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ON TWISTED ORDERED MONOID RINGS OVER QUASI-BAER RINGS

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In this paper we show that if M is an Ordered monoid then the twisted monoid ring $R^T M$ is (left principally) quasi-Baer if and only if R is (left principally) quasi-Baer. Also if R is (left principally) quasi-Baer and G is an ordered group acting on R we give a necessary and sufficient condition for the crossed product $R * G$ to be (left principally) quasi-Baer.

1. INTRODUCTION

Throughout this paper, R denotes an associative ring with identity. If S is a subset of R, $l_R(S)$ denotes the left annihilator of S in R. A ring R is called *(left principally) quasi-Baer* if the left annihilator of every (principal) left ideal of R is generated as a left ideal by an idempotent. A *Baer ring* is a ring in which the left annihilator of every subset is generated as a left ideal by an idempotent. A ring R is called *left (right) P.P.-ring* if the left (right) annihilator of an element of R is generated by an idempotent. Also a ring R is called *P.P.-ring* if it is both left and right P.P.-ring.

Baer rings were introduced by Kaplanasky [8] to abstract various properties of rings of operators of Hilbert space. Clark [6] introduced the quasi-Baer rings and characterized a finite dimensional quasi-Baer ring over an algebraically

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closed field as a twisted matrix units semigroup algebra. Further work in quasi-Baer rings appeared in [2], [3], [4] and [10]. Recently, Birkenmeier, Kim and Park [5] introduced principally quasi-Baer rings and used them to generalize many results on reduced *P.P.*-rings. In [5], it was proved that a ring R is a (left principally) quasi-Baer ring if and only if the polynomial ring $R[x]$ is a (left principally) quasi-Baer ring. In [11] Hirano generalized this result to ordered monoid rings. This paper is devoted to extend this result to twisted monoid rings.

Let R be a ring and M be a monoid then the *twisted monoid ring* $R^T M$ is an *R-algebra* whose elements are finite sum of the form $\sum r_x x,\ r_x \in R,\ x \in M$ with equality and addition defined component wise and multiplication defined distributively according to the relation $(r_x x)(r_y y) = r_x r_y\ f(x,y)(xy)$, where $f : M \times M \to U(R)$ is called *a twisted function* and $U(R)$ denotes the set of all units of R. Moreover, f must satisfy the following:

$$f(y,z)f(x,yz) = f(x,y)f(xy,z), \quad f(1,x) = f(x,1) = 1 \text{ for every } x \in M.$$

Let G be a group acting on R as an automorphism group of R. We denote by r^g the image of $r \in R$ under $g \in G$.

By *a crossed product* $R *_f G$ we understand the set of finite sums,

$$R *_f G = \left\{ \sum r_g g \mid r_g \in R,\ g \in G \right\}$$

with a *twisted function* (*factor system*) $f : G \times G \to U(R)$ which satisfies

(i) $f^g(h,k)f(g,hk) = f(g,h)f(gh,k)$ for every $g,h,k \in G$,

(ii) $f(1,g) = f(g,1) = 1$ for all $g \in G$.

Equality and addition are defined component wise and for $g,h \in G;\ r \in R$ we have

$$g.h = f(g,h)gh; \qquad gr = r^g g.$$

For simplicity we write $R * G$ to denote the crossed product. If the action of G is trivial then $R * G$ is called a *twisted group ring*.

Note that R may be considered as a left $(R * G)$-module as follows: for any $a \in R$ and any $\sum_{g \in G} r_g g \in (R * G)$ define $(\sum_{g \in G} r_g g)a = \sum_{g \in G} r_g a^g \in R$. Now we can define the following

A ring R is called a *G-quasi-Baer* ring if for any $(R * G)$-submodule I of R the left annihilator of I in R is generated as a left ideal by an idempotent.

A ring R is called a *G-left principally quasi-Baer* ring if for an element $a \in R$, the left annihilator of $(R * G)a = \sum_{g \in G} Ra^g$ is generated as a left ideal by an idempotent.

In [5] it was shown that if R is a left principally quasi-Baer ring, then the left annihilator of any finitely generated left ideal is generated as a left ideal by an idempotent.

Note also that if R is a G-left principally quasi-Baer ring, then for any finitely many elements $a_1, a_2, ..., a_n \in R$, the left annihilator of $(R*G)a_1 + (R*G)a_2 + \cdots + (R*G)a_n$ is also generated by an idempotent. We frequently use these facts without mention.

