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SUBCLASSES OF STARLIKE FUNCTIONS ASSOCIATED WITH A FRACTIONAL CALCULUS OPERATOR INVOLVING CAPUTO'S FRACTIONAL DIFFERENTIATION

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In this paper, we introduce a new class of functions which are analytic and univalent with negative coefficients defined by using certain fractional operators described in the Caputo's sense and obtain coefficient estimates, extreme points, the radii of close to convexity, starlikeness and convexity and neighborhood results for $f \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)$. In particular, we obtain modified Hadamard product results for the function $f(z)$ belongs to the class $\mathcal{T}\mathcal{J}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)$ in the unit disc.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C},\ |z| < 1\}$. Also denote by T a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,\ \ z \in U, \tag{2}$$

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introduced and studied by Silverman [18]. Also denote by $\mathcal{ST}(\sigma)$ the class of starlike functions of order $\sigma (0 \leq \sigma < 1)$ such that $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \sigma$, and $\mathcal{CV}(\sigma)$ is convex of order σ $(0 \leq \sigma < 1)$ satisfying the analytic criteria $\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\}$ $> \sigma$.

For functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ \ z \in U. \tag{3}$$

Now, we recall the Caputo's [4] definition which shall be used throughout the paper. Caputo's definition of the fractional-order derivative is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \tag{4}$$

where $n-1 < Re(\alpha) \leq n, n \in N$, and the parameter α is allowed to be real or even complex, a is the initial value of the function f .

We recall the following definitions [11].

Definition 1.1. Let the function $f(z)$ be analytic in a simply - connected region of the $z-$ plane containing the origin. The fractional integral of f of order μ is defined by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\mu}} d\xi, \quad \mu > 0, \tag{5}$$

where the multiplicity of $(z-\xi)^{1-\mu}$ is removed by requiring $log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 1.2. The fractional derivatives of order μ, is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\mu}} d\xi, \quad 0 \leq \mu < 1, \tag{6}$$

where the function $f(z)$ is constrained, and the multiplicity of the function $(z-\xi)^{-\mu}$ is removed as in Definition 1.1.

Definition 1.3. Under the hypothesis of Definition 1.1, the fractional derivative of order $n + \mu$ is defined by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z), \quad (0 \leq \mu < 1; n \in N_0). \tag{7}$$

With the aid of the above definitions and the generalization operator of Salagean[15] derivative operator and Libera integral operator [8], was given by Owa [11] (their known extensions involving fractional derivative and fractional integrals),

$$\Omega^{\delta} f(z)=\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)=z+\sum_{n=2}^{\infty}\Phi(n,\delta)a_nz^n \tag{8}$$

where

$$\Phi(n,\delta)=\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}. \tag{9}$$

For $f\in\mathcal{A}$ and various choices of δ, we get different operators

$$\Omega^0 f(z):=f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{10}$$

$$\Omega^1 f(z):=zf'(z)=z+\sum_{n=2}^{\infty}na_nz^n \tag{11}$$

$$\Omega^j f(z):=\Omega(\Omega^{j-1}f(z))=z+\sum_{n=2}^{\infty}n^ja_nz^n,(j=1,2,3,...) \tag{12}$$

which is known as Salagean operator[15]. Also note that

$$\Omega^{-1}f(z)=\frac{2}{z}\int_0^z f(t)dt:=z+\sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)a_nz^n$$

and

$$\Omega^{-j}f(z):=\Omega^{-1}(\Omega^{-j+1}f(z)):=z+\sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^j a_nz^n,\ (j=1,2,3,...) \tag{13}$$

called Libera integral operator [8]. We note that the Libera integral operator is generalized as Bernardi integral operator given by Bernardi[3],

$$\frac{1+\nu}{z^{\nu}}\int_0^z t^{\nu-1}f(t)dt:=z+\sum_{n=2}^{\infty}\left(\frac{1+\nu}{n+1}\right)a_nz^n,(\nu=1,2,3,...).$$

