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A NOTE ON MONOTONE SOLUTIONS FOR A NONCONVEX SECOND-ORDER FUNCTIONAL DIFFERENTIAL INCLUSION

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The existence of monotone solutions for a second-order functional differential inclusion with Carathéodory perturbation is obtained in the case when the multifunction that defines the inclusion is upper semicontinuous compact valued and contained in the Fréchet subdifferential of a ϕ-convex function of order two.

1. Introduction

Functional differential inclusions, known also as differential inclusions with memory, express the fact that the velocity of the system depends not only on the state of the system at a given instant but depends upon the history of the trajectory until this instant. The class of differential inclusions with memory encompasses a large variety of differential inclusions and control systems. In particular, this class covers the differential inclusions, the differential inclusions with delay and the Volterra inclusions. A detailed discussion on this topic may be found in [1].

Let $\mathbb{R}^n$ be the n-dimensional euclidean space with the norm $||.||$ and the scalar product $\langle .,. \rangle$. Let $\sigma > 0$ and $\mathcal{C}_\sigma := \mathcal{C}([-\sigma, 0], \mathbb{R}^n)$ the Banach space of

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continuous functions from $[-\sigma,0]$ into $\mathbb{R}^n$ with the norm given by $||x(.)||_\sigma := \sup\{||x(t)||; t \in [-\sigma,0]\}$. For each $t \in [0,\tau]$, we define the operator $T(t) : \mathcal{C}([-\sigma,\tau],\mathbb{R}^n) \to \mathcal{C}_\sigma$ as follows: $(T(t)x)(s) := x(t+s)$, $s \in [-\sigma,0]$. $T(t)x$ represents the history of the state from the time $t-\sigma$ to the present time t.

Let $K \subset \mathbb{R}^n$ be a closed set, $\Omega \subset \mathbb{R}^n$ an open set and P a lower semicontinuous multifunction from K into the family of all nonempty subsets of K with closed graph satisfying the following two conditions

$$\forall\, x \in K, \quad x \in P(x),$$

$$\forall\, x,y \in K, y \in P(x) \quad \Rightarrow \quad P(y) \subseteq P(x).$$

Under these conditions, a preorder (reflexive and transitive relation) on K is defined by $x \preceq y$ iff $y \in P(x)$.

Let $K_0 := \{\varphi \in \mathcal{C}_\sigma; \varphi(0) \in K\}$, let F be a multifunction defined from $K_0 \times \Omega$ into the family of nonempty compact subsets of $\mathbb{R}^n$, let $f : \mathbb{R} \times K \times \Omega \to \mathbb{R}^n$ be a Carathéodory function and $(\varphi_0, y_0) \in K_0 \times \Omega$ be given that define the second-order functional differential inclusion

$$\begin{aligned}
&x'' \in F(T(t)x,x') + f(t,x,x') \quad a.e.\ ([0,\tau])\\
&x(t) = \varphi_0(t) \quad \forall t \in [-\sigma,0], \quad x'(0) = y_0,\\
&x(t) \in P(x(t)) \subset K \quad \forall t \in [0,\tau], \quad x(s) \preceq x(t) \quad \forall\, 0 \le s \le t \le \tau.
\end{aligned} \tag{1.1}$$

The present note is motivated by a recent paper of Ibrahim and Aladsani [10] where the existence of solutions of problem (1.1) is provided when F is an upper semicontinuous multifunction contained in the subdifferential of a proper convex function V.

The aim of the present paper is to relax the convexity assumption on the function V that appears in [10], in the sense that we assume that F is contained in the Fréchet subdifferential of a ϕ-convex function of order two. Since the class of proper convex functions is strictly contained into the class of ϕ- convex functions of order two, our result generalizes the one in [10].

On the other hand, the result in the present paper is an extension of the result in [5] obtained for differential inclusions without memory and without constraints. At the same time, our result is an extension of the result in [7] obtained in the case of differential inclusions without memory, $P(x) \equiv K$ and without Carathéodory perturbation. Finally, our main result generalizes Theorem 3.1 in [6] where a similar result is obtained for monotone solutions of a second-order functional differential inclusion without Carathéodory perturbation.

