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ANALYTIC STUDY ON LINEAR SYSTEMS OF DISTRIBUTED ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

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In this paper we introduce the distributed order fractional differential equations (DOFDE) with respect to the nonnegative density function. We generalize the inertia and characteristics polynomial concepts of pair (A,B) with respect to the nonnegative density function. We also give generalization of the invariant factors of a matrix and some inertia theorems for analyzing the stability of the DOFDE systems.

1. Introduction

The idea of fractional derivative of distributed order was stated by Caputo [4] and was developed by Caputo himself [5] and Bagley and Torvik [2] later. Other researchers used this idea and appeared interesting reviews to describe the related mathematical models of partial fractional differential equation of distributed order.

For example, Diethelm et al. [8] used a numerical technique along with its error analysis to solve the distributed-order differential equation and analyze the

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physical phenomena and engineering problems, see references therein [1, 12, 13].
Based on fractional calculus, many fractional order dynamic systems in applied science and engineering have been gaining increasing attention in research communities [3, 11]. Also the stability results for fractional order differential equations (FODE) have been investigated in recent decades.
For example, Matignon considered the stability results of FODE system in control processing and Deng analyzed the stability of FODE system with multiple time delays[7, 14, 20].
The typical differential equations $\dot{x}(t) = Ax(t) + Bu(t)$, where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$ and $u(t)$ is a control vector, is said to be stabilizable if and only if there exist a linear feedback $u(t) = Yx(t)$, with $Y \in \mathbb{R}^{q\times p}$, such that the system becomes stable, that is, the real parts of all the eigenvalues of $A+BY$ are negative.
The characteristic polynomial of (A,B) is defined the product of the invariant factors of

$$[xI_p - A|B]. \tag{1}$$

The eigenvalues of (A,B) are the roots of the characteristic polynomial of (A,B).
The inertia of pair (A,B) is the triplet $In(A,B) = (\pi(A,B), \nu(A,B), \delta(A,B))$, where $\pi(A,B), \nu(A,B), \delta(A,B)$ are denoted, respectively, the number of roots of the characteristic polynomial of (A,B) with real positive part, real negative part and real part equal to zero. Also (A,B) is stabilizable if and only if the roots of the product of the invariant factors of (4-1) have negative real parts [9, 10].
The main purpose of the present paper is the generalization of the above results in order to study the stabilization of the DOFDE system

$${}^{C}_{\mathrm{do}}D^{\alpha}_{t}x(t) = A\,x(t) + Bu(t), \qquad x(0) = x_0, \qquad 0 < \alpha \le 1, \tag{2}$$

where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$, $u(t)$ is a control vector and

$${}^{C}_{\mathrm{do}}D^{\alpha}_{t}x(t) = \int_0^1 b(\alpha)\, {}^{C}_{so}D^{\alpha}_{t}x(t)d\alpha,$$

is the Caputo fractional derivative operator of distributed order of $x(t)$ respect to order-density function $b(\alpha) \ge 0$. Also ${}^{C}_{so}D^{\alpha}_{t} = \frac{d^\alpha}{dt^\alpha}$ is the Caputo fractional derivative of order α, where $0 < \alpha \le 1$.
Since the solution of the above system is much involved, similar to FODE systems the study of stability for DOFDE is a main task. Our main work in the present paper is study the stability of two classes of DOFDE systems. At first, we introduce a characteristic function of a matrix with respect to the distributed function $B(s)$ where $B(s) = \int_0^1 b(\alpha)s^{\alpha}d\alpha$. Then we establish a general theory based on new inertia concept for analyzing the stability of distributed order fractional differential equations. The concepts and theorems presented in this paper

for DOFDE systems can be considered as generalizations of FODE and ODE systems [6, 7, 15, 16].
In Section 2 we recall some basic definitions of the Caputo fractional derivative operator, the Mittag-Leffler function and their elementary properties used in this paper. Section 3 contain the main definitions and Theorems for checking the stability of DOFDE systems. In section 4, the convergence speed for DOFDE system has been discussed near the asymptotic stable point. Finally the conclusion are given in the last section.