When G is a cyclic group generated by g, a G-(left principally) quasi-Baer ring is simply called a g-(left principally) quasi-Baer ring.

Let M be a multiplicative monoid and $\leq$ be an order relation defind on M. The order relation $\leq$ is said to be compatible if $a \leq b$ in M implies $am \leq bm$ for all $m \in M$. Recall that the order relation $\leq$ strictly ordered monoid if $a < b$ in M implies $am < bm$ for all $m \in M$. Hence fourth, we assume that the relation is a strictly totally order relation.

2. RESULTS

Lemma 2.1. *Let R be a left principally quasi-Baer ring, M be an ordered monoid and R^TM be the twisted monoid ring. Suppose that*

$$(a_0x_0 + a_1x_1 + \cdots + a_mx_m)R^TM(b_0y_0 + b_1y_1 + ... + b_ny_n) = 0$$

with $a_i, b_j \in R$, and that $x_i, y_j \in M$ satisfies $x_i < x_j$ and $y_i < y_j$ if $i < j$.
Then $a_iRb_j = 0$ for all i, j.

Proof. Let c be an arbitrary element of R. Then we have the following equation:

$$(a_0x_0 + a_1x_1 + \cdots + a_mx_m)(c1_M)(b_0y_0 + b_1y_1 + \cdots + b_ny_n) = 0$$

$$\begin{aligned}
&a_0cf(x_0,1)b_0f(x_0,y_0)x_0y_0 + \dots \\
&+\{a_mcf(x_m,1)b_{n-3}f(x_m,y_{n-3})x_my_{n-3} \\
&+a_{m-1}cf(x_{m-1},1)b_{n-2}f(x_{m-1},y_{n-2})x_{m-1}y_{n-2} \\
&+a_{m-2}cf(x_{m-2},1)b_{n-1}f(x_{m-2},y_{n-1})x_{m-2}y_{n-1} \\
&+a_{m-3}cf(x_{m-3},1)b_nf(x_{m-3},y_n)x_{m-3}y_n\} \\
&+\{a_mcf(x_m,1)b_{n-2}f(x_m,y_{n-2})x_my_{n-2} \\
&+a_{m-1}cf(x_{m-1},1)b_{n-1}f(x_{m-1},y_{n-1})x_{m-1}y_{n-1} \\
&+a_{m-2}cf(x_{m-2},1)b_nf(x_{m-2},y_n)x_{m-2}y_n\} \\
&+\{a_mcf(x_m,1)b_{n-1}f(x_m,y_{n-1})x_my_{n-1} \\
&+a_{m-1}cf(x_{m-1},1)b_nf(x_{m-1},y_n)x_{m-1}y_n\} \\
&+a_mcf(x_m,1)b_nf(x_m,y_n)x_my_n = 0
\end{aligned} \tag{1}$$

Since $x_m y_n$ is the element of highest order in $x_i y_j$'s, its coefficient equals zero, that is $a_m c f(x_m, 1) b_n f(x_m, y_n) = 0$ so $a_m c f(x_m, 1) b_n = 0$. Hence $a_m \in l_R(Rf(x_m, 1)b_n) = l_R(Rb_n)$. Since R is left principally quasi-Baer, then we have $l_R(Rb_n) = Re_n$ for some idempotent e_n. Replacing c by ce_n in the Equation (1) we obtain

$$\begin{aligned} 0 &= a_0 ce_n f(x_0, 1) b_0 f(x_0, y_0) x_0 y_0 + \dots \\ &+ \{a_m ce_n f(x_m, 1) b_{n-2} f(x_m, y_{n-2}) x_m y_{n-2} \\ &+ a_{m-1} ce_n f(x_{m-1}, 1) b_{n-1} f(x_{m-1}, y_{n-1}) x_{m-1} y_{n-1}\} \\ &+ a_m ce_n f(x_m, 1) b_{n-1} f(x_m, y_{n-1}) x_m y_{n-1}. \end{aligned}$$