Making use of these results recently Salah and Darus [16] introduced the following operator

$$\mathcal{J}_{\mu}^{\eta}f(z)=\frac{\Gamma(2+\eta-\mu)}{\Gamma(\eta-\mu)}z^{\mu-\eta}\int_0^z\frac{\Omega^{\eta}f(t)}{(z-t)^{\mu+1-\eta}}dt \tag{14}$$

where η(real number) and $(\eta-1<\mu<\eta<2)$. By simple calculations for functions $f(z)\in\mathcal{A}$, we get

$$\mathcal{J}_{\mu}^{\eta}f(z)=z+\sum_{n=2}^{\infty}\frac{(\Gamma(n+1))^2\Gamma(2+\eta-\mu)\Gamma(2-\eta)}{\Gamma(n+\eta-\mu+1)\Gamma(n-\eta+1)}a_nz^n \quad (z\in U), \tag{15}$$

and for the sake of brevity we let

$$C_n(\eta,\mu)=\frac{(\Gamma(n+1))^2\Gamma(2+\eta-\mu)\Gamma(2-\eta)}{\Gamma(n+\eta-\mu+1)\Gamma(n-\eta+1)} \tag{16}$$

and

$$C_2(\eta,\mu)=\frac{4\Gamma(2+\eta-\mu)\Gamma(2-\eta)}{\Gamma(3+\eta-\mu)\Gamma(1-\eta)} \tag{17}$$

unless otherwise stated.

Further, note that $\mathcal{J}_0^0 f(z)=f(z)$ and $\mathcal{J}_1^1 f(z)=zf'(z)$. In recent years, considerable interest in fractional calculus operators has been stimulated due to their applications in the theory of analytic functions. There are many definitions of fractional integration and differentiation can be found in various books (see [10, 14, 21] . In this paper, we introduce a new subclass of analytic functions with negative coefficients and discuss some usual properties of the geometric function theory of this generalized function class.

For fixed $-1\le A\le B\le 1$ and $0<B\le 1$,let $\mathcal{SJ}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1) and satisfying the condition

$$\left|\frac{\frac{z(\mathcal{J}_{\mu}^{\eta}f(z))'}{\mathcal{J}_{\mu}^{\eta}f(z)}-1}{2\gamma(B-A)\left(\frac{z(\mathcal{J}_{\mu}^{\eta}f(z))'}{\mathcal{J}_{\mu}^{\eta}f(z)}-\alpha\right)-B\left(\frac{z(\mathcal{J}_{\mu}^{\eta}f(z))'}{\mathcal{J}_{\mu}^{\eta}f(z)}-1\right)}\right|<\beta,\ z\in U \tag{18}$$

where $\mathcal{J}_{\mu}^{\eta}f(z)$ is given by (15), $0\le\alpha<1$, $0<\beta\le 1$,

$$\frac{B}{2(B-A)}<\gamma\le\begin{cases}\frac{B}{2(B-A)\alpha} & \alpha\neq 0,\\ 1 & \alpha=0.\end{cases}$$

We also let $\mathcal{TJ}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)=\mathcal{SJ}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)\cap T$.

For convenience in entire paper we consider $0\le\alpha<1$, $0<\beta\le 1$,

$$\frac{B}{2(B-A)}<\gamma\le\begin{cases}\frac{B}{2(B-A)\alpha} & \alpha\neq 0,\\ 1 & \alpha=0.\end{cases}$$

for fixed $-1\le A\le B\le 1$ and $0<B\le 1$, one or otherwise stated.

By suitably specializing the values of A, B, α, β and γ the class $\mathcal{TJ}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)$ leads to known subclasses studied in [1, 9] and [12] and various new subclasses.

The main object of this paper is to determine the coefficient bound, extreme points, radii of close to convexity, starlikeness and convexity for functions in the class $\mathcal{TJ}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)$. Further, we obtain modified Hadamard product and Neighbourhood results for aforementioned class.

2. Characterization Properties

We now obtain the characterization property for functions $f(z)$ to belong to the class $\mathcal{TJ}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)$ there by obtaining coefficient bounds.

Theorem 2.1. *Let the function $f(z)$ be defined by (2) is in the class $\mathcal{TJ}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)$ if and only if*

$$\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]C_n(\eta,\mu)|a_n| \leq 2\beta\gamma(1-\alpha)(B-A), \tag{19}$$

where $C_n(\eta,\mu)$ is given by (16).

Proof. The proof of Theorem 2.1 is much akin to the proof of theorems on coefficient bounds established in [6, 19], so we skip the details in this regard. □

Corollary 2.2. *Let the function $f(z)$ defined by (2) be in the class*

$$\mathcal{TJ}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B).$$

Then we have

$$a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]C_n(\eta,\mu)} \tag{20}$$

The equation (20) is attained for the function

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]C_n(\eta,\mu)}z^n \quad (n \geq 2) \tag{21}$$

where $C_n(\eta,\mu)$ is given by (16).

