_L\mathcal{M}}$.

(2) Let $P(m) = m_0S(m_1)$, $m \in M$. Then $P(m) \in M^{coH}$. If $n \in M^{coH}$ and $h \in H$, Then $\rho_M(nh) = nh_1 \otimes h_2$ and $P(nh) = n\varepsilon_t(h)$.

(3) The map $F: M^{coH} \otimes_t H \longrightarrow M$, $F(n \otimes h) = nh$ is an isomorphism of Hopf modules. The inverse map is given by $G(m) = P(m_0)m_1$.

Proof. Let $n \in M^{coH}$. Then

$$\begin{aligned}\rho_M(l \longrightarrow n) &= (l_1 \longrightarrow n1_1) \otimes (l_2 \longrightarrow 1_2),\\ &= (l_1 \longrightarrow n)(l_2 \longrightarrow 1_1) \otimes (l_3 \longrightarrow 1_2),\\ &= [(l_1 \longrightarrow n) \otimes 1]\Delta(\varepsilon_t(l_2) \longrightarrow 1),\\ &= ((1_{(1)}l \longrightarrow n) \otimes 1)\Delta(1_{(2)} \longrightarrow 1),\\ &= [1_{(1)} \longrightarrow (l \longrightarrow n)][1_{(2)} \longrightarrow (1_{(1')} \longrightarrow 1_1)] \otimes 1_{(2')} \longrightarrow 1_2,\\ &= (l \longrightarrow n)1_1 \otimes 1_2.\end{aligned}$$

Hence $l \longrightarrow n \in M^{coH}$. So $M^{CoH} \in {}_L\mathcal{M}$.

(2) Applying (2.12) and $x_1 \otimes \varepsilon_s(x_2) = x1_1 \otimes S(1_2)$ we have

$$\begin{aligned}\rho_M(P(m)) =&m_0(R^2 \longrightarrow [S(m_2)]_1) \otimes (R^1 \longrightarrow m_1)[S(m_2)]_2,\\ =&m_0(R^2r^2 \longrightarrow S(m_3)) \otimes (R^1 \longrightarrow m_1)(r^1 \longrightarrow S(m_2)),\\ =&m_0(R^2 \longrightarrow S(m_3)) \otimes ((R^1)_1 \longrightarrow m_1)((R^1)_2 \longrightarrow S(m_2)),\\ =&m_0(R^2 \longrightarrow S(m_2)) \otimes R^1 \longrightarrow \varepsilon_t(m_1),\\ =&m_0(S(m_1)_1) \otimes [S^{-1} \circ \varepsilon_s](S(m_1)_2),\\ =&m_0S(m_1)1_1 \otimes 1_2.\end{aligned}$$

If $n \in M^{coH}$ and $h \in H$, then

$$\begin{aligned}\rho(nh) =&n1_1(R^2 \longrightarrow h_1) \otimes (R^1 \longrightarrow 1_2)h_2 = nh_1 \otimes h_2.\\ P(nh) =&nh_1S(h_2) = n\varepsilon_t(h).\end{aligned}$$

(2) Since

$$\begin{aligned}F(l \longrightarrow (n \otimes h)) =&F((l_1 \longrightarrow n) \otimes (l_2 \longrightarrow h)),\\ =&(l_1 \longrightarrow n)(l_2 \longrightarrow h),\\ =&l \longrightarrow nh,\\ =&l \longrightarrow F(n \otimes h).\end{aligned}$$

Then F is a left H-linear map. It is also right H-colinear by (1). Now we have

$$\begin{aligned}GF(n \otimes h) =&P(nh_1) \otimes h_2 = n\varepsilon_t(h_1) \otimes h_2,\\ =&n \otimes \varepsilon_t(h_1)h_2 = n \otimes S(1_1)1_2h,\\ =&n \otimes h.\end{aligned}$$

$$\begin{aligned}FG(m) =& m_0S(m_1)m_2 = m_0\varepsilon_s(m_1),\\ =&[m_0\varepsilon_s(m_1)]_0\varepsilon([m_0\varepsilon_s(m_1)]_1),\\ =&m_0(m_1^{-1} \longrightarrow 1_1)\varepsilon(m_1^0 1_2\varepsilon_s(m_2)),\\ =&m_0\varepsilon(m_1\varepsilon_s(m_2)),\\ =&m_0\varepsilon(m_1 1_1 S(1_2)),\\ =&m_0\varepsilon(m_1) = m.\end{aligned}$$

□

Example 4.4. Let H be a weak Hopf algebra in ${}_L\mathcal{M}$. $M = H$ is defined as a right H-Hopf module by Δ. Then $M^{CoH} = \{\varepsilon_t(h)|h \in H\}$.

