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NEW STRONG COLOURINGS OF HYPERGRAPHS

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We define a new colouring for a hypergraph, in particular for a graph. Such a method is a partition of the vertex-set of a hypergraph, in particular of a graph. However, it is more intrinsically linked to the geometric structure of the hypergraph and therefore enables us to obtain stronger results than in the classical case. For instance, we prove theorems concerning 3-colourings, 4-colourings and 5-colourings, while we have no analogous results in the classical case. Moreover, we prove that there are no semi-hamiltonian regular simple graphs admitting a hamiltonian 1-colouring. Finally, we characterize the above graphs admitting a hamiltonian 2-colouring and a hamiltonian 3-colouring.

1. Introduction

A hypergraph [2] is a pair $(\mathcal{P},\mathcal{B})$ where $\mathcal{P}$ is a non-empty finite set whose elements are called *vertices* and $\mathcal{B}$ is a non-empty family of non-empty subsets of $\mathcal{P}$, whose elements are called *edges*, such that $\mathcal{B}$ is a covering of $\mathcal{S}$. We denote by $\deg P$, degree of P, the number of edges through the vertex P. A hypergraph is also called *geometric space*. In this case, the vertices are called *points* and the edges are called *blocks*.
Let $|\mathcal{P}| = v, |\mathcal{B}| = b$.
From now on we use the terminology of the geometric spaces, considering that

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everything can be couched using hypergraph-theoretic terminology.
Let
$$r = \max_{P \in \mathcal{P}} \deg P, \quad k = \min_{B \in \mathcal{B}} |B|, \quad k' = \max_{B \in \mathcal{B}} |B|.$$
Let
$$\mathcal{I} = \{1, 2, \ldots, v\}$$
and φ be a bijection
$$\varphi : \mathcal{I} \longrightarrow \mathcal{P}.$$

A block B gives rise to the set
$$\left\{\varphi^{-1}(P)\right\}_{P \in B} = \left\{n_1, n_2, \ldots, n_{|B|}\right\},$$
with $n_1 < n_2 < \ldots < n_{|B|}$.

We call *i-th point* of B, $i = 1, 2, \ldots, |B|$, the point $P \in B$ such that $\varphi^{-1}(P) = n_i$.

For every $j = 0, 1, \ldots, r$ and for every $i = 1, 2, \ldots, k'$, we get the set
$$I_\varphi(j, i) = \left\{ \begin{array}{c} P \in \mathcal{P} : \text{there are } j \text{ blocks through } P \\ \text{such that } P \text{ is their } i\text{-th point} \end{array} \right\}$$

For any i, $1 \le i \le k'$, we get the set of indices
$$J_\varphi(i) = \left\{j,\ 0 \le j \le r : I_\varphi(j, i) \ne \emptyset\right\}.$$
Obviously the family $\left\{I_\varphi(j, i)\right\}_{j \in J_\varphi(i)}$ is a partition of $\mathcal{P}$.

We call the pair
$$\left(\left\{I_\varphi(j, i)\right\}_{j \in J_\varphi(i)}, J_\varphi(i)\right)$$
strong colouring of base φ *and index i of the geometric space* $(\mathcal{P}, \mathcal{B})$ or simply *strong colouring* of $(\mathcal{P}, \mathcal{B})$ and we denote it by $c(\varphi, i)$. The indices $j \in J_\varphi(i)$ are called the *colours of* $c(\varphi, i)$, hence every vertex of $I_\varphi(j, i)$ is said to have the colour j.

Now let $(\mathcal{P}, \mathcal{B})$ be a graph $G = (V(G), E(G))$.
Then $\mathcal{P} = V(G)$, $\mathcal{B} = E(G)$, $v = |V(G)|$, $s = |E(G)|$, $k = k' = 2$, $i \in \{1, 2\}$.
We call strong colouring of a graph G the colouring $c(\varphi, i)$ just defined for the geometric space. Thus, every bijection gives rise to two strong colourings, since $i = \{1, 2\}$.
According to this definition, the colour of a vertex V, that is the number of edges through V admitting V as i-th vertex, is determined by the geometric structure of the graph around V and consequently we get deeper results than in the classical case, where the colour of a vertex is arbitrarily assigned, with the only condition that two vertices have different colours.
The following results hold.

- If G is a simple graph, that is a graph without loops and multiedges, every strong colouring of G has the colour $j = 0$.
- A simple graph G is strongly 1-colorable,if and only if, G is a null graph (that is $E(G) = \emptyset$).
- A regular simple graph is strongly 2-colorable if, and only if, G is a bipartite graph.
- If G is a regular simple graph, of degree $r = 2p,\ p$ a prime, $v = |G|$ even, $v < 2r$, strong 3-colourings of G do not exist.

A graph G is called *semi-hamiltonian*, if it contains a path through all the vertices of G, called *semi-hamiltonian path*. If the path is closed, the graph G is called *hamiltonian*.

Consider the following semi-hamiltonian path

$$\ell = V_1 \to V_2 \to \cdots \to V_v.$$

We define the bijection

$$\varphi_\ell : n \in \mathcal{I} = \{1, 2, \ldots, v\} \longrightarrow V_n \in V(G).$$

For any $i \in \{1, 2\}$ we get the strong colouring $c(\varphi_\ell, i)$, which we call *strong hamiltonian colouring of index i associated with the path ℓ.*
Let G denote a semi-hamiltonian regular simple graph of positive degree. We prove that the only graph G admitting a hamiltonian strong 2-colouring is K_2. The only graphs G admitting a hamiltonian strong 3-colouring, are the circuit-graphs.
If G has a hamiltonian strong 4-colouring, then $r \geq 3$ and the colours of $c(\varphi_\ell, i)$, are $0, 1, r-1, r$.

Moreover, the number of vertices of color 1 equals the number of vertices of color $r-1$, which is $\dfrac{v}{2} - 1$, hence v is even.

The following theorem holds:

Theorem 1.1 (cubic simple semi-hamiltonian graphs theorem)**.**
If G is a simple regular semi-hamiltonian graph with $\deg G = 3$ *and if* $c(\varphi_\ell, i)$ *is a hamiltonian strong colouring of G, then* $c(\varphi_\ell, i)$ *is a strong 4-colouring with colours 0,1,2,3. Moreover the number of vertices of colour 1 equals the number of vertices of color 2, which is* $v/2 - 1$. *Hence v is even.*
Finally if G has a hamiltonian strong 5-colouring, we get $r \geq 4$ *and the colours of* $c(\varphi_\ell, i)$ *are* $0, 1, j, r-1, r,\ 1 < j < r-1$.
The number of vertices of colour 1 and the number of vertices of colour $r-1$

are both less than $\frac{v}{2}-1$. *If* v *is even, there are at least two vertices of colour* j *and, if such vertices are two, we get* $j=\frac{r}{2}$, *hence* r *is even.*
Moreover, the number of vertices of colour 1 and the number of vertices of colour $r-1$ *are both equal to* $\frac{v}{2}-2$.