For the sake of brevity, we let

$$\Phi_n(\alpha,\beta,\gamma,A,B) = [2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)] \tag{22}$$

and

$$\Phi_2(\alpha,\beta,\gamma,A,B) = [1+2\beta\gamma(B-A)(2-\alpha)-B\beta] \tag{23}$$

unless otherwise stated.

Theorem 2.3. *Let the function $f(z)$ defined by (2) belong to $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$. Then*

$$|f(z)| \geq |z|\left\{1-\frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2(\eta,\mu)}\;|z|\right\} \tag{24}$$

and

$$|f(z)| \leq |z|\left\{1+\frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2(\eta,\mu)}\;|z|\right\}, \tag{25}$$

where $C_2(\eta,\mu)$ given by (17).

Proof. In the view of (19) and the fact that $C_n(\eta,\mu)$ is non-decreasing for $n \geq 2$, $0 \leq \alpha < 1$ we have

$$\begin{aligned}
&[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)]C_2(\eta,\mu)\sum_{n=2}^{\infty}a_n\\
&\leq \sum_{n=2}^{\infty}\Phi_n(\alpha,\beta,\gamma,A,B)C_n(\eta,\mu)a_n\\
&\leq 2\beta\gamma(1-\alpha)(B-A)
\end{aligned}$$

which readily yields,

$$\sum_{n=2}^{\infty}a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta)]C_2(\eta,\mu)}. \tag{26}$$

Theorem 2.3 follows readily from (2) and (26). □

Theorem 2.4. (Extreme Points) *Let $f_1(z)=z$; $f_n(z)=z-\frac{2\beta\gamma(1-\alpha)(B-A)}{\Phi_n(\alpha,\beta,\gamma,A,B)C_n(\eta,\mu)}z^n$, $(n \geq 2)$ where $C_n(\eta,\mu)$ is given by (16).*
Then $f(z)$ is in the class $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$ if and only if it can be expressed in the form $f(z)=\sum\limits_{n=1}^{\infty}\omega_n f_n(z)$ where $\omega_n \geq 0$ $(n \geq 1)$ and $\sum\limits_{n=1}^{\infty}\omega_n = 1$.

We shall prove the following results for the closure of functions in the class $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$. Let the functions $f_j(z) (j=1,2)$ be defined by

$$f_j(z) = z-\sum_{n=2}^{\infty}a_{n,j}\,z^n \text{ for } a_{n,\,j} \geq 0,\ z \in U. \tag{27}$$

Theorem 2.5. *(Closure Theorem) Let the functions $f_j(z)(j=1,2,\dots m)$ defined by (27) be in the classes $\mathcal{TJ}_\mu^\eta(\alpha_j,\beta,\gamma,A,B)$ $(j=1,2,\dots m)$ respectively. Then the function $h(z)$ defined by $h(z)=z-\frac{1}{m}\sum\limits_{n=2}^{\infty}\left(\sum\limits_{j=1}^{m}a_{n,\,j}\right)z^n$ is in the class $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$, where $\alpha=\min\limits_{1\leq j\leq m}\{\alpha_j\}$ where $0 \leq \alpha_j \leq 1$.*

Proof. Since $f_j \in \mathcal{TJ}_\mu^\eta(\alpha_j,\beta,\gamma,A,B)$, $(j=1,2,\dots m)$ by applying Theorem 2.1, to (27) we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\Phi_n(\alpha,\beta,\gamma,A,B)C_n(\eta,\mu)\left(\frac{1}{m}\sum_{j=1}^{m}a_{n,\,j}\right)\\
&=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}\Phi_n(\alpha,\beta,\gamma,A,B)C_n(\eta,\mu)a_{n,\,j}\right)\\
&\le\frac{1}{m}\sum_{j=1}^{m}2\beta\gamma(1-\alpha_j)(B-A)\le 2\beta\gamma(1-\alpha)(B-A)
\end{aligned}$$

which in view of Theorem 2.1, again implies that $h \in \mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$ and so the proof is complete. □

Next we obtain the radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$.

Theorem 2.6. *Let the function $f(z)$ defined by (2) belong to the class $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$. Then $f(z)$ is close-to-convex of order σ $(0\le\sigma<1)$ in the disc $|z|<r_1$, where*

$$r_1 := \inf\left[\frac{(1-\sigma)\Phi_n(\alpha,\beta,\gamma,A,B)C_n(\eta,\mu)}{2n\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{n-1}} \quad (n\ge 2), \tag{28}$$

where $C_n(\eta,\mu)$ is given by (16). The result is sharp, with extremal function $f(z)$ given by (2.4).

