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SUBORDINATION AND SUPERORDINATION PROPERTIES FOR ANALYTIC FUNCTIONS INVOLVING WRIGHT'S FUNCTIONS

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In the present investigation, we obtain some subordination and superordination results for the Hadamard product of certain normalized analytic functions in the open unit disk involving the linear operator introduced in [J. Dziok and R. K. Raina, Demonstratio Math., 37 (3) (2004), 533–542]. Several consequences of the results are presented. It is also pointed out that one of the main results (Theorem 2.8 below) provides a corrected form of the proof stated in two recent known results.

1. Introduction

Let $\mathscr{H}$ denote the class of analytic functions in $\mathbb{U} := \{z : |z| < 1\}$, and $\mathscr{H}[a,n]$ be the subclass of $\mathscr{H}$ comprising of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$, and $\mathscr{A}$ be another subclass of $\mathscr{H}$ which consists of functions of the form

$$f(z) = z + a_2 z^2 + \ldots. \tag{1}$$

Suppose $p, h \in \mathscr{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{2}$$

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then p is a solution of the differential superordination (2). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (2) is said to be the best subordinant. Recently Miller and Mocanu[15] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z).$$

For positive real parameters $\alpha_1, A_1, \ldots, \alpha_l, A_l$ and $\beta_1, B_1, \ldots, \beta_m, B_m$ $(l, m \in \mathbb{N} = 1, 2, 3, \ldots)$ such that

$$1 + \sum_{i=1}^{m} B_i - \sum_{j=1}^{l} A_j \geq 0, \tag{3}$$

the Wright's generalization [24] given by

$${}_l\Psi_m[(\alpha_1, A_1), .., (\alpha_l, A_l); (\beta_1, B_1), .., (\beta_m, B_m); z] = {}_l\Psi_m[(\alpha_i, A_i)_{1,l}; (\beta_j, B_j)_{1,m}; z]$$

of the generalized hypergeometric function ${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$ is defined by

$${}_l\Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] = \sum_{k=0}^{\infty} \{\prod_{t=0}^{l} \Gamma(\alpha_t + kA_t)\}\{\prod_{t=0}^{m} \Gamma(\beta_t + kB_t)\}^{-1} \frac{z^k}{k!} \ (z \in \mathbb{U}).$$

If $A_t = 1 (t = 1, ..., l)$ and $B_t = 1 (t = 1, ..., m)$, we have the relationship :

$$\begin{aligned} \Omega_l \Psi_m[(\alpha_t, 1)_{1,l}(\beta_t, 1)_{1,m}; z] &\equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_n}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!}, \end{aligned} \tag{4}$$

where ${}_lF_m$ is the generalized hypergeometric function (see, for example the details in [19]) such that $l \leq m + 1$ $(l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$, $z \in \mathbb{U}$; $\mathbb{N}$ denotes the set of all positive integers, $(\lambda)_n = \lambda(\lambda + 1)...(\lambda + n - 1)$ is the Pochhammer symbol, and Ω is given by

$$\Omega = \left(\prod_{t=0}^{l} \Gamma(\alpha_t) \right)^{-1} \left(\prod_{t=0}^{m} \Gamma(\beta_t) \right). \tag{5}$$

By using the generalized hypergeometric function (4), Dziok and Srivastava [7] introduced a linear operator which was subsequently extended by Dziok and Raina [8] by using the Wright's generalized hypergeometric function (defined above). For the purpose of this paper, we recall the Dziok-Raina linear operator as follows:

$$\Theta[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] : \mathscr{A} \to \mathscr{A}$$

is a linear operator which is defined (in terms of the convolution) by

$$\Theta[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] f(z) := z\, {}_l\Psi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] * f(z).$$

We observe for the function $f(z)$ of the form (1) that

$$\Theta[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] f(z) = z + \sum_{k=2}^{\infty} \sigma_k(\alpha_1)\, a_k z^k, \tag{6}$$

where $\sigma_k(\alpha_1)$ is defined by

$$\sigma_k(\alpha_1) = \frac{\Omega\Gamma(\alpha_1 + A_1(k-1)) \ldots \Gamma(\alpha_l + A_l(k-1))}{(k-1)!\Gamma(\beta_1 + B_1(k-1)) \ldots \Gamma(\beta_m + B_m(k-1))} \tag{7}$$

and Ω is given by (5).
For notational convenience, we write

$$\Theta[\alpha_1] f(z) = \Theta[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m)] f(z), \tag{8}$$

which in view of (6) gives (see [8])

$$z A_1 (\Theta[\alpha_1] f(z))' = \alpha_1 \Theta[\alpha_1 + 1] f(z) - (\alpha_1 - A_1) \Theta[\alpha_1] f(z). \tag{9}$$