One may consider that the result in the present paper extends and unifies all the results quoted above. The proof follows the general ideas in [5] and [10].

For the motivation, discussion on existence results in the literature and a consistent bibliography on this topic we refer to [10] and the references therein.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2. Preliminaries

We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all subsets of $\mathbb{R}^n$, by $cl(A)$ we denote the closure of the set $A \subset R^n$ and by $co(A)$ we denote the convex hull of A. For $x \in \mathbb{R}^n$ and $r > 0$ let $B(x,r) := \{y \in \mathbb{R}^n; ||y-x|| < r\}$ be the open ball centered at x with radius r, and let $\overline{B}(x,r)$ be its closure. For $\varphi \in \mathcal{C}_\sigma$ let $B_\sigma(\varphi,r) := \{\psi \in \mathcal{C}_\sigma; ||\psi-\varphi||_\sigma < r\}$ and $\overline{B}_\sigma(\varphi,r) := \{\psi \in \mathcal{C}_\sigma; ||\psi-\varphi||_\sigma \leq r\}$.

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V : \Omega \to \mathbb{R} \cup \{+\infty\}$ be a function with domain $D(V) = \{x \in \mathbb{R}^n;\ V(x) < +\infty\}$.

Definition 2.1. The multifunction $\partial_F V : \Omega \to \mathcal{P}(\mathbb{R}^n)$, defined as

$$\partial_F V(x) = \{\alpha \in \mathbb{R}^n, \liminf_{y\to x} \frac{V(y)-V(x)-<\alpha, y-x>}{||y-x||} \geq 0\} \text{ if} V(x) < +\infty$$

and $\partial_F V(x) = \emptyset$ if $V(x) = +\infty$ is called the *Fréchet subdifferential* of V.

We also put $D(\partial_F V) = \{x \in \mathbb{R}^n; \partial_F V(x) \neq \emptyset\}$.

According to [9] the values of $\partial_F V(.)$ are closed and convex.

Definition 2.2. Let $V : \Omega \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. We say that V is a *ϕ-convex of order 2* if there exists a continuous map $\phi_V : (D(V))^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ such that for every $x, y \in D(\partial_F V)$ and every $\alpha \in \partial_F V(x)$ we have

$$V(y) \geq V(x) + <\alpha, x-y> - \phi_V(x,y,V(x),V(y))(1+||\alpha||^2)||x-y||^2.$$

In [4], [9] there are several examples and properties of such maps. For example, according to [4], if $K \subset \mathbb{R}^2$ is a closed and bounded domain, whose boundary is a C^2 regular Jordan curve, the indicator function of K

$$V(x) = I_K(x) = \begin{cases} 0, & \text{if} \quad x \in K \\ +\infty, & \text{otherwise} \end{cases}$$

is ϕ- convex of order 2.

The second-order contingent set of a closed subset $C \subset \mathbb{R}^n$ at $(x,y) \in C \times \mathbb{R}^n$ is defined by:

$$T_C^2(x,y) = \{v \in \mathbb{R}^n; \quad \liminf_{h\to 0+} \frac{d(x+hy+\frac{h^2}{2}v, C)}{h^2} = 0\}.$$

For properties of second-order contingent set see, for example, [2].

A multifunction $F : K_0 \to \mathcal{P}(\mathbb{R}^n)$ is upper semicontinuous at $(\varphi, y) \in K_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(\psi, z) \subset F(\varphi, y) + B(0, \varepsilon), \quad \forall (\psi, z) \in B_\sigma(\varphi, \delta) \times B(y, \delta).$$

We recall that a continuous function $x(.) : [-\sigma, \tau] \to \mathbb{R}^n$ is said to be a solution of (1.1) if $x(.)$ is absolutely continuous on $[0, \tau]$ with absolutely continuous derivative $x'(.)$, $T(t)x \in K_0, \forall t \in [0, \tau]$, $x'(t) \in \Omega$ a.e. $[0, \tau]$ and (1.1) is satisfied.