Proof. Given $f\in T$, and f is close-to-convex of order σ, we have

$$|f'(z)-1|<1-\sigma. \tag{29}$$

For the left hand side of (29) we have $|f'(z)-1|\le\sum_{n=2}^{\infty}na_n|z|^{n-1}$. The last expression is less than $1-\sigma$ if

$$\sum_{n=2}^{\infty}\frac{n}{1-\sigma}a_n|z|^{n-1}<1,$$

that is, if

$$\frac{n}{1-\sigma}|z|^{n-1}\le\frac{\Phi_n(\alpha,\beta,\gamma,A,B)C_n(\eta,\mu)}{2\beta\gamma(B-A)(1-\alpha)}.$$

where we have made use of the assertion (19) of Theorem 2.1.The last inequality leads immediately to the disk $|z|<r_1$ where r_1 given by (28,which completes the proof. □

Theorem 2.7. *Let* $f \in \mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$. *Then*

(i) f *is starlike of order* $\sigma(0 \leq \sigma < 1)$ *in the disc* $|z| < r_2$; *where*

$$r_2 = \inf\left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{\Phi_n(\alpha,\beta,\gamma,A,B)C_n(\eta,\mu)}{2\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{n-1}} \quad (n \geq 2) \tag{30}$$

(ii) f *is convex of order* σ $(0 \leq \sigma < 1)$ *in the disc* $|z| < r_3$, *where*

$$r_3 = \inf\left[\left(\frac{1-\sigma}{n(n-\sigma)}\right)\frac{\Phi_n(\alpha,\beta,\gamma,A,B)C_n(\eta,\mu)}{2\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{n-1}} \quad (n \geq 2) \tag{31}$$

where $C_n(\eta,\mu)$ *is given by (16). Each of these results are sharp for the extremal function* $f(z)$ *given by (2.4).*

Proof. Following the techniques employed in [19],we can easily prove (i)

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii). □

3. Modified Hadamard Products

Let the functions $f_j(z)(j = 1,2)$ be defined by (27). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) \;=\; z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}\, z^n.$$

Using the techniques of Schild and Silverman [17], we prove the following results.

Theorem 3.1. *For functions* $f_j(z)$ $(j = 1,2)$ *defined by (27), let* $f_1 \in \mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$ *and* $f_2 \in \mathcal{TJ}_\mu^\eta(\delta,\beta,\gamma,A,B)$. *Then* $(f_1 * f_2) \in \mathcal{TJ}_\mu^\eta(\xi,\beta,\gamma,A,B)$, *where*

$$\xi = 1 - \frac{2\beta\gamma(B-A)(1-\alpha)(1-\delta)(1+2\beta\gamma(B-A)-B\beta)}{\Phi_2(\alpha,\beta,\gamma,A,B)\Phi_2(\delta,\beta,\gamma,A,B)C_2(\eta,\mu)-4\beta^2\gamma^2(B-A)^2(1-\alpha)(1-\delta)} \tag{32}$$

and $\Phi_2(\alpha,\beta,\gamma,A,B)$ *is given by (23),* $C_2(\eta,\mu)$ *is given by (17) and* $\Phi_2(\delta,\beta,\gamma,A,B,2) = [2\beta\gamma(B-A)(2-\delta)+(1-B\beta)]$.

Proof. In view of Theorem 2.1, it suffice to prove that

$$\sum_{n=2}^{\infty} \frac{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]C_n(\eta,\mu)}{2\beta\gamma(1-\xi)(B-A)} a_{n,1}a_{n,2} \leq 1, \ (0 \leq \xi < 1)$$

where ξ is defined by (32). On the other hand, under the hypothesis, it follows from (19) and the Cauchy's-Schwarz inequality that

$$\sum_{n=2}^{\infty} \frac{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}}{\sqrt{(1-\alpha)(1-\delta)}(C_n(\eta,\mu))^{-1}} \sqrt{a_{n,1}a_{n,2}} \leq 1, \tag{33}$$

where $\Phi_n(\alpha,\beta,\gamma,A,B)$ is given by (22) and $\Phi_n(\delta,\beta,\gamma,A,B,n) = [2\beta\gamma(B-A)(n-\delta)+(1-B\beta)(n-1)]$. Thus we need to find the largest ξ such that