2. Strong colourings of a geometric space

Let $(\mathcal{P},\mathcal{B})$ be a finite geometric space and $c(\varphi,i)$ a strong colouring of $(\mathcal{P},\mathcal{B})$, that is the pair $\left(\{I_\varphi(j,i)\}_{j\in J_\varphi(i)}, J_\varphi(i)\right)$. The indices $j\in J_\varphi(i)$ are the colours of $c(\varphi,i)$. We say that $j\in J_\varphi(i)$ is the colour of $I_\varphi(j,i)$ and that $P\in I_\varphi(j,i)$ has the colour j.
Obviously the number of colours $|J_\varphi(i)|$ satisfies the condition $1\le |J_\varphi(i)|\le r+1$. For any integer k, $1\le k\le r+1$, we say that $(\mathcal{P},\mathcal{B})$ is strongly k-colourable, if there is a strong colouring $c(\varphi,i)$ of $(\mathcal{S},\mathcal{B})$ with k colours. Such $c(\varphi,i)$ is called *strong k-colouring* of $(\mathcal{P},\mathcal{B})$.
Let $t(j,i)=|I_\varphi(j,i)|$. Obviously

$$\sum_{j=0}^{r} t(j,i)=v, \quad i=1,2,\dots,k'. \tag{1}$$

Moreover we get:

$$\sum_{j=0}^{r} jt(j,i)=b, \quad \forall i=1,2,\dots,k. \tag{2}$$

We remark that (1) and (2) hold for any bijection $\varphi: I\to\mathcal{P}$.

3. The strong colourings of a graph

Let us prove the following

Theorem 3.1. *Let* G *be a simple graph, then every strong colouring of* G *has the colour* $j=0$.

Proof. If G is the null graph, the theorem is obvious. Then assume that G is not the null graph and then it has two distinct vertices. Let $c(\varphi,i)$ be a strong colouring of G. Let V_M and V_m be the vertices such that $\varphi^{-1}(V_M)=|G|=v$, $\varphi^{-1}(V_m)=1$. Such vertices are distinct, since $|G|\ge 2$. If $i=1$, there is no edge through V_M admitting V_M as first vertex, therefore $V_M\in I_\varphi(0,1)$ and so $I_\varphi(0,1)\ne\emptyset$. It follows that $j=0\in J_\varphi(1)$. If $i=2$, there is no edge through V_m admitting V_m as second vertex, therefore $V_m\in I_\varphi(0,2)$ and so $I_\varphi(0,2)\ne\emptyset$. It follows $j=0\in J_\varphi(2)$. $\square$

Theorem 3.2. *A simple graph G is strongly* 1*-colourable if, and only if, G is the null graph.*

Proof. Obviously, if G is the null graph, it is strongly 1-colourable, with the colour $j = 0$. Conversely, let G be strongly 1-colorable and let $c(\varphi, i)$ be a strong 1-colouring of G. Then, by Theorem 3.1, the colour of $c(\varphi, i)$ is $j = 0$. Assume now G is not the null graph. Then in G there is an edge $\{V', V''\}$. In this case either V', or V'' cannot have the colour 0, a contradiction. $\square$

The following theorem holds

Theorem 3.3. *Let G be a non-null simple graph and let $c(\varphi, i)$ be a strong colouring of G. Then there is at least a colour $j \neq 0$ of $c(\varphi, i)$ such that*

$$j \leq \left|I_\varphi(0, i)\right|.$$

Proof. By Theorems 3.1 and 3.2 it follows that the strong colouring $c(\varphi, i)$ has at least two distinct colours and one of them is $j = 0$. Then, there is a vertex V_1 of colour $j \neq 0$ and so $V_1 \notin I_\varphi(0, i)$. Assume that every colour $j \neq 0$ satisfies the condition $j > \left|I_\varphi(0, i)\right|$. Then there is an edge $\{V_1, V_2\}$, with $V_2 \notin I_\varphi(0, i)$, which admits V_1 as i-th vertex. Since V_2 has a colour different from zero, there is an edge $\{V_2, V_3\}$, with $V_3 \notin I_\varphi(0, i)$, which admits V_2 as i-th vertex. Moreover we get $V_3 \neq V_1$, since

$$\begin{aligned} &\varphi^{-1}(V_1) > \varphi^{-1}(V_2) > \varphi^{-1}(V_3), \quad \text{if } i = 2,\\ &\varphi^{-1}(V_1) < \varphi^{-1}(V_2) < \varphi^{-1}(V_3), \quad \text{if } i = 1. \end{aligned}$$

Similarly, since V_3 has a colour different from zero, there is an edge $\{V_3, V_4\}$, with $V_4 \notin I_\varphi(0, i)$, which admits V_3 as i-th vertex and such that $V_4 \neq V_1$, $V_4 \neq V_2$, $V_4 \neq V_3$. This procedure continues indefinitely and so the set $V(G) - I_\varphi(0, i)$ is not finite: a contradiction, since G is finite. The contradiction proves that $j > \left|I_\varphi(0, i)\right|$, for every colour $j \neq 0$ of $c(\varphi, i)$ is impossible. $\square$

Now let G be a strongly 2-colourable graph and let $c(\varphi, i)$ be a strong 2-colouring of G with colours 0 and j, $j \leq r$. Obviously one of the two vertices of an edge ℓ is the i-th vertex of ℓ. It follows that ℓ cannot have both the vertices in $I_\varphi(0, i)$ and that, if both the vertices of ℓ are in $I_\varphi(j, i)$, there is at least a vertex of $I_\varphi(j, i)$ which is the i-th vertex of ℓ. Let $\ell = \{V', V''\}$ with $V' \in I_\varphi(j, i)$ and $V'' \in I_\varphi(0, i)$. Then V' is the i-th vertex of ℓ, since there is no edge admitting V'' as i-th vertex. It follows that for any such an edge ℓ of G, there is a vertex $V \in I_\varphi(j, i)$, which is the i-th vertex of ℓ. Then, any edge ℓ of G has a vertex $V \in I_\varphi(j, i)$. Thus it follows that

$$s = j\left|I_\varphi(j, i)\right|,$$

as it can be proved also by (2). So the following theorem holds

Theorem 3.4. *Let $c(\varphi, i)$ be a strong 2-colouring of a simple graph G. Then the colours of G are 0 and j, $j > 0$, and the following holds:*

a) *two distinct vertices of $I_\varphi(0,i)$ are not adjacent;*

b) *for any edge ℓ of G, there is a vertex of $I_\varphi(j,i)$ which is i-th vertex of ℓ;*

c) $\left|I_\varphi(j,i)\right| = \dfrac{s}{j}$, *where s is the number of edges of G.*

Example 3.5. We provide an example of a strongly 2-colourable graph whose colours are $j_1 = 0$ and $j_2 = 3$.