Hypothesis. Let $T > 0$, $K \subset \mathbb{R}^n$ be a nonempty closed set, $\Omega \subset \mathbb{R}^n$ be an open set and $P : K \to \mathcal{P}(K)$ a lower semicontinuous multifunction with nonempty closed values satisfying $\forall\, x \in K, \quad x \in P(x)$ and $\forall\, x, y \in K, y \in P(x) \quad \Rightarrow \quad P(y) \subseteq P(x)$. Let $K_0 := \{\varphi \in \mathcal{C}_\sigma; \varphi(0) \in K\}$, $F : K_0 \times \Omega \to \mathcal{P}(\mathbb{R}^n)$ be upper semicontinuous with nonempty compact values and $f : \mathbb{R} \times K \times \Omega \to \mathbb{R}^n$ that satisfy

i) For all $(t, \varphi, y) \in [0, T] \times K_0 \times \Omega$, there exists $z \in F(\varphi, y)$ such that

$$\liminf_{h \to 0+} \frac{1}{h^2} d(\varphi(0) + hy + \frac{h^2}{2} z + \int_t^{t+h} (t + h - s) f(s, \varphi(0), y) ds, P(\varphi(0))) = 0,$$

ii) f is a Carathéodory function, i.e., for each $(x, y) \in \Omega$, $f(., x, y)$ is measurable; for all $t \in \mathbb{R}$, $f(t, ., .)$ is continuous and there exists $m(.) \in L^2(\mathbb{R}, \mathbb{R})$ such that $||f(t, x, y)|| \le m(t)\ \forall (t, x, y) \in \mathbb{R} \times K \times \Omega$.

The next technical result is proved in [10] (Lemma 3.1).

Lemma 2.3. *Assume that Hypothesis is satisfied, let* $(\varphi_0, y_0) \in K_0 \times \Omega$ *and there exist* $r, M \ge 0$ *such that* $\sup\{||z||; \quad z \in F(\psi, y)\} \le M\ \forall (\psi, y) \in (K_0 \cap B_\sigma(\varphi_0, r)) \times \overline{B}(y_0, r)$.

Then there exists $\tau > 0$ *such that for any* $m \in \mathbb{N}$ *there exist* $l_m \in \mathbb{N}$*, a set of points* $\{t_0^m = 0 < t_1^m < \ldots < t_{l_m - 1}^m \le \tau < t_{l_m}^m\}$*, the points* $x_p^m, y_p^m, z_p^m \in \mathbb{R}^n$*,* $p = 0, 1, \ldots, l_m - 1$ *with* $x_0^m = \varphi_0(0)$ *and* $y_0^m = y_0$*, a continuous function* $x_m(.) : [-\sigma, \tau] \to \mathbb{R}^n$ *with* $x_m(t) = \varphi_0(t)\ \forall t \in [-\sigma, 0]$ *and with the following properties for* $p = 0, 1, \ldots, l_m - 1$

(i) $h_{p+1}^m := t_{p+1}^m - t_p^m < \frac{1}{m}$,

(ii) $z_p^m = u_p^m + w_p^m$, *with* $u_p^m \in F(T(t_p^m) x_m, y_p^m)$ *and* $w_p^m \in B(0, \frac{1}{m})$,

(iii) $x_m(t) = x_p^m + (t - t_p^m) y_p^m + \frac{1}{2}(t - t_p^m)^2 z_p^m + \int_{t_p^m}^t (t - s) f(s, x_p^m, y_p^m) ds$,
$t \in [t_p^m, t_{p+1}^m]$,

(iv) $x_{p+1}^m = x_p^m + h_{p+1}^m y_p^m + \frac{1}{2}(h_{p+1}^m)^2 z_p^m + \int_{t_p^m}^{t_p^m + h_{p+1}^m} (t_p^m + h_{p+1}^m - s) f(s, x_p^m, y_p^m) ds$,

(v) $x_{p+1}^m \in P(x_p^m) \cap B(\varphi_0(0), r) \subset K$, $y_{p+1}^m = y_p^m + h_{p+1}^m z_p^m \in \overline{B}(y_0, r) \subset \Omega$,

(vi) $T(t_{p+1}^m) x_m \in B_\sigma(\varphi_0, r) \cap K_0$.