2. Elementary Definitions and Theorems

The fractional derivative of single order of $f(t)$ in the Caputo sense is defined as [11]

$$ {}_{so}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \tag{3}$$

for $m-1 < \alpha \leq m, m \in N, t > 0$.
Now, we generalize the above definition in the fractional derivative of distributed order in the Caputo sense with respect to order-density function $b(\alpha) \geq 0$ as follows

$$ {}_{do}^{C}D_t^{\alpha} f(t) = \int_{m-1}^{m} b(\alpha) \, {}_{so}^{C}D_t^{\alpha} f(t) d\alpha, \tag{4}$$

and the Laplace transform of the Caputo fractional derivative of distributed order satisfies

$$\begin{aligned} \mathcal{L}\{{}_{do}^{C}D_t^{\alpha} f(t)\} &= \int_{m-1}^{m} b(\alpha)[s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} f^{(k)}(0^{+})] d\alpha \\ &= B(s)F(s) - \sum_{k=0}^{m-1} \frac{1}{s^{k+1}} B(s) f^{(k)}(0^{+}), \end{aligned} \tag{5}$$

where

$$ B(s) = \int_{m-1}^{m} b(\alpha) s^{\alpha} d\alpha. $$

3. Stabilization of DOFDE Systems with Control Vector

Stability of linear distributed order fractional systems is one of the main interest in control theory. In [14], Matignon introduced the stability properties for some linear fractional order systems. In this section, we generalize the main stability properties for the linear system of distributed order fractional differential equations (6). In this section we consider the following linear system of distributed fractional order differential equations,

$$ {}_{\text{do}}^{C}D_t^{\alpha}x(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \qquad 0 < \alpha \leq 1, \tag{6} $$

where $x \in \mathbb{R}^n$, the matrix $A \in \mathbb{R}^{n\times n}$ and ${}_{\text{do}}^{C}D_t^{\alpha} = \int_0^1 b(\alpha)\, {}_{\text{so}}^{C}D_t^{\alpha}x(t)d\alpha$ is the Caputo fractional derivative operator of distributed order of $x(t)$ respect to order-density function $b(\alpha) \geq 0$.

Theorem 3.1. *The linear distributed order fractional system (6) is stabilizable if and only if there exist a linear feedback $u(t) = Yx(t)$, with $Y \in \mathbb{R}^{q\times p}$, such that $A + BY$ is stable respect to the order density function $b(\alpha) \geq 0$.*

Proof. By applying the Laplace transform on the above system and using the initial condition and relations, we have

$$ B(s)X(s) \quad = \quad AX(s) + \frac{1}{s}B(s)x(0) + BU(s), \tag{7} $$

where $X(s)$ is the Laplace transform of $x(t)$, $U(s)$ is the Laplace transform of $u(t)$ and $B(s) = \int_0^1 b(\alpha)s^{\alpha}d\alpha$. We can write (7) as follows:

$$ [B(s)I - A]sX(s) \quad = \quad B(s)x(0) + sBU(s) \tag{8} $$

Now, we suppose that there exist a linear feedback $u(t) = Yx(t)$, with $Y \in F^{q\times p}$, such that $A + BY$ is stable respect to the order density function $b(\alpha) \geq 0$. Thus according to the (8), we have

$$ [B(s)I - (A + BY)]sX(s) \quad = \quad B(s)x(0). \tag{9} $$

Thus all roots of $\det[B(s)I - (A + BY)] = 0$ have negative real parts. Then we consider (3-4) in $\Re(s) \geq 0$. In this restricted area, (8) has a unique solution $sX(s)$. Since $\lim_{s\to 0} B(s) = 0$, so by using final-value theorem of Laplace transform, we have

$$ \lim_{t\to\infty} x(t) = \lim_{s\to 0, \Re(s)\geq 0} sX(s) = 0. $$

□

The description of the possible characteristic function of $A + BY$, when Y varies, presented in the Theorem 3-4. In this sense, we need the new concept of invariant factor of a matrix with respect to a distributed function and some preliminaries theorems.

Definition 3.2. Let square matrix A be equivalent to a "Rational Canonical Form", that is, there exists an invertible matrix Q such that

$$Q^{-1}AQ=\begin{pmatrix} K(p_1) & & & \\ & K(p_1) & & \\ & & \ddots & \\ & & & K(p_r) \end{pmatrix},$$

where $K(p)$ is the companion matrix for the monic polynomial

$$p(s)=s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0.$$

We define the functions $p_i(B(s))$ the invariant factors of A with respect to the distributed function $B(s)$, where $B(s)=\int_0^1 b(\alpha)s^{\alpha}d\alpha$, and satisfy $p_i|p_{i+1}$ for $(i=1,2,\ldots,r-1)$.

Lemma 3.3. *Suppose A is the companion matrix of a monic polynomial*

$$p(s)=s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0.$$

Then $det(B(s)I-A)=p(B(s))$.