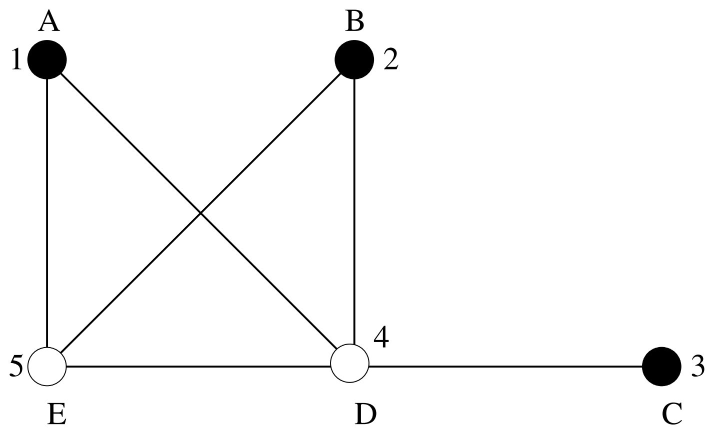

Figure 1: Example 3.5

$$\varphi : (1,2,3,4,5) \to (A,B,C,D,E), \quad I_\varphi(0,2) = \{A,B,C\}, \quad I_\varphi(3,2) = \{D,E\}.$$

We remark that this strong colouring is not classical, since the two adjacent vertices D and E have both the colour 3. Moreover $c(\varphi,1)$ is a strong 3-colouring of G with colours 0,1,2, since $I_\varphi(0,1) = \{E,D\}$, $I_\varphi(1,1) = \{C\}$, $I_\varphi(2,1) = \{A,B\}$. This confirms that the strong colouring depends on i.

4. Strong colourings of regular simple graphs

A graph is *regular* if all its vertices have the same degree.

Here we consider the strong colourings $c(\varphi, i)$ of a regular simple graph. The following theorem holds

Theorem 4.1. *A strong colouring $c(\varphi,i)$ of a regular simple graph G of positive degree r has at least the colours $j_1 = 0$ and $j_2 = r$.*

Proof. Let $c(\varphi,i)$ be a strong colouring of a regular simple graph G of degree $r > 0$. Let V_M and V_m be the vertices of G such that $\varphi^{-1}(V_M) = |G| = v$, $\varphi^{-1}(V_m) = 1$. We remark that $V_M \neq V_m$, since $|G| \geq 2$ (we have $|G| \geq 2$, since $r > 0$). Then $I_\varphi(r,1) = I_\varphi(0,2) \neq \emptyset$, since $V_m \in I_\varphi(r,1)$. Moreover $I_\varphi(0,1) = I_\varphi(r,2) \neq \emptyset$, since $V_M \in I_\varphi(0,1)$. It follows that $j_1 = 0$ and $j_2 = r$ are colours of $c(\varphi,i)$. □

Theorem 4.2. *Let G be a regular simple graph of positive degree. Then G is strongly 2-colorable if, and only if, G is a bipartite graph $G(\mathcal{V}_1,\mathcal{V}_2)$, with $|\mathcal{V}_1| = |\mathcal{V}_2| = |G|/2$.*

Proof. Let G be strongly 2-colourable and let $c(\varphi,i)$ be a strong 2-colouring of G. By Theorem 4.1 it follows that the colours of $c(\varphi,i)$ are $j_1 = 0$ and $j_2 = r$. Since the colours are two, we have $I_\varphi(r,1) = I_\varphi(0,2)$, and $I_\varphi(0,1) = I_\varphi(r,2)$. By Theorem 3.4 it follows that two distinct vertices of $I_\varphi(r,i)$ are not adjacent. By (1) and (2) and since in a regular graph of degree r it is $s = vr/2$, we have

$$t(r,i) = t(0,i) = \frac{v}{2}. \tag{3}$$

Then by the previous arguments, it follows that G is a bipartite graph $G(\mathcal{V}_1,\mathcal{V}_2)$, with

$$|\mathcal{V}_1| = t(r,i) = |\mathcal{V}_2| = t(0,i) = v/2.$$

Conversely, let $G = G(\mathcal{V}_1,\mathcal{V}_2)$ be a bipartite regular simple graph of degree $r > 0$.

Let $\mathcal{V}_1 = \{V_1,V_2,\ldots,V_m\}$, $\mathcal{V}_2 = \{V_{m+1},V_{m+2},\ldots,V_v\}$. Let

$$\varphi : n \in \{1,2,\ldots,v\} \longmapsto V_n \in \mathcal{V}_1 \cup \mathcal{V}_2.$$

The strong colouring $c(\varphi,1)$ is a strong 2-colouring of G. For, through any vertex $V \in \mathcal{V}_1$ there are r edges admitting V as first vertex (and then all the vertices of $\mathcal{V}_1$ have the colour r) and as a consequence through any vertex $V' \in \mathcal{V}_2$ there is no edge admitting V' as first vertex (and all the vertices of $\mathcal{V}_2$ have the colour 0). □

This theorem holds also for the classical colourings of graphs.

Example 4.3. An example of a strongly 2-colourable graph of degree 2 (the colours are 0 and 2) is the following.

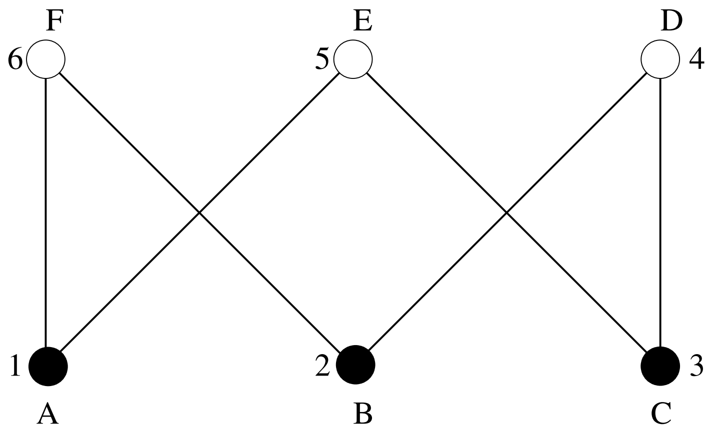

Figure 2: Example 4.3

$$
\begin{aligned}
V(G) &= \{A,B,C,D,E,F\},\\
E(G) &= \{\{A,F\},\{A,E\},\{B,F\},\{B,D\},\{C,E\},\{C,D\}\},\\
\varphi &: (1,2,3,4,5,6) \longrightarrow (A,B,C,D,E,F); \quad i=2,\\
I_\varphi(0,2) &= \{A,B,C\}, \quad I_\varphi(2,2) = \{D,E,F\}.
\end{aligned}
$$

Example 4.4. An example of strongly 2-colourable graph of degree 3 (the colours are 0 and 3) is the following.

$$
\begin{aligned}
V(G) &= \{A,B,C,D,E,F\},\\
E(G) &= \{\{A,F\},\{A,E\},\{A,D\},\{B,F\},\{B,E\},\{B,D\},\{C,F\},\\
&\qquad \{C,E\},\{C,D\}\},\\
\varphi &: (1,2,3,4,5,6) \longrightarrow (A,B,C,D,E,F); \quad i=2\\
I_\varphi(0,2) &= \{A,B,C\}, \quad I_\varphi(3,2) = \{E,F,D\}.
\end{aligned}
$$

By the definition of complete graph and by definition of $c(\varphi,i)$ the following theorem holds

Theorem 4.5. *Every strong colouring $c(\varphi,i)$ of a complete graph K_n is a strong n-colouring, that is distinct vertices of K_n have different colours.*

This theorem holds also for the classical colourings. Let G be a strongly 3-colourable regular simple graph, of degree $r > 0$. Let $c(\varphi,i)$ be a strong 3-colouring of G. By Theorem 4.1, the colours of $c(\varphi,i)$ are $0, j, r$ with $0 < j < r$.