We note that in order to prove Lemma 2.3 is not necessary to assume that P is lower semicontinuous as in Hypothesis.

On the other hand, in Lemma 3.1 in [10] it is used the assumption that there exists a proper lower semicontinuous convex function $V : \mathbb{R}^n \to \mathbb{R}$ with $F(\varphi, y) \subseteq \partial V(y)$ $\forall(\varphi, y) \in K_0 \times \Omega$ in order to deduce (via the fact that the set-valued map $\partial V(.)$ is bounded) that

$$\sup\{||z||; \quad z \in F(\psi, y)\} \leq M \quad \forall(\psi, y) \in (K_0 \cap B_\sigma(\varphi_0, r)) \times \overline{B}(y_0, r). \quad (2.1)$$

In our statement of Lemma 2.3 we assume directly that there exist $r, M \geq 0$ such that (2.1) is satisfied.

3. The main result

We are now able to prove our main result.

Theorem 3.1. *Assume that Hypothesis is satisfied. In addition, assume that K_0 is locally compact, $P(.)$ has closed graph and there exists a proper lower semicontinuous function ϕ- convex of order two $V : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ with $F(\varphi, y) \subseteq \partial_F V(y)$ $\forall(\varphi, y) \in K_0 \times \Omega$.*

Then for any $(\varphi_0, y_0) \in K_0 \times \Omega$ there exists $\tau > 0$ and $x(.) : [0, \tau] \to K$ a solution to problem (1.1).

Proof. Let $(\varphi_0, y_0) \in K_0 \times \Omega$. Since K_0 is locally compact there exists $r > 0$ such that $K_0 \cap B_\sigma(\varphi_0, r)$ is compact and $\overline{B}(y_0, r) \subset \Omega$. Using the fact that $F(.,.)$ is upper semicontinuous with compact values, by Proposition 1.1.3 in [1], $F((K_0 \cap B_\sigma(\varphi_0, r)) \times \overline{B}(y_0, r))$ is compact. Take $M := \sup\{||z||; z \in F(\psi, y); (\psi, y) \in (K_0 \cap B_\sigma(\varphi_0, r)) \times \overline{B}(y_0, r)\}$.

Let ϕ_V the continuous function appearing in Definition 2.2. Since $V(.)$ is continuous on $D(V)$ (e.g. [9]), by possibly decreasing r one can assume that for all $y \in B_r(y_0) \cap D(V)$, $|V(y) - V(y_0)| \leq 1$. Set $S := \sup\{\phi_V(y_1, y_2, z_1, z_2);$
$y_i \in \overline{B}_r(y_0), z_i \in [V(y_0) - 1, V(y_0) + 1], i = 1, 2\}$.

One may apply Lemma 2.3 and according to the definition of x_m for all $m \geq 1$, all $p = 0, 1, ..., l_m - 1$ and all $t \in [t_p^m, t_{p+1}^m]$ we have

$$x'_m(t) = y_p^m + (t - t_p^m) z_p^m + \int_{t_p^m}^t f(s, x_p^m, y_p^m) ds,$$

$$x''_m(t) = z_p^m + f(t, x_p^m, y_p^m).$$

Therefore, from (ii) and (v) of Lemma 2.3 one has

$$||x'_m(t)|| \leq ||y_0|| + \frac{3r}{4} \quad \forall t \in [0, \tau], \quad (3.1)$$