Since $x_m y_{n-1}$ is the element of highest order in $\{x_i y_j \mid 1 \leq i \leq m, 1 \leq j \leq n\} \setminus \{x_{m-1} y_n, x_m y_n\}$, then $a_m ce_n f(x_m, 1) b_{n-1} f(x_m, y_{n-1}) = 0$. Hence we have $a_m ce_n f(x_m, 1) b_{n-1} = 0$. Since Re_n is an ideal of R, and $e_n \in Re_n$, we have $e_n c \in Re_n$. So $e_n c = e_n ce_n$ for any element $c \in R$. Also since $a_m \in l_R(Rb_n) = Re_n$, then $a_m = a_m e_n$. Hence

$$\begin{aligned} a_m c f(x_m, 1) b_{n-1} &= a_m e_n c f(x_m, 1) b_{n-1} \\ = a_m e_n ce_n f(x_m, 1) b_{n-1} &= a_m ce_n f(x_m, 1) b_{n-1} = 0, \end{aligned}$$

therefore $a_m \in l_R(Rf(x_m, 1)b_{n-1}) = l_R(Rb_{n-1})$. So $a_m \in l_R(Rb_n + Rb_{n-1})$. Since R is left principally quasi-Baer, $l_R(Rb_n + Rb_{n-1}) = Re_{n-1}$ for some idempotent $e_{n-1} \in R$. Next, replacing c by ce_{n-1} in the Equation (1), we obtain

$$a_m ce_{n-1} f(x_m, 1) b_{n-2} f(x_m, y_{n-2}) = 0$$

in the same way as above. Hence we have $a_m \in l_R(Rb_n + Rb_{n-1} + Rb_{n-2})$. Continuing this process we obtain $a_m Rb_k = 0$ for all $k = 0, 1, ..., n$. Thus we get $(a_0 x_0 + a_1 x_1 + ... + a_{m-1} x_{m-1}) R^T M (b_0 y_0 + b_1 y_1 + ... + b_n y_n) = 0$. Using induction on $m + n$ we obtain $a_i Rb_j = 0$ for all i, j. □

Lemma 2.2. *Let M be an ordered monoid and consider the twisted monoid ring $R^T M$. Let I be a (principal) left ideal of $R^T M$ and let I_0 denote the set of all coefficients of elements of I, then*

- *(i) I_0 is a (finitely generated) ideal of R;*
- *(ii) $l_R(I) = l_R(I_0)$;*
- *(iii) If J is a left ideal of R , then $l_{R^T M}(J) = l_{R^T M}((R^T M)J)$.*

Proof. (i) The proof is clear.
(ii) Let $a \in l_R(I_0)$ then $aI = 0$ and $a \in l_R(I)$. Hence, $l_R(I_0) \subset l_R(I)$. Conversely,

let $a \in l_R(I)$ then $a \sum_{x \in M} b_x x = \sum_{x \in M} ab_x x = 0$ and $ab_x = 0$, for each $x \in M$. Therefore $a \in l_R(I_0)$. Hence $l_R(I) \subset l_R(I_0)$.
(iii) Let J be a left ideal of R, since $J \subset (R^T M)J$ then $l_{R^T M}((R^T M)J) \subset l_{R^T M}(J)$. Conversely, let $x \in l_{R^T M}(J)$ then $x((R^T M)J) = x((RJ)^T M) = x(J^T M) = 0$. So $x \in l_{R^T M}((R^T M)J)$ and $l_{R^T M}(J) \subset l_{R^T M}((R^T M)J)$. Then we can conclude that $l_{R^T M}(J) = l_{R^T M}((R^T M)J)$. □

Now we will use these lemmas to prove the following theorem.

Theorem 2.3. *Let M be an ordered monoid. Then the twisted monoid ring $R^T M$ is a (left principally) quasi-Baer ring if and only if R is a (left principally) quasi-Baer ring.*

Proof. Suppose R is a (left principally) quasi-Baer. Let I be a (principal) left ideal of $R^T M$ and I_0 denote the set of all coefficients of elements of I. Since R is (left principally) quasi-Baer, there exist an idempotent $e \in R$ such that $l_R(I) = l_R(I_0) = Re$. Now it is sufficient to show that $l_{R^T M}(I) \subseteq (R^T M)e$. Let $\alpha = \sum_{x \in M} a_x x \in l_{R^T M}(I)$ then $\alpha I = (\sum_{x \in M} a_x x)I = 0$, by Lemma 2.1 we get $a_x I_0 = 0$ for all a_x. Therefore $a_x \in l_R(I_0)$ which implies that $a_x = a_x e$. Consequently $\alpha = \sum_{x \in M} a_x e x = (\sum_{x \in M} a_x x)e \in (R^T M)e$. Hence $l_{R^T M}(I) = (R^T M)e$ and $R^T M$ is (left principally) quasi-Baer.