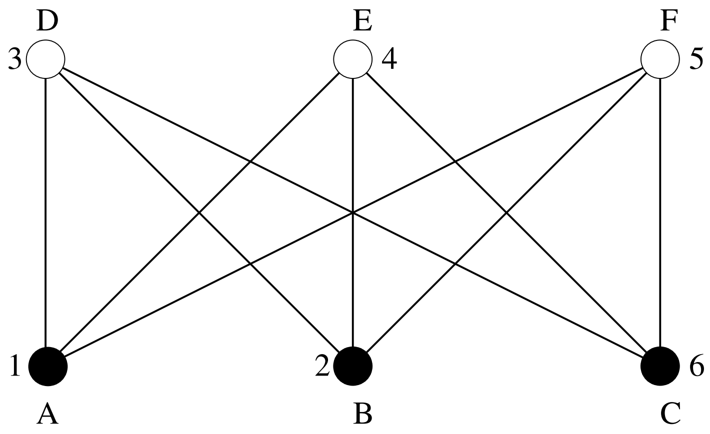

Figure 3: Example 4.4

It is

$$c(\varphi,i)=\left(\left\{I_\varphi(0,i),I_\varphi(j,i),I_\varphi(r,i)\right\},\{0,j,r\}\right).$$

Let us prove the following

Theorem 4.6. *Let G be a regular simple graph of degree $r>0$. Let $c(\varphi,i)$ be a strong 3-colouring of G. Then the following inequalities hold:*

$$\begin{aligned} r-\left|I_\varphi(r,i)\right| &\le j\le \left|I_\varphi(0,i)\right|,\\ \left|I_\varphi(0,i)\right|+\left|I_\varphi(r,i)\right| &\ge r,\\ \left|I_\varphi(j,i)\right| &\le v-r.\end{aligned}$$

If in the last inequality the equality holds, then

$$j=\left|I_\varphi(j,i)\right|.$$

Proof. Let us prove that $j\le\left|I_\varphi(0,i)\right|$. If $r\le\left|I_\varphi(0,i)\right|$, we get $j<\left|I_\varphi(0,i)\right|$. If $r>\left|I_\varphi(0,i)\right|$, by Theorem 3.3 it immediately follows that $j\le\left|I_\varphi(0,i)\right|$. The strong colouring $c(\varphi,i')$ with $i'=\{1,2\}-\{i\}$, has obviously the colours $0,r-j,r$. Therefore

$$c\left(\varphi,i'\right)=\left(\left\{I_\varphi(0,i'),I_\varphi(r-j,i'),I_\varphi(r,i')\right\},\{0,r-j,r\}\right),$$

where

$$\begin{aligned} I_\varphi(0,i') &= I_\varphi(r,i),\\ I_\varphi(r-j,i') &= I_\varphi(j,i),\\ I_\varphi(r,i') &= I_\varphi(0,i).\end{aligned}$$

Applying to $c(\varphi, i')$ the arguments of $c(\varphi, i)$, we get

$$r - j \leq \left| I_\varphi(0, i') \right| = \left| I_\varphi(r, i) \right|. \tag{4}$$

By (4) it follows

$$j \geq r - \left| I_\varphi(r, i) \right|.$$

Thus

$$r - \left| I_\varphi(r, i) \right| \leq j \leq \left| I_\varphi(0, i) \right| \tag{5}$$

and so

$$\left| I_\varphi(0, i) \right| + \left| I_\varphi(r, i) \right| \geq r,$$

hence

$$\left| I_\varphi(j, i) \right| \leq v - r.$$

If $I_\varphi(j, i) = v - r$ we get

$$j = \left| I_\varphi(0, i) \right|.$$

□

Theorem 4.7. *Let G be a regular simple graph of degree r, r an odd prime, $c(\varphi, i)$ a strong 3-colouring of G, then*

$$\left| I_\varphi(j, i) \right| \equiv 0 \mod r.$$

Proof. By (2) we get:

$$j \left| I_\varphi(j, i) \right| + r \left| I_\varphi(r, i) \right| = \frac{vr}{2}. \tag{6}$$

By (6) and since r is odd, it follows

$$j \left| I_\varphi(j, i) \right| \equiv 0 \mod r.$$

The integers j and r are coprime, since r is prime and $0 < j < r$. It follows that $\left| I_\varphi(j, i) \right| \equiv 0 \mod r$ and so the theorem is proved. □

Theorem 4.8. *Let G be a regular simple graph of degree $r = p^h$, $h \geq 1$, p a prime, $|G| = v$ even, $v \leq 2r$. Let $c(\varphi, i)$ be a strong 3-colouring of G with colours $0, j, r$, $0 < j < r$. We get either:*

i) $v = 2r$, $\left| I_\varphi(j, i) \right| = r$, $j = \left| I_\varphi(0, i) \right|$,

or

ii) $j = p^{h'}$, $1 \leq h' < h$.

It follows that if $h = 1$, only i) occurs.

Proof. By (2) it follows

$$j\left|I_\varphi(j,i)\right| \equiv 0 \mod r. \tag{7}$$

a) $\left|I_\varphi(j,i)\right| = kr$, with k positive integer;

b) $\left|I_\varphi(j,i)\right| \neq kr$.

In the case a) we remark that $k = 1$. For, assume $k \geq 2$. By (1), since $\left|I_\varphi(0,i)\right| \geq 1$, $\left|I_\varphi(r,i)\right| \geq 1$, it follows

$$v \geq 2r+2,$$

a contradiction, since $v \leq 2r$. Therefore

$$\left|I_\varphi(j,i)\right| = r. \tag{8}$$

By (8) and by the third inequality of Theorem 4.6 we get $r \leq v-r$, that is

$$v \geq 2r, \tag{9}$$

hence

$$v = 2r. \tag{10}$$

By (8) and (10) it follows

$$\left|I_\varphi(j,i)\right| = r = v-r. \tag{11}$$

By (11) and Theorem 4.6 it follows $j = \left|I_\varphi(0,i)\right|$.
In the case b), by (7) it follows $\gcd(j,r) \neq 1$, since both the integers j and $r = p^h$ have at least the factor p in common. Then, since $0 < j < r$, it follows $j = p^{h'}$, $1 \leq h' < h$. □

Example 4.9. We provide some examples concerning Theorems 4.7 and 4.8.

$$\begin{aligned}
V(G) &= \{A,B,C,D,E,F\},\\
E(G) &= \{\{A,D\},\{A,B\},\{A,C\},\{D,E\},\{D,F\},\\
&\qquad \{F,E\},\{C,F\},\{B,E\},\{B,C\}\},\\
&\varphi : (1,2,3,4,5,6) \longrightarrow (A,D,F,B,E,C), \quad i=2.
\end{aligned}$$

The colouring $c(\varphi,2)$ is a strong 3-colouring of G with colours $0,1,3$. For,

$$I_\varphi(0,2) = \{A\}, \quad I_\varphi(1,2) = \{B,D,F\}, \quad I_\varphi(3,2) = \{C,E\}.$$

This colouring is not classical, since the adjacent vertices D and E have the same colour.