$$||x_m''(t)|| \le M + 1 + m(t) \quad a.e. \ ([0,\tau]). \tag{3.2}$$

Then the sequences $\{x_m\}$ and $\{x_m'\}$ are equicontinuous in $C([0,\tau],\mathbb{R}^n)$. Applying Arzela-Ascoli theorem, there exists a subsequence (again denoted) $\{x_m\}$ and an absolutely continuous function $x(.) : [0,\tau] \to \mathbb{R}^n$ with absolutely continuous derivative $x'(.)$ such that $x_m(.)$ converges uniformly to $x(.)$ on $[0,\tau]$, $x_m'(.)$ converges uniformly to $x'(.)$ on $[0,\tau]$ and $x_m''(.)$ converges weakly to $x''(.)$ in $L^2([0,\tau],\mathbb{R}^n)$. Furthermore, since all the functions $x_m(.)$ are equal with $\varphi_0(.)$ on $[-\sigma,0]$, then $x_m(.)$ converges uniformly to $x(.)$ on $[-\sigma,\tau]$, where $x_m = \varphi_0$ on $[-\sigma,0]$.

For each $t \in [0,\tau]$ and each $m \ge 1$ let $\delta_m(t) = t_p^m$, $\theta_m(t) = t_{p+1}^m$ if $t \in (t_p^m, t_{p+1}^m]$ and $\delta_m(0) = \theta_m(0) = 0$. If $t \in (t_p^m, t_{p+1}^m]$ we get

$$x_m''(t) = z_p^m + f(t, x_p^m, y_p^m) \in F(T(t_p^m)x_m, y_p^m) + B(0, \frac{1}{m}) + f(t, x_p^m, y_p^m)$$

and for all $m \ge 1$ and a.e. on $[0,\tau]$

$$x_m''(t) \in F(T(\delta_m(t))x_m, x_m'(\delta_m(t))) + B(0, \frac{1}{m}) + f(t, x_p^m, y_p^m).$$

Also for all $m \ge 1$ and a.e. on $[0,\tau]$ $T(\theta_m(t))x_m \in B_\sigma(\varphi_0, r) \cap K_0$, $x_m(t) \in B(\varphi_0(0), r)$, $x_m(\theta_m(t)) \in P(x_m(\delta_m(t)) \subset K$.

As in the proof of Theorem 3.2 in [10] $\forall t \in [0,\tau]$, $\lim_{m\to\infty} T(\theta_m(t))x_m = T(t)x$ in $\mathcal{C}_\sigma$ and $\lim_{m\to\infty} x_m'(\delta_m(t)) = x'(t)$.

Taking into account the upper semicontinuity of $F(.,.)$, Theorem 1.4.1 in [1] and (3.1) one deduces

$$x''(t) \in \mathrm{co}F(T(t)x, x'(t)) + f(t, x(t), x'(t)) \quad a.e. \ ([0,\tau]),$$

which implies

$$x''(t) - f(t, x(t), x'(t)) \subset \partial_F V(x'(t)) \quad a.e. \ ([0,\tau]). \tag{3.3}$$

The next step of the proof shows that $x_m''(.)$ has a subsequence that converges pointwise to $x''(.)$. From property (ii) of Lemma 2.3

$$z_p^m - w_p^m \in F(T(t_p^m)x_m, y_p^m) \subset \partial_F V(y_p^m) = \partial_F V(x_m'(t_p^m))$$

for $p = 0, 1, 2, ..., l_m - 2$.

From the definition of the Fréchet subdifferential for $p = 0, 1, 2, ..., l_m - 2$ one has

