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ON GRÜSS TYPE INEQUALITY FOR A HYPERGEOMETRIC FRACTIONAL INTEGRAL

SHYAM L. KALLA - ALKA RAO

Aim of the present paper is to investigate a new integral inequality of Grüss type for a hypergeometric fractional integral. Two main results are proved, the first one deals with Grüss type inequality using the hypergeometric fractional integral. The second result states another inequality regarding two synchronous functions.

1. Introduction

The well known Grüss inequality [4] (see also [12], p. 296) is stated as follows: Let f and g be two functions defined and integrable on $[a,b]$. Further, let

$$\ell \leq f(x) \leq L, \qquad m \leq g(x) \leq M, \tag{1}$$

for each $x \in [a,b]$, where ℓ, m, L, M are given real constants, then

$$\left| \frac{1}{(b-a)} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)} \int_a^b f(x)dx \cdot \frac{1}{(b-a)} \int_a^b g(x)dx \right| \leq \frac{1}{4}(L-\ell)(M-m) \tag{2}$$

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and the constant $1/4$ is the best possible.

The inequality (2) has various generalizations, that have appeared in the literature, we refer to a few, e.g. [1, 3, 11, 12, 13, 14] and the references cited therein.

Besides the Reimann-Liouville definition of fractional operators many more modifications and generalizations have been studied in the literature. Several authors e.g. [5, 6, 7, 9, 10, 15] have studied and used different modifications of hypergeometric fractional integral operators, for further details we refer to [8, 9, 16]. Recently, in [1] a generalization of Grüss inequality for Reimann-Liouville fractional integral has been studied.

In the present work we shall investigate a fractional integral over the space C_λ introduced in [2] and defined as follows:

Definition 1.1. The space of functions C_λ, $\lambda \in \mathbb{R}$ consists of all functions $f(x)$, $x > 0$, that can be represented in the form $f(x) = x^p f_1(x)$ with $p > \lambda$ and $f_1 \in C[0,\infty)$, where $C[0,\infty)$ is the set of continuous functions in the interval $[0,\infty)$.

We define a fractional integral $K^{\alpha,\beta,\eta}$ associated with the Gauss hypergeometric function as follows:

Definition 1.2. Let $f \in C_\lambda$; for $\alpha > \max\{0, -(\eta+1)\}$, $\eta - \beta > -1$, $\beta < 1$, we define a fractional integral $K^{\alpha,\beta,\eta} f$ as follows:

$$\left(K^{\alpha,\beta,\eta} f\right)(x) = \frac{\Gamma(1-\beta)\Gamma(\alpha+\eta+1)}{\Gamma(\eta-\beta+1)}\, x^\beta \left(I_{0+}^{\alpha,\beta,\eta} f\right)(x), \tag{3}$$

where $I_{0+}^{\alpha,\beta,\eta} f$ is the right -hand sided Gauss hypergeometric fractional integral of order α and defined as:

Definition 1.3. Let $\alpha > 0$, $\beta, \eta \in R$ then the right-hand sided Gauss hypergeometric fractional integral of order α for a real valued continuous function $f(x)$ on $(0,\infty)$ is defined as [15]:

$$I_{0+}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt, \tag{4}$$

The above integral has following property:

$$I_{0+}^{\alpha,\beta,\eta} I_{0+}^{\gamma,\delta,\zeta} f(x) = I_{0+}^{\gamma,\delta,\zeta} I_{0+}^{\alpha,\beta,\eta} f(x)\,. \tag{5}$$

Erdélyi-Kober and Riemann-Liouville fractional integrals of order α, denoted by $I_{0+}^{\alpha,\eta}$ *and* I_{0+}^{α} respectively, are obtained by the following relation:

$$I_{0+}^{\alpha,0,\eta} f(x) = I_{0+}^{\alpha,\eta} f(x) \quad \text{and} \quad I_{0+}^{\alpha,-\alpha,\eta} f(x) = I_{0+}^{\alpha} f(x), \tag{6}$$

where Erdélyi-Kober fractional integral of order α is defined as:

$$\left(I_{0+}^{\alpha,\eta} f\right)(x) = \frac{x^{-(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, dt, \ \alpha > 0, \eta \in R\,. \tag{7}$$