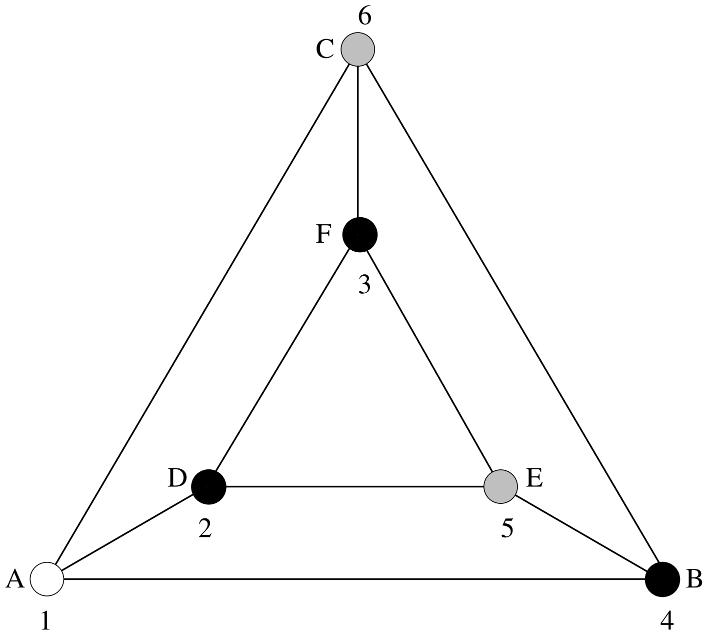

Figure 4: Example 4.9

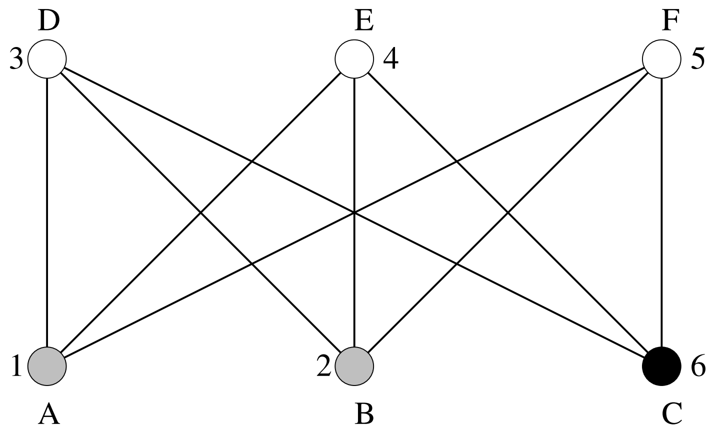

Figure 5: Example 4.10

Example 4.10. This graph G is the complete bipartite graph $K_{3,3}$.

$$V(G) = \{A,B,C,D,E,F\},$$
$$E(G) = \{\{A,D\},\{A,E\},\{A,F\},\{B,D\},\{B,E\},$$
$$\{B,F\},\{C,D\},\{C,E\},\{C,F\}\},$$
$$\varphi : (1,2,3,4,5,6) \longrightarrow (A,B,D,E,F,C), \quad i=1.$$

The strong colouring $c(\varphi,1)$ is a strong 3-colouring of colours $0,1,3$.

$$I_\varphi(0,1) = \{C\}, \quad I_\varphi(1,1) = \{D,E,F\}, \quad I_\varphi(3,1) = \{A,B\}.$$

Example 4.11. Cubic Petersen Graph.

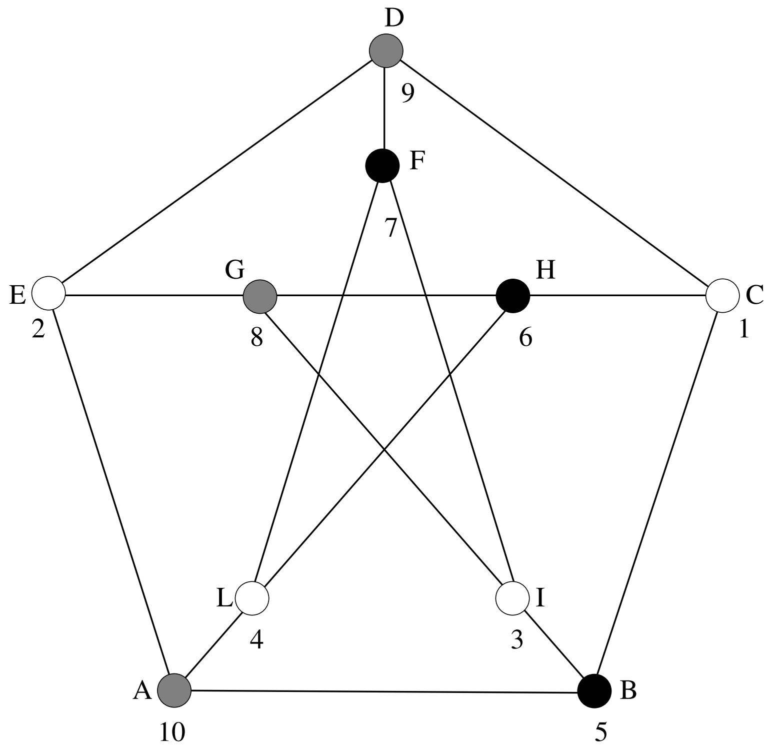

Figure 6: Example 4.11

$$V(G) = \{A,B,C,D,E,F,G,H,I,L\},$$
$$E(G) = \{\{A,B\},\{B,C\},\{C,D\},\{D,E\},\{E,A\},\{E,G\},\{A,L\},\{B,I\},$$
$$\{C,H\},\{D,F\},\{G,I\},\{G,H\},\{F,L\},\{F,I\},\{L,H\}\},$$
$$\varphi : (1,2,3,4,5,6,7,8,9,10) \longrightarrow (C,E,I,L,B,H,F,G,D,A); \quad i=2.$$

The strong colouring $c(\varphi, 2)$ is a strong 3-colouring with colours $0, 2, 3$, since

$$\begin{aligned} I_\varphi(0,2) &= \{C, E, L, I\}, \\ I_\varphi(2,2) &= \{B, F, H\}, \\ I_\varphi(3,2) &= \{A, D, G\}. \end{aligned}$$

Example 4.12. Graph with 3-strong colourings of colours 0, 2, 3.