$$
\begin{aligned}
&V(x'_m(t^m_{p+1})) - V(x'_m(t^m_p)) \geq \\
&\langle z^m_p - w^m_p, x'_m(t^m_{p+1}) - x'_m(t^m_p)\rangle - \phi_V(x'_m(t^m_{p+1}), x'_m(t^m_p), V(x'_m(t^m_{p+1})), V(x'_m(t^m_p))) \\
&\cdot(1 + ||z^m_p - w^m_p||^2)||x'_m(t^m_{p+1}) - x'_m(t^m_p)||^2 \geq \\
&\langle x''_m(t) - f(t, x_m(t^m_p), x'_m(t^m_p)) - w^m_p, x'_m(t^m_{p+1}) - x'_m(t^m_p)\rangle \\
&-\phi_V(x'_m(t^m_{p+1}), x'_m(t^m_p), V(x'_m(t^m_{p+1})), V(x'_m(t^m_p)))(1 + ||z^m_p - w^m_p||^2) \\
&\cdot||x'_m(t^m_{p+1}) - x'_m(t^m_p)||^2 = \\
&\int_{t^m_p}^{t^m_{p+1}} ||x''_m(t)||^2 dt - \int_{t^m_p}^{t^m_{p+1}} \langle f(t, x_m(t^m_p), x'_m(t^m_p)), x''_m(t)\rangle dt - \int_{t^m_p}^{t^m_{p+1}} \langle x''_m(t), w^m_p\rangle dt \\
&-\phi_V(x'_m(t^m_{p+1}), x'_m(t^m_p), V(x'_m(t^m_{p+1})), V(x'_m(t^m_p)))(1 + ||z^m_p - w^m_p||^2) \\
&\cdot||x'_m(t^m_{p+1}) - x'_m(t^m_p)||^2
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
&V(x'_m(\tau)) - V(x'_m(t^m_{l_m-1})) \geq \\
&\langle z^m_p - w^m_p, x'_m(\tau) - x'_m(t^m_{l_m-1})\rangle - \phi_V(x'_m(\tau), x'_m(t^m_{l_m-1}), V(x'_m(\tau)), V(x'_m(t^m_{l_m-1}))) \\
&\cdot(1 + ||z^m_p - w^m_p||^2)||x'_m(\tau) - x'_m(t^m_{l_m-1})||^2 \geq \\
&\langle x''_m(t) - f(t, x_m(t^m_{l_m-1}), x'_m(t^m_{l_m-1})) - w^m_p, x'_m(\tau) - x'_m(t^m_{l_m-1})\rangle \\
&-\phi_V(x'_m(\tau), x'_m(t^m_{l_m-1}), V(x'_m(\tau)), V(x'_m(t^m_{l_m-1})))(1 + ||z^m_p - w^m_p||^2) \\
&\cdot||x'_m(\tau) - x'_m(t^m_{l_m-1})||^2 = \\
&\int_{t^m_{l_m-1}}^{\tau} ||x''_m(t)||^2 dt - \int_{t^m_{l_m-1}}^{\tau} \langle f(t, x_m(t^m_{l_m-1}), x'_m(t^m_{l_m-1})), x''_m(t)\rangle dt \\
&-\int_{t^m_{l_m-1}}^{\tau} \langle x''_m(t), w^m_p\rangle dt - \phi_V(x'_m(\tau), x'_m(t^m_{l_m-1}), V(x'_m(\tau)), V(x'_m(t^m_{l_m-1}))) \\
&\cdot(1 + ||z^m_p - w^m_p||^2)||x'_m(\tau) - x'_m(t^m_{l_m-1})||^2
\end{aligned}
\tag{3.5}
$$

By adding the $l_m - 1$ inequalities from (3.4) and the inequality from (3.5), one has

$$V(x'_m(\tau)) - V(x'_m(0)) \geq \int_0^\tau ||x''_m(t)||^2 dt - \sum_{p=0}^{l_m-2} \int_{t_p^m}^{t_{p+1}^m} < f(t, x_m(t_p^m), x'_m(t_p^m)),$$
$$x''_m(t) > dt - \int_{t_{l_m-1}^m}^\tau < f(t, x_m(t_{l_m-1}^m), x'_m(t_{l_m-1}^m)), x''_m(t) > dt + \alpha(m) + \beta(m),$$

where

$$\alpha(m) = -\sum_{p=0}^{l_m-2} < w_p^m, \int_{t_p^m}^{t_{p+1}^m} x''_m(t)dt > - < w_{l_m-1}^m, \int_{t_{l_m-1}^m}^\tau x''_m(t)dt >,$$

$$\beta(m) = -\sum_{p=0}^{l_m-2} \phi_V(x'_m(t_{p+1}^m), x'_m(t_p^m), V(x'_m(t_{p+1}^m)), V(x'_m(t_p^m)))$$
$$\cdot (1 + ||z_p^m - w_p^m||^2)||x'_m(t_{p+1}^m) - x_k(t_p^m)||^2 - \phi_V(x'_m(\tau), x'_m(t_{l_m-1}^m), V(x'_m(\tau)),$$
$$V(x'_m(t_{l_m-1}^m)))(1 + ||z_{l_m-1}^m - w_{l_m-1}^m||^2)||x'_m(\tau) - x'_m(t_{l_m-1}^m)||^2.$$