$$\begin{aligned}&\sum_{n=2}^{\infty} \frac{[\Phi_n(\xi,\beta,\gamma,A,B)]C_n(\eta,\mu)}{2\beta\gamma(1-\xi)(B-A)} a_{n,1}a_{n,2} \\ &\leq \sum_{n=2}^{\infty} \frac{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}}{\sqrt{(1-\alpha)(1-\delta)}(C_n(\eta,\mu))^{-1}} \sqrt{a_{n,1}a_{n,2}}\end{aligned}$$

or, equivalently that

$$\sqrt{a_{n,1}a_{n,2}} \quad \leq \quad \frac{1-\xi}{\sqrt{(1-\alpha)(1-\delta)}} \frac{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}}{[\Phi_n(\xi,\beta,\gamma,A,B)]}, \ (n \geq 2)$$

where $\Phi_n(\xi,\beta,\gamma,A,B) = 2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)$. By view of (33) it is sufficient to find largest ξ such that

$$\begin{aligned}&\frac{2\beta\gamma(B-A)\sqrt{(1-\alpha)(1-\delta)}(C_n(\eta,\mu))^{-1}}{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}} \\ &\leq \frac{1-\xi}{\sqrt{(1-\alpha)(1-\delta)}} \frac{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}}{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]}\end{aligned}$$

which yields

$$\begin{aligned}&\xi = \Psi(n) = 1- \\ &\frac{2\beta\gamma(B-A)(1-\alpha)(1-\delta)(n-1)(1+2\beta\gamma(B-A)-B\beta)}{[\Phi_n(\alpha,\beta,\gamma,A,B)\Phi_n(\delta,\beta,\gamma,A,B)]C_n(\eta,\mu)-4\beta^2\gamma^2(B-A)^2(1-\alpha)(1-\delta)}\end{aligned} \tag{34}$$

for $n \geq 2$ is an increasing function of n $(n \geq 2)$ and letting $n = 2$ in (34), we get the desired result. □

By using arguments similar to those in proof of Theorem 3.1, and employing the techniques of [19] we can easily prove the following results, hence we state the following theorems without proof.

Theorem 3.2. *Let the functions* $f_j(z)(j=1,2)$ *defined by (27), be in the class* $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$ *then* $(f_1 * f_2) \in \mathcal{TJ}_\mu^\eta(\rho,\beta,\gamma,A,B)$, *where*

$$\rho = 1 - \frac{2\beta\gamma(B-A)(1-\alpha)^2(1+2\beta\gamma(B-A)-B\beta)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]^2 C_2(\eta,\mu) - 4\beta^2\gamma^2(B-A)^2(1-\alpha)^2}$$

and $C_2(\eta,\mu)$ *is given by (17).*

Proof. By taking $\delta = \alpha$, in the above theorem, the result follows. □

Theorem 3.3. *Let the function* $f(z)$ *defined by (2) be in the class* $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$. *Also let* $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ *for* $|b_n| \le 1$.

Then $(f * g) \in \mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$.

Theorem 3.4. *Let the functions* $f_j(z)(j=1,2)$ *defined by (27) be in the class* $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$. *Then the function* $h(z)$ *defined by* $h(z) = z - \sum_{n=2}^{\infty}(a_{n,1}^2 + a_{n,2}^2)z^n$ *is in the class* $\mathcal{TJ}_\mu^\eta(\xi,\beta,\gamma,A,B)$, *where*

$$\xi = 1 - \frac{4\beta\gamma(1-\alpha)^2(B-A)}{C_2(\eta,\mu)[\Phi_2(\alpha,\beta,\gamma,A,B)]^2 - 8\beta^2\gamma^2(B-A)^2(1-\alpha)^2}$$

and $C_2(\eta,\mu)$ *is given by (17).*

4. Inclusion relations involving $N_\delta(e)$

Following [7, 13], we define the $\delta-$ neighbourhood of function $f \in T$ by

$$N_\delta(f) := \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} d_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - d_n| \le \delta \right\}. \tag{35}$$

Particulary for the identity function $e(z) = z$, we have

$$N_\delta(e) := \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} d_n z^n \text{ and } \sum_{n=2}^{\infty} n|d_n| \le \delta \right\}. \tag{36}$$

Now we obtain inclusion relations of the class $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B)$.