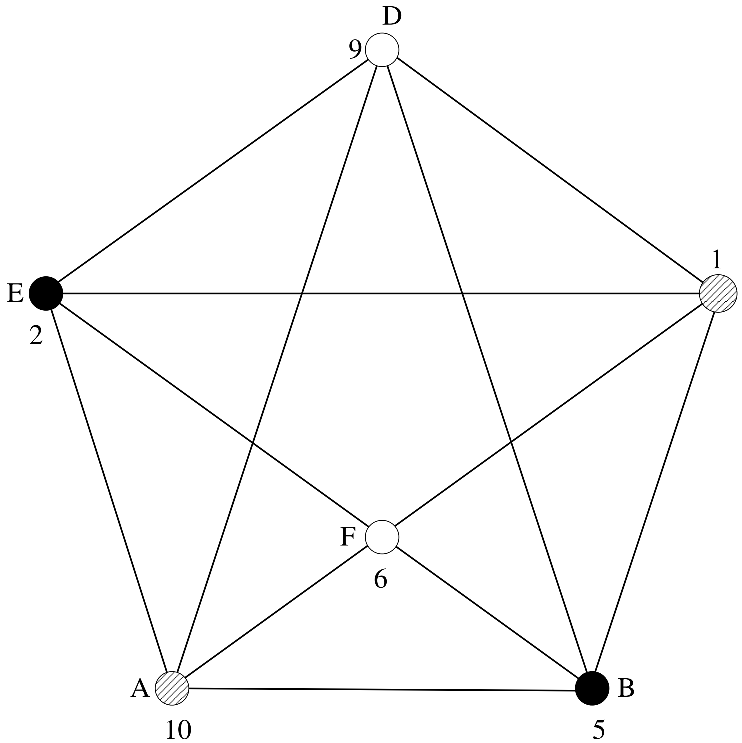

Figure 7: Example 4.12

$$\begin{aligned} V(G) &= \{A, B, C, D, E, F\}, \\ E(G) &= \{\{A, B\}, \{B, C\}, \{C, D\}, \{D, E\}, \{A, E\}, \{E, F\}, \\ &\qquad \{C, F\}, \{B, F\}, \{A, F\}, \{A, D\}, \{B, D\}\}, \\ &\varphi : (1, 2, 3, 4, 5, 6) \longrightarrow (A, C, B, E, F, D); \quad i = 1. \end{aligned}$$

We have a 3-colouring of colours $0, 2, 3$, with

$$I_\varphi(0,1) = \{D, F\},$$
$$I_\varphi(2,1) = \{B, E\},$$
$$I_\varphi(3,1) = \{A, C\}.$$

This example satisfies the hypotheses of Theorem 4.8 and ii) holds, but not i).

Example 4.13. An example of a non-standard colouring.

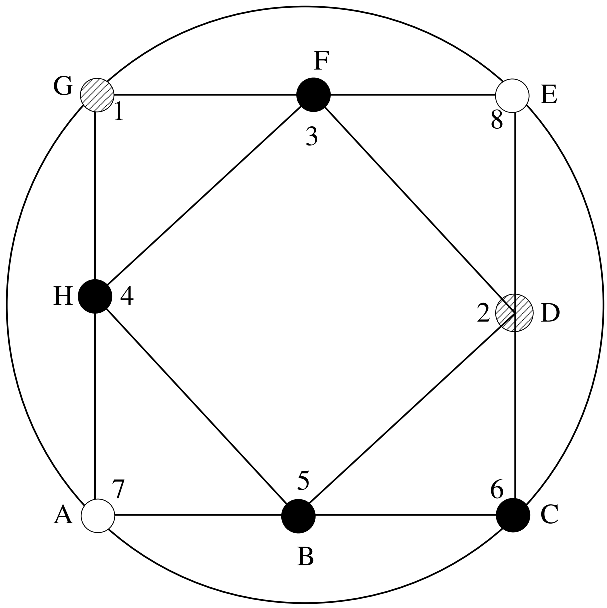

Figure 8: Example 4.13

$$V(G) = \{A, B, C, D, E, F, G, H\},$$
$$E(G) = \{\{A,B\},\{B,C\},\{C,D\},\{D,E\},\{E,F\},\{F,G\},\{G,H\},\{H,A\},$$
$$\{H,B\},\{B,D\},\{D,F\},\{F,H\},\{A,C\},\{C,E\},\{E,G\},\{G,A\}\},$$
$$\varphi : (1,2,3,4,5,6,7,8) \longrightarrow (G,D,F,H,B,C,A,E); \quad i = 1,$$

$$I_\varphi(0,1) = \{A, E\},$$
$$I_\varphi(2,1) = \{B, C, F, H\},$$
$$I_\varphi(3,1) = \{D, G\}.$$

This graph satisfies the hypotheses of Theorem 4.8 and both i) and ii) hold. Moreover this strong colouring is not classical, since the adjacent vertices B and C have the same colour.

By Theorem 4.8 it follows

Theorem 4.14. *Let G be a simple regular graph of degree $r = p$, p a prime, $|G| = v$, $v < 2r$. Then strong 3-colourings of G do not exist.*

Example 4.15. We provide an example of a graph satisfying the hypoteses of Theorem 4.14 and therefore not admitting a strong 3-colouring.

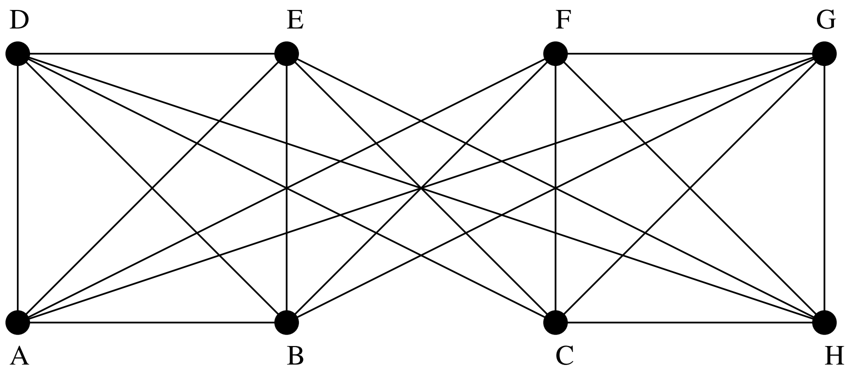

Figure 9: Example 4.15

$$\begin{aligned} V(G) &= \{A,B,C,D,E,F,G,H\}, \\ E(G) &= \{\{A,H\},\{A,G\},\{A,F\},\{A,E\},\{B,H\},\{B,G\},\{B,F\}, \\ &\quad \{B,E\},\{C,H\},\{C,G\},\{C,F\},\{C,E\},\{D,H\},\{D,G\}, \\ &\quad \{D,F\},\{D,E\},\{A,B\},\{C,D\},\{F,E\},\{G,H\}\}. \end{aligned}$$

5. Hamiltonian strong colourings of regular simple graphs

A *path* of a graph G is a finite sequence of edges such as

$$V_1V_2, V_2V_3, \dots, V_mV_{m+1},$$

denoted also

$$V_1 \to V_2 \to \cdots \to V_m \to V_{m+1},$$

where the edges and the vertices are distinct (may be, evenctually, $V_1 = V_{m+1}$).