As in the proof of Theorem 3.2 in [10] , since f is a Carathéodory map we have that

$$\lim_{m\to\infty} [\sum_{p=0}^{l_m-2} \int_{t_p^m}^{t_{p+1}^m} \langle f(t, x_m(t_p^m), x'_m(t_p^m)), x''_m(t)\rangle dt$$
$$+ \int_{t_{l_m-1}^m}^\tau \langle f(t, x_m(t_{l_m-1}^m), x'_m(t_{l_m-1}^m)), x''_m(t)\rangle dt] =$$
$$\int_0^\tau \langle f(t, x(t), x'(t)), x''(t)\rangle dt.$$

One may write

$$|\alpha(m)| \leq$$
$$(M+1)\left[\sum_{p=0}^{l_m-2} ||w_p^m||(t_{p+1}^m - t_p^m) + ||w_{l_m-1}^m||(\tau - t_{l_m-1}^m)\right] \leq \frac{\tau(M+1)}{m},$$

$$|\beta(m)| \leq S(1+M^2)\left[\sum_{p=0}^{l_m-2} ||\int_{t_p^m}^{t_{p+1}^m} x''_m(t)dt||^2 + ||\int_{t_{l_m-1}^m}^\tau x''_m(t)dt||^2\right] \leq$$
$$S(1+M^2)\left[\sum_{p=0}^{l_m-2} \frac{1}{m}\int_{t_p^m}^{t_{p+1}^m} ||x''_m(t)||^2 dt + \frac{1}{m}\int_{t_{l_m-1}^m}^\tau ||x''_m(t)||^2 dt\right] \leq$$
$$\frac{1}{m}S(1+M^2)\int_0^\tau ||x''_m(t)||^2 dt \leq \frac{1}{m}S(1+M^2)\tau(M+1)^2.$$

Therefore, $\lim_{m\to\infty}\alpha(m) = \lim_{m\to\infty}\beta(m) = 0$ and thus

$$V(x'_m(\tau)) - V(y_0) \geq \limsup_{m\to\infty} \int_0^\tau ||x''_m(t)||^2 dt - \int_0^\tau < f(t,x(t),x'(t)),x''(t) > dt. \tag{3.6}$$

From (3.3) and Theorem 2.2 in [4] we deduce that there exists $\tau_1 > 0$ such that the mapping $t \to V(x'(t))$ is absolutely continuous on $[0,\min\{\tau,\tau_1\}]$ and

$$(V(x'(t)))' =< x''(t), x''(t) - f(t,x(t),x'(t)) > \quad a.e.\ ([0,\min\{\tau,\tau_1\}]).$$

Without loss of generality we may assume that $\tau = \min\{\tau,\tau_1\}$. Hence,

$$V(x'(\tau)) - V(x'(0)) = \int_0^\tau ||x''(t)||^2 dt - \int_0^\tau < f(t,x(t),x'(t)),x''(t) > dt;$$

therefore from (3.6) one has

$$\int_0^\tau ||x''(t)||^2 dt \geq \limsup_{m\to\infty} \int_0^\tau ||x''_m(t)||^2 dt$$

and, since $x''_m(.)$ converges weakly in $L^2([0,\tau],\mathbb{R}^m)$ to $x''(.)$, by the lower semicontinuity of the norm in $L^2([0,\tau],\mathbb{R}^n)$ (e.g., Proposition III 30 in [3]), we obtain that $x''_m(.)$ converges strongly in $L^2([0,\tau],\mathbb{R}^m)$ to $x''(.)$, hence a subsequence (again denote by) $x''_m(.)$ converges pointwise a.e. to $x''(.)$.