Conversely assume that $R^T M$ is a (left principally) quasi-Baer ring. Let I be a (principal) left ideal of R, then $(R^T M)I$ is a left ideal of $R^T M$. By hypothesis there exists an idempotent $e \in R^T M$ such that $l_{R^T M}((R^T M)I) = (R^T M)e$. We may write $e = a_0 1_M + a_1 x_1 + \cdots + a_n x_n \in R^T M$ where $a_i \in R$ and $1, x_1, \dots, x_n$ are distinct elements of M. We show that $l_R(I) = Ra_0$ where a_0 is an idempotent of R. Since $l_{R^T M}(I) = (R^T M)e$, then $(a_0 1_M + a_1 x_1 + \dots + a_n x_n)I = 0$ and $a_i x_i \in l_{R^T M}(I) = (R^T M)e$ for each $i = 0, 1, 2, \dots, n$, $x_0 = 1$. In particular $a_0 1 = (a_0 1)e = (a_0 1)(a_0 1_M + a_1 x_1 + \dots + a_n x_n) = a_0^2 f(1,1)1 + a_0 a_1 f(1, x_1)x_1 + \dots + a_0 a_n f(1, x_n)x_n$. Since $f(1,1) = 1$ it follows that $a_0^2 = a_0$ is an idempotent element of R. Obviously $Ra_0 \subset l_R(I)$. Now, let $a \in l_R(I)$, then $a1 \in l_{R^T M}(I) = (R^T M)e$ and we get $a1 = (a1)e = (a1)(a_0 1_M + a_1 x_1 + \dots + a_n x_n) = aa_0 f(1,1)1 + aa_1 f(1, x_1)x_1 + \dots + aa_n f(1, x_n)x_n$. $a = aa_0 \in Ra_0$. Consequently $Ra_0 = l_R(I)$ and R is a (left principally) quasi-Baer ring. □

It is well-known that torsion-free groups and free groups are ordered groups (see [9, Lemma 13.1.6 and 13.2.8], [7, Theorem 3.1]). Hence the following corollary easily follows.

Corollary 2.4. *Let M be a submonoid of a free group or a torsion-free group. Then the twisted monoid ring $R^T M$ is a (left principally) quasi-Baer ring if and only if R is a (left principally) quasi-Baer ring.*

A ring R is called *reduced* if it has no nonzero nilpotent elements. In a reduced ring R left and right annihilators coincide for any subset S of R. Hence if R is a reduced ring, then R is a P.P.-ring (a Baer ring) if and only if R is a left principally quasi-Baer ring (a quasi-Baer ring). Hence we can deduce that the following corollary,

Corollary 2.5. *Let R be a reduced ring and M be an ordered monoid; then the twisted monoid ring R^TM is a P.P.-(Baer) ring if and only if R is a P.P.-(Baer) ring.*

Proof. Let R^TM be a reduced P.P.(Baer) ring which is equivalent to R^TM is a left principally quasi-Baer(quasi-Baer) ring. Hence by using Theorem 2.3 R is a reduced left principally quasi-Baer (quasi-Baer) ring if and only if R is a reduced P.P.-(Baer) ring. □

Theorem 2.6. *Let R be a ring and G be an ordered group acting on R. If $R*G$ is a (left principally) quasi-Baer ring then R is a G-(left principally) quasi-Baer ring.*