A graph G is called *semi-hamiltonian* if there is a path through every vertex of G. If the path is closed, G is called *hamiltonian.*
Let G be a simple semi-hamiltonian graph and ℓ be a path through every vertex of G.

Let $\mathcal{V}$ be the set of vertices of G and let $v = |\mathcal{V}|$. Let

$$\ell = V_1 \to V_2 \to \cdots \to V_v.$$

The following bijection arises

$$\varphi_\ell : n \in \mathcal{I} = \{1,2,\ldots,v\} \longmapsto V_n \in \mathcal{V}.$$

For every $i \in \{1,2\}$, we get the strong colouring $c(\varphi_\ell, i)$ which is called *hamiltonian strong colouring of index i associated with* ℓ.

By Theorem 4.1 we have that, if G is regular of degree $r > 0$, the strong colouring $c(\varphi_\ell, i)$ has the colours 0 and r. If $c(\varphi_\ell, i)$ is a hamiltonian strong colouring, the following theorem holds

Theorem 5.1. *Let G be a semi-hamiltonian regular simple graph of positive degree r and let $c(\varphi_\ell, i)$ a hamitonian strong colouring of G, with $\ell = V_1 \to V_2 \to \cdots \to V_v$, $v = |G|$. Then there is a unique vertex of colour 0, which is V_1 and a unique vertex of colour r, which is V_r.*

Proof. Let $i = 1$. Then V_1 has the colour r, and V_v has the colour 0. Any vertex V_n, $1 < n < v$, has a colour which is neither 0, nor r. Therefore, V_1 and V_v are the only vertices with colours r and 0, respectively. The same result holds in the case $i = 2$, but V_1 has the colour 0. and V_v has the colour r. □

Theorem 5.2. *Semi-hamiltonan regular simple graphs of positive degree having a hamiltonian strong 1-colouring do not exists.*

Proof. This result follows by Theorem 3.2, since a hamltonian graph cannot be the null graph. □

Theorem 5.3. *The only semi-hamiltonian regular simple graph of positive degree admitting a hamiltonian strong 2-colouring is K_2.*

Proof. Let G be a regular simple graph of positive degree having a hamiltonian strong 2-colouring $c(\varphi_\ell, i)$ and let $\ell = V_1 \to V_2 \to \cdots \to V_v$, where $v = |G|$. By Theorem 5.1 it follows

$$\ell = V_1 \to V_2$$

then $G = K_2$. Conversely, K_2 is a regular simple graph of degree 1 having a hamiltonian strong 2-colouring with colours 0 and 1. □

Theorem 5.4. *The only semi-hamiltonian regular simple graphs of positive degree having a hamiltonian strong 3-colouring are the circuit-graphs.*

Proof. Let G be a semi-hamiltonian regular simple graph of positive degree r having a hamiltonian strong 3-colouring $c(\varphi_\ell, i)$, with $\ell = V_1 \to V_2 \to \cdots \to V_v$, where $v = |G|$. We get $r > 1$, since $r = 1$ implies $\ell = V_1 \to V_2$ and then $v = 2$: a contradiction, since G admits a strong 3-colouring, then $r \geq 2$. By Theorem 5.1 it follows that $\left|I_\varphi(r,i)\right| = \left|I_\varphi(0,i)\right| = 1$. By Theorem 4.6 it follows that $r \leq \left|I_\varphi(0,i)\right| + \left|I_\varphi(r,i)\right| = 2$. Then $r = 2$ and G is a connected regular simple graph of degree 2 and then a circuit-graph. The converse is obvious, since a simple circuit-graph admits a hamiltonian strong 3-colouring with colours $0,1,2$. $\square$

We remark that in the case of classical colourings there is no characterization of strongly 3-colourable graphs.

Theorem 5.5. *Let G, $v = |G|$, be a semi-hamiltonian regular simple graph of positive degree r admitting a hamiltonian strong 4-colouring $c(\varphi_\ell, i)$. Then $r \geq 3$ and the colours of $c(\varphi_\ell, i)$ are $0, 1, r-1, r$. Moreover, the number of vertices with colour 1 equals that of vertices of colour $r-1$. This number is $\frac{v}{2} - 1$, hence v is even.*

Proof. Let G be a semi-hamiltonian regular simple graph of degree $r > 0$ admitting a hamiltonian strong 4-colouring $c(\varphi_\ell, i)$ and let $\ell = V_1 \to V_2 \to \cdots \to V_v$, where $v = |G|$. Obviously $v \geq 4$. Let $0, j_1, j_2, r$ be the colours of $c(\varphi_\ell, i)$, $0 < j_1 < j_2 < r$. It is $r \geq 3$. By Theorem 5.1 it follows $\left|I_{\varphi_\ell}(0,i)\right| = 1$. By Theorem 3.3 it follows the existence of a colour $j \neq 0$ such that $j \leq \left|I_{\varphi_\ell}(0,i)\right| = 1$. Therefore $j_1 = 1$. Let us consider the strong colouring $c(\varphi_\ell, i')$, $i' = \{1,2\} - \{i\}$. The colours of $c(\varphi_\ell, i')$ are $0, r-j_2, r-j_1 = r-1, r$, with $0 < r-j_2 < r-j_1 = r-1 < r$. By Theorem 5.1 it follows $\left|I_{\varphi_\ell}(0,i')\right| = 1$. By Theorem 3.3 it follows the existence of a colour $j \neq 0$ such that $j \leq \left|I_{\varphi_\ell}(0,i')\right| = 1$. Therefore $r - j_2 = 1$, that is $j_2 = r-1$. So the colours of $c(\varphi_\ell, i)$ are $0, 1, r-1, r$. By (1) and (2), we get:

$$\begin{aligned} t(0,i) + t(1,i) + t(r-1,i) + t(r,i) &= v, \\ t(1,i) + (r-1)t(r-1,i) + rt(r,i) &= \frac{vr}{2}. \end{aligned} \tag{12}$$

By Theorem 5.1 it follows

$$t(0,i) = t(r,i) = 1. \tag{13}$$

By (12) and (13) we get:

$$\begin{aligned} t(1,i) + t(r-1,i) &= v-2, \\ t(1,i) + (r-1)t(r-1,i) &= \frac{vr}{2} - r. \end{aligned} \tag{14}$$

By (14) we get

$$(r-2)t(r-1,i) = \frac{v(r-2)}{2} - (r-2).$$

Since $r-2 \neq 0$ (it is $r \geq 3$), we get $t(r-1,i) = v/2-1$. By previous conditions we get $t(1,i) = v/2-1$. Since

$$t(j,i) = \left|I_{\varphi_\ell}(j,i)\right|, \quad j = 0,1,\dots r,$$

the Theorem is proved. □

By Theorems 5.2, 5.3, 5.4, 5.5 it follows immediately

Theorem 5.6 (Theorem of cubic simple graphs)**.** *Let G be a semi-hamiltonian regular simple graph of degree 3 and let $c(\varphi_\ell,i)$ be a hamiltonian strong colouring of G. Then $c(\varphi_\ell,i)$ is a strong 4-colouring with colours $0,1,2,3$. Moreover, the number of vertices of colour 1 equals the number of vertices of colour 2. This number is $\frac{v}{2}-1$, hence $v = |G|$ is even.*

Example 5.7. This example is an explanation of Theorem 5.6.