The above Erdélyi-Kober integral operator has been used by many authors, in particular, to obtain solutions of the single, dual and triple integral equations possessing special functions of mathematical physics as their kernels. For the theory and applications of the Erdélyi-Kober (E-K) fractional integrals, we refer to [5, 8, 9, 16, 17].
The Riemann-Liouville fractional integral I_{0+}^{α} of order α is defined as (see [9, 10, 16]):

$$\left(I_{0+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \ \alpha > 0\,. \tag{8}$$

Remark 1.4. We obtain the following special cases of the operator $K^{\alpha,\beta,\eta}$

When $\beta = 0$, it reduces to an Erdélyi-Kober type fractional integral of order α and defined for $\alpha > 0$; $\eta > -1$ as:

$$K^{\alpha,\eta} = \frac{\Gamma(\eta+\alpha+1)}{\Gamma(\eta+1)} \left(I_{0+}^{\alpha,\eta} f\right)(x).$$

When $\beta = -\alpha$, then it is described for $\alpha > 0$ as:

$$\left(K^{\alpha} f\right)(x) = \frac{\Gamma(1+\alpha)}{x^{\alpha}} \left(I_{0+}^{\alpha} f\right)(x),$$

where $I_{0+}^{\alpha,\eta}$ and I_{0+}^{α} are given by (7) and (8) respectively.

Definition 1.5. Two functions f and g are said to be synchronous functions on $[0,\infty)$ if

$$A(u,v) = (f(u) - f(v))(g(u) - g(v)) \geq 0; u, v \in [0,\infty)\,. \tag{9}$$

Here we discuss some results regarding the fractional integral $K^{\alpha,\beta,\eta}$ which have been used in the present work.

Lemma 1.6. *For* $\mu > \max\{0, -(\eta-\beta)\} - 1,\ \alpha > \max\{0, -(\eta+1)\}$; *and* $\eta - \beta > -1$, $\beta < 1$

$$K^{\alpha,\beta,\eta}(x^{\mu}) = \frac{\Gamma(\mu+1)\Gamma(\mu+\eta-\beta+1)\Gamma(1-\beta)\Gamma(\alpha+\eta+1)}{\Gamma(\mu-\beta+1)\Gamma(\alpha+\mu+\eta+1)\Gamma(\eta-\beta+1)}\, x^{\mu}. \tag{10}$$

Thus

$$K^{\alpha,\beta,\eta}(C) = C,$$

where C is a constant.

Lemma 1.7. *Let $h \in C_\lambda$ and $m, M \in R$: $m \leq h(x) \leq M$, for all $x \in [0,\infty)$; $\alpha >$ max $\{0,-(\eta+1)\}$; $\eta-\beta > -1$, $\beta < 1$, we have:*

$$K^{\alpha,\beta,\eta}h^2(x) - \left(K^{\alpha,\beta,\eta}h(x)\right)^2 =$$
$$\left(M - K^{\alpha,\beta,\eta}h(x)\right)\left(K^{\alpha,\beta,\eta}h(x) - m\right) - K^{\alpha,\beta,\eta}(M-h(x))(h(x)-m). \quad (11)$$

Proof. Let $h \in C_\lambda$ and $m, M \in R$: $m \leq f(x) \leq M$, for every x in $[0,\infty)$, then for any $u, v \in [0,\infty)$, we have

$$(M-h(u))(h(v)-m) + (M-h(v))(h(u)-m) - (M-h(u))(h(u)-m)$$
$$-(M-h(v))(h(v)-m) = h^2(u) + h^2(v) - 2h(u)h(v). \quad (12)$$