Proof. Suppose that $R*G$ is a (left principally) quasi-Baer ring, and that I is a (cyclic) $(R*G)$-submodule of R. First, we show that $I = I^g$, for all $g \in G$. Since I is a $R*G$-submodule of R, $(1g)I \subset I$ for every $g \in G$. Hence $I^g \subset I$ for every $g \in G$. To prove the other inclusion, let $a \in I$; then for every $g \in G$ we have $a = r^g$ for some $r \in R$. Hence $r = a^{g^{-1}} \in I$, which implies that $a \in I^g$, and it follows that $I = I^g$, for all $g \in G$. Now we show that $l_R(I)$ is generated by an idempotent. By hypothesis there exists an idempotent $e \in R*G$ such that $l_{R*G}((R*G)I) = (R*G)e$. We may write $e = a_0 1_G + a_1 g_1 + ... + a_n g_n \in R*G$, where $a_i \in R$ and $1, g_1, ..., g_n$ are distinct elements of G. Since $e \in l_{R*G}((R*G)I)$, then $(a_0 1_G + a_1 g_1 + ... + a_n g_n)b1 = 0$ for each $b \in I$. Hence $a_0 b f(1,1) 1_G + a_1 b^{g_1} f(g_1,1) g_1 + ... + a_n b^{g_n} f(g_n,1) g_n = 0$ for all $b \in I$, which implies that $a_i \in l_R(I)$ for each $i = 1,2,...,n$. Therefore $a_i 1 \in l_{R*G}(I(R*G)) = l_{R*G}((R*G)I) = (R*G)e$, for each i. In particular, $a_0 1 = (a_0 1)e = (a_0 1)(a_0 1_G + a_1 g_1 + \cdots + a_n g_n) = a_0^2 f(1,1)1 + a_0 a_1 f(1,g_1) g_1 + \cdots + a_0 a_n f(1,g_n) g_n$. So $a_0^2 = a_0$ is an idempotent element of R. Obviously $Ra_0 \subset l_R(I)$. To prove the inverse inclusion, let $a \in l_R(I)$, then $a1 \in l_{R*G}(I(R*G)) = (R*G)e$. So $a1 = (a1)e = (a1)(a_0 1_G + a_1 g_1 + \cdots + a_n g_n) = aa_0 f(1,1)1 + aa_1 f(1,g_1) g_1 + ... + aa_n f(1,g_n) g_n$. This implies that $a = aa_0 \in Ra_0$. Thus we obtain $Ra_0 = l_R(I)$, and R is a G- (left principally) quasi-Baer ring. □

The following example shows that there exists a crossed product $R*G$ which is Quasi-Baer while R is not Quasi-Baer .

Example 2.7. Consider the ring $R=\{(a,b)\in\mathbb{Z}\oplus\mathbb{Z}\mid a\equiv b\pmod 2\}$. With the usual operations of component wise addition and multiplication R is clearly a commutative reduced ring and the only idempotents of R are $(0,0)$, $(1,1)$. Let $G=\langle g\rangle$ be an infinite cyclic group and let the action of G be defined by $(a,b)^g=(b,a)$. Now we claim that $R*G$ is Quasi-Baer. To prove this claim, let I be a non Zero ideal of $R*G$, hence there exist a non zero element $x\in I$. Suppose $x=\sum_j (a_j,b_j)\,g^j$ and $g^i<g^j$ if $i<j$, letg^i be the smallest element with non zero coefficient (a_i,b_i). Let $y=(1,1)\,f^{-1}(2k-i,i)g^{2k-i}$ and $Z=(1,1)\,f^{-1}(2k-i+1,i)g^{2k-i+1}$. Hence $yx\in I$ and $zx\in I$, clearly the smallest order with non zero coefficient in both of them is $2k$ and one of them has the coefficient (a_i,b_i) for the smallest term and the other has (b_i,a_i). Suppose that $0\neq q=\sum_s (u_s,v_s)\,g^s\in l_{R*G}(I)$, with g^j be the smallest element with non zero coefficient (u_j,v_j), Hence $q\,(yx)=0$ and $q\,(zx)=0$. The coefficients of the smallest term in both of them are $(u_j,v_j)\,(a_i,b_i)\,f\left(g^j,g^{2k}\right)$ and $(u_j,v_j)\,(b_i,a_i)\,f\left(g^j,g^{2k}\right)$. Therefore, we get $(u_ja_i,v_jb_i)=(0,0)$ and $(u_jb_i,v_ja_i)=(0,0)$, since $(a_i,b_i)\neq(0,0)$ this means that a_i or b_i are non zero. Consequently, $(u_j,v_j)=(0,0)$ which is a contradiction. Therefore, $l_{R*G}(I)=\{(0,0)\}$ and $R*G$ is a Quasi-Baer.

Conversely, R is not Quasi-Baer ring. For $(2,0)\in R$, we get $l_R(\langle(2,0)\rangle)=\{(0,2n)\,|n\in\mathbb{Z}\}$. Consequently, $l_R((2,0))$ doesn't contain any non zero idempotent. Hence R is not Quasi-Baer.