Theorem 4.1. *If*

$$\delta := \frac{4\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2(\eta,\mu)} \tag{37}$$

where $C_2(\eta,\mu)$ *is given by (17). Then* $\mathcal{TJ}_\mu^\eta(\alpha,\beta,\gamma,A,B) \subset N_\delta(e)$.

Proof. For $f \in \mathcal{T}\mathcal{J}_\mu^\eta(\alpha,\beta,\gamma,A,B)$, Theorem 2.1 immediately yields

$$[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2(\eta,\mu)\sum_{n=2}^{\infty} a_n \leq 2\beta\gamma(1-\alpha)(B-A),$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2(\eta,\mu)}. \tag{38}$$

On the other hand, from (19) and (38) that

$$\begin{aligned}
&[2\beta\gamma(B-A)+(1-B\beta)]C_2(\eta,\mu)\sum_{n=2}^{\infty} na_n \\
&\leq 2\beta\gamma(1-\alpha)(B-A)+[2\beta\gamma\alpha(B-A)+(1-B\beta)]C_2(\eta,\mu) \\
&\times\left[\frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2(\eta,\mu)}\right] \\
&= \frac{2[2\beta\gamma(1-\alpha)(B-A)][2\beta\gamma(B-A)+(1-B\beta)]}{[\Phi_2(\alpha,\beta,\gamma,A,B)]}
\end{aligned}$$

that is

$$\sum_{n=2}^{\infty} na_n \leq \frac{4\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2(\eta,\mu)} := \delta \tag{39}$$

which, in view of the (36) which complete the proof of Theorem 4.1. □

Next we determine the neighborhood for the class $\mathcal{T}\mathcal{J}_\mu^\eta(\rho,\alpha,\beta,\gamma,A,B)$ which we define as follows. A function $f \in T$ is said to be in the class $\mathcal{T}\mathcal{J}_\mu^\eta(\rho,\alpha,\beta,\gamma,A,B)$ if there exists a function $h \in \mathcal{T}\mathcal{J}_\mu^\eta(\rho,\alpha,\beta,\gamma,A,B)$ such that

$$\left|\frac{f(z)}{h(z)}-1\right| < 1-\rho,\ (z \in U,\ \ 0 \leq \rho < 1).$$

Theorem 4.2. *If $h \in \mathcal{T}\mathcal{J}_\mu^\eta(\rho,\alpha,\beta,\gamma,A,B)$ and*

$$\rho = 1-\frac{[\Phi_2(\alpha,\beta,\gamma,A,B)]\delta C_2(\eta,\mu)}{[2+4\beta\gamma(B-A)(2-\alpha)-B\beta]C_2(\eta,\mu)-4\beta\gamma(1-\alpha)(B-A)} \tag{40}$$

then $N_\delta(h) \subset \mathcal{T}\mathcal{J}_\mu^\eta(\rho,\alpha,\beta,\gamma,A,B)$.

Proof. Suppose that $f \in N_\delta(g)$ we then find from (35) that $\sum_{n=2}^{\infty} n|a_n-b_n| \leq \delta$ which implies that the coefficient inequality $\sum_{n=2}^{\infty} |a_n-b_n| \leq \frac{\delta}{2}$. Since $h \in$

$\mathcal{TJ}_{\mu}^{\eta}(\alpha,\beta,\gamma,A,B)$,

we have $\sum\limits_{n=2}^{\infty} b_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2(\eta,\mu)}$ so that

$$\left|\frac{f(z)}{h(z)}-1\right| < \frac{\sum\limits_{n=2}^{\infty}|a_n-b_n|}{1-\sum\limits_{n=2}^{\infty}b_n} \leq \frac{\frac{\delta}{2}}{1-\frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2(\eta,\mu)}}$$

$$= \frac{[\Phi_2(\alpha,\beta,\gamma,A,B)]\delta C_2(\eta,\mu)}{[2+4\beta\gamma(B-A)(2-\alpha)-B\beta]C_2(\eta,\mu)-4\beta\gamma(1-\alpha)(B-A)} = 1-\rho.$$

provided that ρ is given precisely by (40). Thus by definition, $f \in \mathcal{TJ}_{\mu}^{\eta}(\rho,\alpha,\beta,\gamma,A,B)$ for ρ given by (40), which completes the proof. $\square$

Concluding Remarks: By suitably specializing the various parameters involved in Theorem 2.1 to Theorem 4.2, one can state the corresponding results for many relatively more familiar function classes.

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