If $h \in C_\lambda$, then h is integrable on $[0, x]$;$x > 0$, thus, multiplying the above equation by $\frac{(x-u)^{\alpha-1}}{\Gamma(\alpha)}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{u}{x}\right)$; $u \in (0,x)$; $x > 0$ and integrating with respect to u from 0 to x, then using definition 1.2 and Lemma 1.6, we obtain

$$\left(M - K^{\alpha,\beta,\eta}h(x)\right)(h(v)-m) + (M-h(v))\left(K^{\alpha,\beta,\eta,}h(x) - m\right)$$
$$- K^{\alpha,\beta,\eta}(M-h(x))(h(x)-m) - (M-h(v))(h(v)-m)$$
$$= K^{\alpha,\beta,\eta}h^2(x) + h^2(v) - 2K^{\alpha,\beta,\eta}h(x)\,h(v) \quad (13)$$

Again, multiplying the above equation by

$$\frac{1}{\Gamma(\alpha)}(x-v)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;\,1-\frac{v}{x}\right); v \in (0,x)\,; x > 0,$$

then integrating with respect to v from 0 to x , we obtain (11). This proves the lemma. □

2. Main Results

Theorem 2.1. *Let f, $g \in C_\lambda$ satisfying the condition 1 on $[0,\infty)$. Then for all $x > 0$; $\alpha >$ max $\{0,-(\eta+1)\}$; $\eta-\beta > -1$; $\beta < 1$, we have*

$$\left|\, K^{\alpha,\beta,\eta}fg(x) - K^{\alpha,\beta,\eta}f(x)K^{\alpha,\beta,\eta}g(x)\,\right| \leq \frac{1}{4}(L-\ell)(M-m). \quad (14)$$

Proof. Let us define a function

$$A(u,v) = (f(u)-f(v))(g(u)-g(v))\,; u, v \in [0,x). \quad (15)$$

First we multiply the above expression by

$$\frac{(x-u)^{\alpha-1}(x-v)^{\alpha-1}}{(\Gamma(\alpha))^2}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{u}{x}\right){}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{v}{x}\right);$$

and then integrate twice with respect to u and v from 0 to x , we obtain the following result by applying (3), (4) and the property (5)

$$\frac{1}{(\Gamma(\alpha))^2}\int_0^x\int_0^x (x-u)^{\alpha-1}(x-v)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{u}{x}\right)$$
$$\times{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{v}{x}\right)A(u,v)\,du\,dv$$
$$=2K^{\alpha,\beta,\eta}fg(x)-2K^{\alpha,\beta,\eta}f(x)K^{\alpha,\beta,\eta}g(x). \quad (16)$$

Making use of the well known Cauchy-Schwarz inequality for linear operator [12, eq. (1.3), p. 296], we find that

$$\left(K^{\alpha,\beta,\eta}fg(x)-K^{\alpha,\beta,\eta}f(x)K^{\alpha,\beta,\eta}g(x)\right)^2\le$$
$$\left(K^{\alpha,\beta,\eta}f^2(x)-\left(K^{\alpha,\beta,\eta}f(x)\right)^2\right)\left(K^{\alpha,\beta,\eta}g^2(x)-\left(K^{\alpha,\beta,\eta}g(x)\right)^2\right) \quad (17)$$

Since

$$(L-f(x))(f(x)-\ell)\ge 0 \text{ and } (M-g(x))(g(x)-m)\ge 0,$$

therefore,

$$K^{\alpha,\beta,\eta}(L-f(x))(f(x)-\ell)\ge 0 \text{ and } K^{\alpha,\beta,\eta}(M-g(x))(g(x)-m)\ge 0 \quad (18)$$

Thus, by Lemma 1.7

$$K^{\alpha,\beta,\eta}f^2(x)-\left(K^{\alpha,\beta,\eta}f(x)\right)^2\le\left(L-K^{\alpha,\beta,\eta}f(x)\right)\left(K^{\alpha,\beta,\eta}f(x)-\ell\right), \quad (19)$$

and

$$K^{\alpha,\beta,\eta}g^2(x)-\left(K^{\alpha,\beta,\eta}g(x)\right)^2\le\left(M-K^{\alpha,\beta,\eta}g(x)\right)\left(K^{\alpha,\beta,\eta}g(x)-m\right). \quad (20)$$