5. Application

Let $V \in {}_L\mathcal{M}$ we have constructed a left L-coaction over V via

$$\sigma_V : V \longrightarrow L\otimes V,\ \ v \longmapsto v^{-1} \otimes v^0 = R^2 \otimes R^1 \longrightarrow v.$$

Applying (2.11) and (2.12) we can verify

$$\begin{aligned}(id\otimes\sigma_V)\circ\sigma_V(v) =&(id\otimes\sigma_V)(R^2\otimes(R^1 \longrightarrow v)),\\ =&R^2\otimes r^2\otimes r^1R^1 \longrightarrow v,\\ =&(R^2)_1\otimes(R^2)_2\otimes R^1 \longrightarrow v,\\ =&(\Delta\otimes id)\circ\sigma_V(v).\\ (\varepsilon\otimes id)\circ\sigma_V(v) =&\varepsilon(R^2)R^1 \longrightarrow v = v.\end{aligned}$$

So V is a left L-comodule with σ_V.

Next we check the compatibility conditions for V. Since $R = R^1 \otimes R^2 \in \Delta^{op}(1)(L\otimes L)\Delta(1)$ we immediately get $\sigma_V(v) \in L\otimes_t V = \{1_{(1)}l\otimes 1_{(2)} \longrightarrow v|\forall l \in L,\ v \in V\}$. Using (2.10) one can obtain that

$$\begin{aligned}l_1v^{-1}\otimes l_2 \longrightarrow v^0 =&l_1R^2\otimes l_2R^1 \longrightarrow v,\\ =&R^2l_2\otimes R^1l_1 \longrightarrow v,\\ =&(l_1 \longrightarrow v)^{-1}l_2\otimes(l_1 \longrightarrow v)^0.\end{aligned}$$

Therefore $V \in {}^L_L\mathcal{YD}$. It is clearly that the matrix R give rise to a natural braiding for ${}_LM$ to ${}^L_L\mathcal{YD}$. Applying (2.12), for $\forall g,\ h \in H$ we have

$$\begin{aligned}g^{-1}h^{-1}\otimes g^0h^0 =&R^2r^2\otimes(R^1 \longrightarrow g)(r^1 \longrightarrow h),\\ =&R^2\otimes((R^1)_1 \longrightarrow g)((R^1)_2 \longrightarrow h),\\ =&R^2\otimes R^1 \longrightarrow gh,\\ =&(gh)^{-1}\otimes(gh)^0.\end{aligned}$$

Furthermore

$$
\begin{aligned}
\varepsilon_s(R^2)\otimes R^1 \longrightarrow 1 =& 1_{(1)}\otimes S^{-1}(1_{(2)}),\\
=& 1_{(1)}\otimes \varepsilon_t(S^{-1}(1_{(2)})) \longrightarrow 1,\\
=& 1_{(1)}\otimes S^{-1}(\varepsilon_s(1_{(2)})) \longrightarrow 1,\\
=& 1_{(1)}\otimes S^{-1}(S(1_{(2)})) \longrightarrow 1,\\
=& 1_{(1)}\otimes 1_{(2)} \longrightarrow 1,\\
=& R^2\otimes \varepsilon_t(R^1) \longrightarrow 1,\\
=& R^2\otimes R^1 \longrightarrow 1,\\
=& 1^{-1}\otimes 1^0.
\end{aligned}
$$