Lemma 2.8. *Let G be an ordered group acting on R and consider the crossed product $(R*G)$, then*

(i) $\sum_{g\in G} Rb^g$ *is an invariant under the action of elements of G where $b\in R$;*

(ii) I is a left $R*G$ - submodule of R if and only if I is an invariant left ideal of R.

Proof. (i) Let h be an arbitrary element in G; then, $(\sum_{g\in G} Rb^g)^h=\sum_{g\in G}(Rb^g)^h=\sum_{g\in G}R^h(b^g)^h=\sum_{g\in G}Rb^{gh}=(\sum_{g'=gh\in G}Rb^{g'})=(\sum_{g\in G}Rb^g)$. Hence $\sum_{g\in G}Rb^g$ is an invariant under the action of elements of G.
(ii) Let I be a left $R*G$-submodule of R, then it is clear that I is an abelian group with addition. We will show that I is closed under multiplication by elements of R from the left; let $r\in R,\ i\in I$ we have $(r1_G)i=ri^1=ri$, but I is a left $R*G$-submodule of R then $ri\in I$. Now we will prove that I is invariant. Since $(1_Rg)i=1i^g\in I$ then $I^g\subseteq I$. Therefore I an invariant left ideal of R.

On the other hand, let I be an invariant left ideal of R, then it is sufficient to show that I is closed under multiplication by elements of $R*G$ from the left, let

$\sum_{g\in G} a_g g \in R*G$, then $(\sum_{g\in G} a_g g)I = \sum_{g\in G} a_g I^g \subset \sum_{g\in G} a_g I \subset I$. Therefore I is a left $R*G$-submodule of R. □

Remark 2.9. Using Lemma 2.8 (ii) we can deduce that a (left principally) quasi-Baer ring is a G-(left principally) quasi-Baer ring.

Lemma 2.10. *Let R be a G- left principally quasi-Baer ring, G be an ordered group acting on R and $(R*G)$ be the crossed product. If $(a_0g_0 + a_1g_1 + ... + a_mg_m)(R*G)(b_0h_0 + b_1h_1 + ... + b_nh_n) = 0$ with $a_i, b_j \in R, g_i, h_j \in G$ satisfying $g_i < g_j$ and $h_i < h_j$ if $i < j$, then $a_i(\sum_{g\in G} Rb_j^g) = 0$ for all i, j.*

Proof. Let x be an arbitrary element of $R*G$ and suppose that

$$(a_0g_0 + a_1g_1 + ... + a_mg_m)x(b_0h_0 + b_1h_1 + ... + b_nh_n) = 0. \tag{2}$$

Let c be an arbitrary element of R and g be an arbitrary element of G. Substitute $x = cg_m^{-1}g$ in (2) and consider the coefficient of the highest order g_mh_n in the g_ih_j's, i.e. the coefficient of the term

$$\begin{aligned} a_mg_m(cg_m^{-1}g)b_nh_n &= a_mc^{g_m}f(g_m, g_m^{-1}g)(g_mg_m^{-1}g)b_nh_n \\ &= a_mc^{g_m}f(g_m, g_m^{-1}g)b_n^g f(g, h_n)gh_n, \end{aligned}$$

so we obtain $a_mc^{g_m}f(g_m, g_m^{-1}g)b_n^g f(g, h_n) = 0$ then $a_mc^{g_m}f(g_m, g_m^{-1}g)b_n^g = 0$. This implies $a_mRf(g_m, g_m^{-1}g)b_n^g = a_mRb_n^g = 0$, so $a_m \in l_R(\sum_{g\in G} Rb_n^g)$. Since $I = (\sum_{g\in G} Rb_n^g)$ is a left $R*G$-submodule of R. By hypothesis we have $l_R(\sum_{g\in G} Rb_n^g) = Re_n$ for some idempotent $e_n \in R$. We show that $Re_n = Re_n^h$, let $x \in Re_n^h$, therefore

$$\begin{aligned} x(\sum_{g\in G} Rb_n^g) &= ae_n^h(\sum_{g\in G} Rb_n^g) = [(ae_n^h(\sum_{g\in G} Rb_n^g))^{h^{-1}}]^h \\ &= [a^{h^{-1}}e_n \sum_{g\in G}(Rb_n^g)^{h^{-1}}]^h = [a^{h^{-1}}e_n \sum_{g\in G}(Rb_n^g)]^h = 0^h = 0. \end{aligned}$$