Now let G be a semi-hamiltonian regular simple graph of degree $r > 0$ admitting a hamiltonian strong 5-colouring $c(\varphi_\ell,i)$. Like in Theorem 5.5, we prove that $r \geq 4$ and that the colours of $c(\varphi_\ell,i)$ are $0,1,j,r-1,r$, with $1 < j < r-1$. By (1) and (2), we get:

$$\begin{aligned} t(0,i)+t(1,i)+t(j,i)+t(r-1,i)+t(r,i) &= v, \\ t(1,i)\, + jt(j,i)+(r-1)t(r-1,i)+rt(r,i) &= \frac{vr}{2}, \end{aligned} \tag{15}$$

where $v = |G|$. By Theorem 5.1 it follows

$$t(0,i) = t(r,i) = 1. \tag{16}$$

By (15) and (16) we have

$$\begin{aligned} t(1,i)+t(j,i)+t(r-1,i) &= v-2, \\ t(1,i)\, + jt(j,i)+(r-1)t(r-1,i) &= \frac{vr}{2} - r. \end{aligned} \tag{17}$$

By (17), we have:

$$j\left[t(1,i)+t(r-1,i)-(v-2)\right] = t(1,i)+(r-1)t(r-1,i) - \frac{r}{2}(v-2).$$

Since $t(1,i)+t(r-1,i) \leq v-3$, the integer $t(1,i)+t(r-1,i)-(v-2)$ is negative, then we get

$$j = \frac{t(1,i)+(r-1)t(r-1,i)-\frac{r}{2}(v-2)}{t(1,i)+t(r-1,i)-(v-2)} > 1. \tag{18}$$

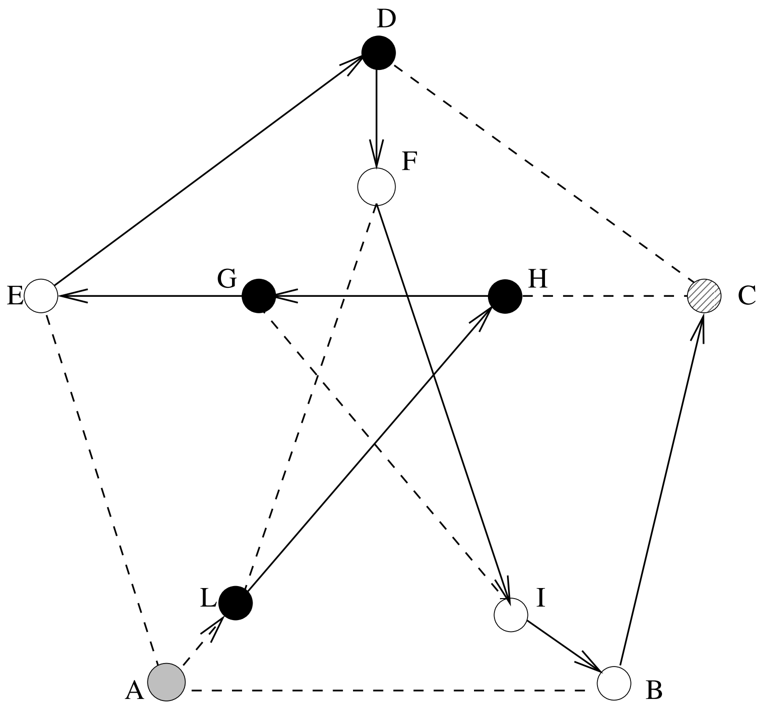

Figure 10: Example 5.7

By (18), since $r-2>0$, we get

$$t(r-1,i) < \frac{v}{2} - 1. \tag{19}$$

We denote by $c(\varphi_\ell, i')$ the hamiltonian strong 5-colouring, with $i' = \{1,2\} - \{i\}$, whose colours are $0, 1, r-j, r-1, r$. Applying (19), to this strong colouring, since $t(r-1,i') = t(1,i)$, we get

$$t(1,i) < \frac{v}{2} - 1. \tag{20}$$

Now assume v even. By (19) and (20) we have

$$\begin{aligned} t(r-1,i) \leq & \quad \frac{v}{2} - 2, \\ t(1,i) \leq & \quad \frac{v}{2} - 2. \end{aligned} \tag{21}$$

By the first of (17) and by (21) we get

$$v - 2 - t(j,i) = t(1,i) + t(r-1,i) \leq v - 4. \tag{22}$$

Then
$$t(j,i) \geq 2.$$
If $t(j,i) = 2$, by (21) and (22) we get
$$t(1,i) = t(r-1,i) = \frac{v}{2} - 2. \tag{23}$$
By (18) and (23) it follows
$$j = \frac{r}{2}.$$

Then the following theorem holds

Theorem 5.8. *Let G be a semi-hamiltonian regular simple graph of positive degree r admitting a hamiltonian strong 5-colouring $c(\varphi_\ell, i)$ and let $v = |G|$. Then $r \geq 4$, the colours of $c(\varphi_\ell, i)$ are $0, 1, j, r-1, r$, with $1 < j < r-1$. The number of the vertices of colour 1 and that of the vertices of colour $r-1$ are both less than $v/2-1$. If v is even, the number of vertices of colour j is greater than or equal 2 and if this number equals 2, we get $j = r/2$. Therefore r is even and the number of vertices of colour 1 and that of vertices of colour $r-1$ are both equal to $v/2-2$.*

Example 5.9. This example provides a hamiltonian strong 5-colouring with $i = 1$ of a regular simple graph of degree 4 with 6 vertices. The colours are

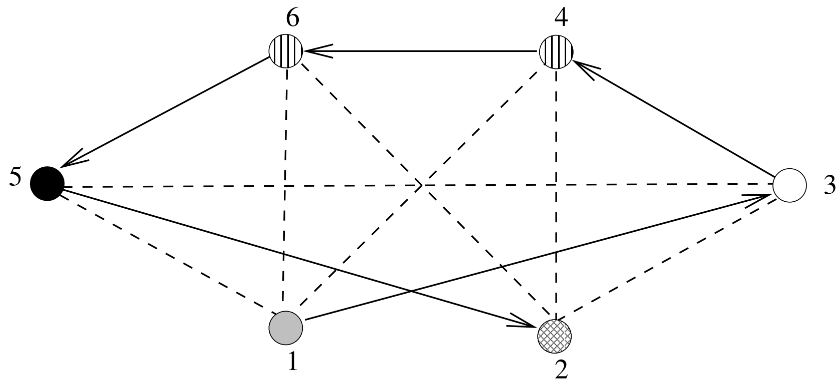

Figure 11: Example 5.9

$0, 1, 2, 3, 4$ and $j = 2 = r/2$.
This strong colouring is not classical, since there are two adjacent vertices having the same colour 2. We remark that in the case of classical colourings, we have no result concerning 4-colourings and 5-colourings.

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