Now, by using Lemma 3.3, we can show that the characteristic function of matrix A with respect to the distributed function $B(s)$ is the product of the invariant factors of A with respect to the distributed function $B(s)$. It immediately follows the theorem below.

Theorem 3.4. *Let* $A\in R^{n\times n}$ *be a square matrix and* $p_1(B(s)),p_2(B(s)),\ldots,p_r(B(s))$ *the invariant factors of A with respect to the distributed function* $B(s)$*, where* $B(s)=\int_0^1 b(\alpha)s^{\alpha}d\alpha$*. Then the matrix* $B(s)I-A$ *is equivalent to the* $n\times n$*-diagonal matrix with entries* $p_1(B(s)),p_2(B(s)),\ldots,p_r(B(s)),1,\ldots,1,\ldots,1$.

Proof. There is invertible Q such that $Q^{-1}AQ$ is in rational form with companion matrices $K1,..,K_r$ in block-diagonals. Thus

$$Q^{-1}(B(s)I-A)Q=\begin{pmatrix} B(s)I-K_1 & & & \\ & B(s)I-K_2 & & \\ & & \ddots & \\ & & & B(s)I-K_r \end{pmatrix},$$

Now, according to the Lemma 3.3, $B(s)I-K_i$ is equivalent to a diagonal matrix with entries $p_iB(s),1,,1$. On the other hand there are invertible matrices M and N such that

$$M(B(s)I-K_i)N=\begin{pmatrix} p_i(B(s)) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

rearrange to get the desired diagonal matrix.

□

Remark 3.5. Suppose $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $B(s)$ is the distributed function with respect to the density function $b(\alpha) \geq 0$. Notice that

$$\begin{pmatrix} I_n & -B \\ 0 & I_m \end{pmatrix} \begin{pmatrix} B(s)I_n - A & B \\ Y & I_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -Y & I_m \end{pmatrix} = \begin{pmatrix} B(s)I_n - (A+BY) & 0 \\ 0 & I_m \end{pmatrix},$$

hence

$$\begin{pmatrix} B(s)I_n - A & B \\ Y & I_m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B(s)I_n - (A+BY) & 0 \\ 0 & I_m \end{pmatrix},$$

are equivalent characteristic function matrices, and so the invariant factors of

$$\begin{pmatrix} B(s)I_n - A & B \\ Y & I_m \end{pmatrix},$$

with respect to the distributed function $B(s)$, are those of $A+BY$ and m invariant factors equal to 1.

Now, we are ready to generalize the concept of inertia of the pair (A,B) with respect to the distributed function $B(s)$ in order to study stabilization of the distributed order fractional system (6).

Definition 3.6. Let $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $B(s)$ is the distributed function with respect to the density function $b(\alpha) \geq 0$. The characteristic function of (A,B) with respect to the distributed function $B(s)$ is the product the invariant factors of (10). The eigenvalues of (A,B) with respect to the distributed function $B(s)$ are the roots of characteristic function of (A,B).

Definition 3.7. Let $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $B(s)$ is the distributed function with respect to the density function $b(\alpha) \geq 0$. The inertia of pair (A,B) with respect to the distributed function $B(s)$ is the triplet

$$In_{B(s)}(A,B) = (\pi_{B(s)}(A,B), \nu_{B(s)}(A,B), \delta_{B(s)}(A,B))$$

where $\pi_{B(s)}(A,B)$, $\nu_{B(s)}(A,B)$ and $\delta_{B(s)}(A,B)$ are, respectively, the number of roots of characteristic function of (A,B) with positive, negative and zero real parts.

Theorem 3.8. *The linear distributed order fractional system (6) is stabilizable if and only if any of the following equivalent conditions holds:*

1. *The pair (A,B) is stabilizable with respect to the distributed function $B(s)$ where $B(s) = \int_0^1 b(\alpha)s^\alpha d\alpha$.*
2. $\pi_{B(s)}(A,B) = \delta_{B(s)}(A,B) = 0$.
3. *All roots s of the characteristic function of pair (A,B) with respect to the distributed function $B(s)$ satisfy $|\arg(s)| > \frac{\pi}{2}$.*

Proof. According to the Theorem 3.1 and above definitions, the proof can be easily obtained. □