From (17) and the inequalities (19), (20) we deduce that

$$\left(K^{\alpha,\beta,\eta}fg(x)-K^{\alpha,\beta,\eta}f(x)K^{\alpha,\beta,\eta}g(x)\right)^2\le$$
$$\left(L-K^{\alpha,\beta,\eta}f(x)\right)\left(K^{\alpha,\beta,\eta}f(x)-\ell\right)\left(M-K^{\alpha,\beta,\eta}g(x)\right)\left(K^{\alpha,\beta,\eta}g(x)-m\right). \quad (21)$$

Applying the well known inequality $4ab\le(a+b)^2$; $a,b\in R$ in the right - hand side of the inequality (21) and simplifying it, we obtain the result (14). This completes the proof. □

Theorem 2.2. *Let f and g be two synchronous functions on $[0,\infty)$, then the following inequality holds:*

$$K^{\alpha,\beta,\eta} fg(x) \ \geq \ K^{\alpha,\beta,\eta} f(x) K^{\alpha,\beta,\eta} g(x). \tag{22}$$

Proof. For the synchronous functions f and g, the inequality (9) holds for all $u, v \in [0, \infty)$. This implies that

$$f(u)g(u)+f(v)g(v) \geq f(u)g(v)+f(v)g(u). \tag{23}$$

Following the procedure of the Lemma 1.7 for applying the fractional integral $K^{\alpha,\beta,\eta}$, we arrive at the result (22). This completes the proof. □

3. Corollaries and applications

The Grüss inequality for the Erdélyi-Kober type fractional integral

If we put $\beta = 0$, in the Theorems 2.1 and 2.2, we obtain the following Grüss inequality for the Erdélyi-Kober type fractional integral $K^{\alpha,\eta}$ of order α given in Remark 1.4 (i).

Corollary 3.1. *Let f, $g \in C_\lambda$ satisfying the condition (1) on $[0,\infty)$. Then for all $x > 0;\ \alpha > 0;\ \eta > -1$, we have*

$$|\, K^{\alpha,\eta} fg(x) \ - \ K^{\alpha,\eta} f(x) K^{\alpha,\eta} g(x)\, | \ \leq \ \frac{1}{4}(L-\ell)(M-m). \tag{24}$$

Corollary 3.2. *Let f and g be two synchronous functions on $[0,\infty)$, then for all $x > 0;\ \alpha > 0,\ \eta > -1$, the following inequality holds:*

$$K^{\alpha,\eta} fg(x) \ \geq \ K^{\alpha,\eta} f(x) K^{\alpha,\eta} g(x). \tag{25}$$

Remark 3.3. If we take $\beta = -\alpha$ and apply the relations (6) and (8) for Riemann-Liouville fractional integral I^{α}_{0+}, then $K^{\alpha,\beta,\eta}$ reduces to the following form:

$$K^{\alpha,-\alpha,\eta} f(x) = \frac{\Gamma(1+\alpha)}{x^{\alpha}} I^{\alpha}_{0+} f(x), \alpha > 0; \tag{26}$$

and the Theorem 2.1 reduces to the result obtained in [1, Theorem 3.1].
Further, if we take $\alpha = 1$ in (26) and Theorem 2.1, we obtain the Grüss inequality (2) on $[0, x]$.
Also from Theorem 2.2 we obtain the following inequality for two synchronous functions on $[0,\infty)$ and Riemann-Liouville fractional integral:

$$I^{\alpha}_{0+} fg(x) \ \geq \ \frac{\Gamma(1+\alpha)}{x^{\alpha}} I^{\alpha}_{0+} f(x) I^{\alpha}_{0+} g(x);\ \alpha > 0.$$

Further, in Theorem 2.2 taking $\alpha = 1$, we obtain the following inequality for the two synchronous functions on interval [0, x]; $x > 0$ as:

$$\int_0^x f(x)\, g(x)\, dx \geq \frac{1}{x} \int_0^x f(x)\, dx \int_0^x g(x)\, dx.$$

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SHYAM L. KALLA
Vyas Institute of Higher Education,
Jodhpur 342008, INDIA.
e-mail: shyamkalla@gmail.com

ALKA RAO
Department of Mathematics,
Govt. P.G. College
Chittorgarh, Rajasthan, INDIA.
e-mail: alkrao.ar@rediffmail.com