Hence $x \in l_R(\sum_{g\in G} Rb_n^g)$, and $Re_n^h \subset Re_n$ for each $h \in G$. Now let $x \in Re_n$ so, $x = ce_n = c(e_n^{h^{-1}})^h = c(re_n)^h$ for some $c \in R$. Hence $x = cr^he_n^h = c'e_n^h \in Re_n^h$, then $Re_n \subset Re_n^h$ and we get $Re_n = Re_n^h$ for any $h \in G$. Note that Re_n is an ideal of R. Hence substituting $x = ce_ng_m^{-1}g$ in (2) we have

$$\begin{aligned} &(a_0g_0 + a_1g_1 + \cdots + a_mg_m)(ce_ng_m^{-1}g)(b_0h_0 + b_1h_1 + \cdots + b_nh_n) \\ =& a_0c^{g_0}e_n^{g_0}f(g_0, g_m^{-1}g)b_0^{g_0g_m^{-1}g}f(g_0g_m^{-1}g, h_0)g_0g_m^{-1}gh_0 + \ldots \\ &+ a_mc^{gm}e_n^{g_m}f(g_m, g_m^{-1}g)b_{n-1}^g f(g, h_{n-1})gh_{n-1} = 0. \end{aligned}$$

Thus $a_m c^{g_m} e_n^{g_m} f(g_m, g_m^{-1} g) b_{n-1}^g = 0$. But $a_m = a_m e_n$ and $e_n c^{g_m} e_n^{g_m} = e_n c^{g_m}$, therefore $a_m c^{g_m} f(g_m, g_m^{-1} g) b_{n-1}^g = a_m c^{g_m} e_n^{g_m} f(g_m, g_m^{-1} g) b_{n-1}^g = 0$. Hence

$$a_m R f(g_m, g_m^{-1} g) b_{n-1}^g = a_m R b_{n-1}^g = 0,$$

so that $a_m \in l_R(\sum_{g\in G} R b_n^g) \cap l_R(\sum_{g\in G} R b_{n-1}^g)$. Continuing this process, we obtain $a_m \in \bigcap_{i=1}^{n} l_R(\sum_{g\in G} R b_i^g)$. Therefore $(a_0 g_0 + a_1 g_1 + ... + a_{m-1} g_{m-1})(R * G)(b_0 h_0 + b_1 h_1 + ... + b_n h_n) = 0$. Using induction on $m+n$, we can complete the proof of this lemma. □

Now we will use the proceding lemmas to prove the following without mention.

Theorem 2.11. *Let R be a ring and G be an ordered group acting on R. If R is a G-(left principally) quasi-Baer ring, then $R * G$ is a (left principally) quasi-Baer ring.*

Proof. Suppose that R is a G-(left principally) quasi-Baer ring, and that I is a (principal) left ideal of $R * G$. Let I_0 denote the set of all coefficients of elements of I then, I_0 is a left ideal of R, hence I_0 is a left $(R * G)$-submodule of R. But R is G-(left principally) quasi-Baer then, there exists an idempotent $e \in R$ such that $l_R(I) = l_R(I_0) = Re$ then by Lemma 2.10 we deduce that $l_{R*G}(I) = (R * G)e$. Therefore $R * G$ is (left principally) quasi-Baer. □

Corollary 2.12. *Let R be a ring such that every ideal of R is a G-invariant ideal and G be an ordered group acting on R, then $R * G$ is a (left principally) quasi-Baer ring if and only if R is a G-(left principally) quasi-Baer ring.*

Corollary 2.13. *(Similar to Corollary 2.5) Let R be a ring and G be an ordered group acting on R. Then the crossed product $R * G$ is a reduced P.P.-(Baer) ring if and only if R is a reduced G-left principally quasi-Baer (G- quasi-Baer) ring.*

Proof. Since $R * G$ is a reduced P.P.- (Baer) ring if and only if $R * G$ is left principally quasi-Baer (a quasi-Baer) ring which using Corollary 2.12 is equivalent to say that R is a reduced G-left principally quasi-Baer (G-quasi-Baer) ring $\Longleftrightarrow$ R is a reduced P.P.-(Baer) ring. □

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