4. The Convergence Speed of DOFDE System

In this section we consider the convergence speed near stable point for a DOFDE system. For this purpose, we use the final value theorem for the Laplace transform for the function $\frac{x_1(t)}{x_2(t)}$

$$\lim_{t\to\infty} \frac{x_1(t)}{x_2(t)} = \lim_{s\to 0} \frac{sX_1(s)}{sX_2(s)}. \tag{10}$$

Now, if we set $b(\alpha) = \delta(\alpha - \alpha_1)$ in the following homogeneous DOFDE system

$${}_{\mathrm{do}}^{C}D_t^{\alpha} x_1(t) = A\,x_1(t), \qquad x_1(0) = x_0, \qquad 0 < \alpha \leq 1, \tag{11}$$

then, we arrive at the following fractional differential system

$${}^{C}D_t^{\alpha_1} x_1(t) = A\,x_1(t), \qquad x(0) = x_0. \tag{12}$$

Also, if we set $b(\alpha) = \delta(\alpha - \beta_1)$ in a sperate homogeneous DOFDE system, we get

$${}^{C}D_t^{\beta_1} x_2(t) = A\,x_2(t), \qquad x(0) = x_0. \tag{13}$$

It is obvious that, by applying the final value theorem (10), we obtain the value of limit as

$$\lim_{t\to\infty} \frac{x_1(t)}{x_2(t)} = \lim_{s\to 0} \frac{\frac{s^{\alpha_1}}{s^{\alpha_1}I-A}}{\frac{s^{\beta_1}}{s^{\beta_1}I-A}}, \tag{14}$$

which for $\alpha_1 < \beta_1$, we deduce that

$$\lim_{s\to 0} \frac{s^{\alpha_1}(s^{\beta_1}I - A)}{s^{\beta_1}(s^{\alpha_1}I - A)} = \infty. \tag{15}$$

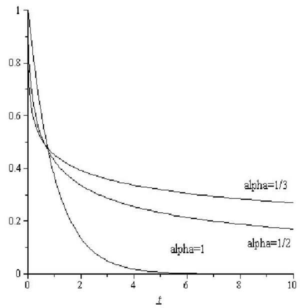

Figure 1: The function $x(t) = E_\alpha(-t^\alpha)$, $\alpha = \frac{1}{3}, \frac{1}{2}, 1$.

The above result shows that in the region of stability, the solution of system (12) decreases much slower than the solution of system (13) near the stable point $x = 0$. In similar way, we can show the solution of the following system

$$^{C}D_t^{\alpha} x(t) = A\,x(t), \qquad x(0) = x_0, \qquad 0 < \alpha \leq 1, \tag{16}$$

decreases much slower than the solution of system $x' = Ax$.
In general, next theorem compares the convergence speed of two DOFDE system by setting the density function $b(\alpha) = \sum_{i=0}^{n} \delta(\alpha - \alpha_i)$. The proof can be easily written by the stated final value theorem.

Theorem 4.1. *For $\alpha_1 < \beta_1$, the convergence speed of the solution of fractional differential system near $x = 0$*

$$\sum_{i=0}^{n} {}^{C}D_t^{\alpha_i} x(t) = A\,x(t), \qquad x(0) = x_0, \qquad 0 < \alpha_n < \cdots < \alpha_2 < \alpha_1 \leq 1, \tag{17}$$

much slower than the solution of fractional differential system

$$\sum_{i=0}^{n} {}^{C}D_t^{\beta_i} x(t) = A\,x(t), \qquad x(0) = x_0, \qquad 0 < \beta_n < \cdots < \beta_2 < \beta_1 \leq 1. \tag{18}$$

Example 4.2. As a simple example for $A_{1\times 1} = -1$, we consider the following fractional differential system

$$^{C}D_t^{\alpha} x(t) = -x(t), \qquad x(0) = 1. \tag{19}$$

If we set $\alpha = \frac{1}{3}, \frac{1}{2}, 1$ and plot the solution $x(t) = E_\alpha(-t^\alpha)$ for these values, we see that the exponential function decreases much faster than other function near origin. See Figure1.

5. Conclusion and Future Works

In this paper we introduced *the distributed order fractional differential equations* with respect to a nonnegative density function.
Then the asymptotical stability for such systems has been investigated. Based on the main theorem (Theorem 3.1) in this paper, several interesting stability criterions were derived. Also, the convergence speed for these system was stated.

For future works, our attention will be focused on the generalization of the numerical methods for computing the eigenvalues of a matrix with respect to the distribute function. In this way the algorithms described in [17–19], which have been used for computing the eigenvalues of a matrix, may be effective.

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