therefore H is a left L-comodule algebra. For $h\in H$ we do a calculation

$$
\begin{aligned}
h^{-1}\otimes (h^0)_1\otimes (h^0)_2 =& R^2\otimes (R^1 \longrightarrow h)_1\otimes (R^1 \longrightarrow h)_2,\\
=& R^2\otimes (R^1)_1 \longrightarrow h_1\otimes (R^1)_2 \longrightarrow h_2,\\
=& R^2 r^2\otimes R^1 \longrightarrow h_1\otimes r^1 \longrightarrow h_2,\\
=& h_1^{-1}h_2^{-1}\otimes h_1^0\otimes h_2^0.\\
\varepsilon(h^0)\varepsilon_t(h^{-1}) =& \varepsilon(R^1 \longrightarrow h)\varepsilon_t(R^2),\\
=& \varepsilon(S_L^{-1}(1_{(1)}) \longrightarrow h)1_{(2)},\\
=& \varepsilon(\varepsilon_s(S_L^{-1}(1_{(1)})) \longrightarrow h)1_{(2)},\\
=& \varepsilon(S_L^{-1}(\varepsilon_t(1_{(1)})) \longrightarrow h)1_{(2)},\\
=& \varepsilon(1_{(1)} \longrightarrow h)1_{(2)},\\
=& \varepsilon(\varepsilon_s(R^1) \longrightarrow h)R^2,\\
=& \varepsilon(R^1 \longrightarrow h)R^2,\\
=& \varepsilon(h^0)h^{-1}.
\end{aligned}
$$

It is clearly that H is a left L-comodule coalgebra. By the above proof, we conclude that

Proposition 5.1. *Suppose H is a weak Hopf algebra in ${}_L\mathcal{M}$ as above. H is also a weak Hopf algebra in ${}_L^L\mathcal{YD}$ with a left L-coaction via $\sigma_H : H \longrightarrow L\otimes H$, $h\longmapsto h^{-1}\otimes h^0 = R^2\otimes R^1 \longrightarrow h$.*

In particular L is a quasitriangular Hopf algebra, similarly we can define Hopf algebra in the braided category ${}_L\mathcal{M}$.

Definition 5.2. Let (L,R) be a quasitriangular Hopf algebra. An object $H\in {}_L\mathcal{M}$ is called a bialgebra in this category if it is both a algebra and a coalgebra satisfying the following conditions:

(1) Δ and ε are homomorphism of algebras such that

$$\Delta(xy) = x_1(R^2 \longrightarrow y_1) \otimes (R^1 \longrightarrow x_2)y_2, \quad \Delta(1) = 1 \otimes 1, \quad \varepsilon(xy) = \varepsilon(x)\varepsilon(y).$$

(2) H is a left L-module algebra. For $\forall x, y \in H, l \in L$

$$l \longrightarrow xy = l_1 \longrightarrow x \otimes l_2 \longrightarrow y, \quad l \longrightarrow 1_H = \varepsilon(l)1_H.$$

(3) H is a left L-module coalgebra. For $\forall h \in H, l \in L$

$$\Delta(l \longrightarrow h) = l_1 \longrightarrow h_1 \otimes l_2 \longrightarrow h_2, \quad \varepsilon_H(l \longrightarrow h) = \varepsilon_L(l)\varepsilon_H(h).$$

(4) Furthermore, H is called a Hopf algebra in ${}_L\mathcal{M}$ if there exist an antipode $S:\ H \longrightarrow H$ (here S is left L-linear i.e., S is a morphism in the category of ${}_L\mathcal{M}$ satisfying

$$x_1S(x_2) = S(x_1)x_2 = \varepsilon(x)1_H, \qquad S(x_1)x_2S(x_3) = S(x), \ \forall x \in H.$$

Furthermore we can prove the following theorem.

Theorem 5.3. *Suppose L is a quasitriangular Hopf algebra and H is a Hopf algebra in ${}_L\mathcal{M}$, then H is also a Hopf algebra in the Yetter-Drinfeld category ${}_L^L\mathcal{YD}$. If M is a right H-Hopf module in ${}_L\mathcal{M}$, then*

(1) $M^{coH} = \{m \in M | \rho(m) = m \otimes 1_H\}$.

(2) The map $F:\ M^{coH} \otimes H \longrightarrow M,\ F(n \otimes h) = nh$ is an isomorphism of Hopf modules, the inverse map is given by $G(m) = m_0S(m_1) \otimes m_2$.

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