Let $t \in [0,\tau]$. By the above construction, there exists p such that $t \in [t^m_p, t^m_{p+1})$ and $\lim_{m\to\infty} t^m_p = t$. Since for all $m \geq 1$ and a.e. $t \in [0,\tau]$

$$x''_m(t) \in F(T(t^m_p)x_m, y^m_p) + B(0,\frac{1}{m}) + f(t, x^m_p, y^m_p),$$

for each $m \in \mathbb{N}$ we have

$$x''_m(t) - f(t, x_m(t^m_p), x'_m(t^m_p)) \in F(T(t^m_p)x_m, x'_m(t^m_p)) + B(0,\frac{1}{m}).$$

From the fact that f is a Catrathéodory function, F is upper semicontinuous, $\lim_{m\to\infty} T(\theta_m(t))x_m = T(t)x$ in $\mathcal{C}_\sigma$ and $\lim_{m\to\infty} x'_m(\delta_m(t)) = x'(t)$ we infer that

$$x''(t) \in F(T(t)x, x'(t)) + f(t,x(t),x'(t)) \quad a.e.\ ([0,\tau]).$$

It remains to prove that

$$(x(t),x'(t)) \in K \times \Omega, \quad \forall t \in [0,\tau],$$

$$x(s) \in P(x(t)) \quad \forall t,s \in [0,\tau], \quad t \leq s.$$

First, from property (iii) of Lemma 2.3 it follows that $x_m(\delta_m(t)) \in \overline{B}(\varphi_0(0), r)$ and $x'_m(\delta_m(t)) \in \overline{B}(y_0, r) \cap \Omega$. Since $\lim_{m\to\infty} x_m(\delta_m(t)) = x(t)$ and $\lim_{m\to\infty} x'_m(\delta_m(t)) = x'(t)$ then $x(t) \in \overline{B}(\varphi_0(0), r)$ and $x'(t) \in \overline{B}(y_0, r) \cap \Omega$.

Secondly, let $t, s \in [0, \tau], \quad t \leq s$. For m large enough we can find $p, q \in \{0, 1, 2, ..., l_m - 2\}$ such that $p > q$, $t \in [t^m_q, t^m_{q+1}]$, $s \in [t^m_p, t^m_{p+1}]$. If $j = p - q$, then property (v) of Lemma 2.3 gives

$$P(x_m(t^m_p)) \subseteq P(x_m(t^m_{p-1})) \subseteq P(x_m(t^m_{p-2})) \subseteq ... \subseteq P(x_m(t^m_q)).$$

This implies $P(x_m(\delta_m(s))) \subseteq P(x_m(\delta_m(t)))$ and since $x_m(\delta_m(s)) \in P(x_m(\delta_m(s)))$ it follows $x_m(\delta_m(s)) \in P(x_m(\delta_m(t)))$ which completes the proof. □

Remark 3.2. If $V(.) : \mathbb{R}^n \to \mathbb{R}$ is a proper lower semicontinuous convex function then (e.g. [9]) $\partial_F V(x) = \partial V(x)$, where $\partial V(.)$ is the subdifferential in the sense of convex analysis of $V(.)$, and Theorem 3.1 yields the main result in [10]. On the other hand, if $P(x) \equiv K$ and $T(t) = I$ then Theorem 3.1 yields the main result in [5]. At the same time, if $f \equiv 0$ then the tangential condition i) in Hypothesis becomes: for all $(t, \varphi, y) \in [0, T] \times K_0 \times \Omega$, there exists $z \in F(\varphi, y)$ such that

$$\liminf_{h\to 0+} \frac{1}{h^2} d(\varphi(0) + hy + \frac{h^2}{2} z, P(\varphi(0))) = 0,$$

which is equivalent with the fact that for all $(t, \varphi, y) \in [0, T] \times K_0 \times \Omega$, there exists $z \in F(\varphi, y)$ such that $F(\varphi, y) \subset T^2_{P(\varphi(0))}(\varphi(0), y)$. Therefore, Theorem 3.1 generalizes to Carathéodory perturbation the result in [6].

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