The Dziok- Raina operator (6) has been studied recently in the theory of analytic functions by many authors and one may refer to [9], [10] for the latest work on the subject. In view of the relationship (4), the linear operator (6) includes (as its special cases) various other linear operators introduced and studied by Bernardi [1], Carlson and Shaffer [4], Cho-Kwon-Srivastava [5], Choi-Saigo-Srivastava [6], Libera [11], Livingston [12], Ruscheweyh [21] and Srivastava-Owa [23].

In [3], Bulboacă (see also [2]) considered certain classes of first order differential superordinations as well as superordination-preserving integral operators by using the results of Miller and Mocanu[15]. Further, using the results in [2] and [15], Shanmugam *et al.* [22] obtained sufficient conditions for the normalized analytic function $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z) \text{ and } q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

Recently, Murugusundaramoorthy and Magesh in their papers [13], [16], [17] and [18] obtained certain sufficient conditions for the normalized analytic function $f(z)$ to satisfy

$$q_1(z) \prec \left(\frac{H_m^l[\alpha_1]f(z)}{(z)}\right)^{\delta} \prec q_2(z), \quad q_1(z) \prec \frac{(f*\Phi)(z)}{(f*\Psi)(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \quad q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

The main aim of the present paper is to find sufficient conditions for the Hadamard product of a fixed normalized analytic function $f(z)$ with the normalized analytic functions $\phi(z)$ and $\psi(z)$ in the open unit disk $\mathbb{U}$, such that $(f*\Psi)(z) \neq 0$, involving the Dziok-Raina operator [8] which satisfy

$$q_1(z) \prec \frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent functions in $\mathbb{U}$, and $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty} \mu_n z^n$ are analytic functions in $\mathbb{U}$ with $\lambda_n \geq 0$, $\mu_n \geq 0$ $(\lambda_n \geq \mu_n)$. Several results are deduced as worthwhile consequences of the main results. In particular, we mention two such special cases of one of our main results which provide the corrected proof details of certain recent results established in [17] and [18].

2. Subordination and superordination results

In our present investigation, we require following key lemmas.

Lemma 2.1. *[14, p.132, Theorem 3.4h] Let q be univalent in the unit disk $\mathbb{U}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{U})$. Set $Q(z) := zq'(z)\phi(q(z))$ and $h(z) := \theta(q(z)) + Q(z)$. Suppose that $Q(z)$ is starlike univalent in $\mathbb{U}$ and Re $\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in \mathbb{U}$. If p is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$ and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{10}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.2. *[3, p.289, Corollary 3.2] Let q be convex univalent in the unit disk $\mathbb{U}$ and ϑ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$. Suppose that, $Re\{\vartheta'(q(z))/\varphi(q(z))\} > 0$ for $z \in \mathbb{U}$ and $\psi(z) = zq'(z)\varphi(q(z))$ is starlike univalent in $\mathbb{U}$. If $p(z) \in \mathbb{H}[q(0),1] \cap Q$, with $p(\mathbb{U}) \subseteq \mathbb{D}$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\mathbb{U}$ and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{11}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Making use of Lemma 2.1, we first prove the following theorem.

Theorem 2.3. *Let $\Phi, \Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, q be convex univalent with $q(0) = 1$, and assume that*

$$Re\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in \mathbb{U}). \tag{12}$$

If $f \in \mathscr{A}$ satisfies

$$\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi) = \Delta(f,\Phi,\Psi,\gamma_1,\gamma_2,\gamma_3,\gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 zq'(z), \tag{13}$$

where

$$\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi) := \begin{cases} \gamma_1 + \gamma_2\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^2 + \gamma_3\frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \\ +\frac{\gamma_4}{A_1}\left(\alpha_1\frac{\Theta[\alpha_1+1](f*\Phi)(z)}{\Theta[\alpha_1](f*\Phi)(z)} - (\alpha_1+1)\frac{\Theta[\alpha_1+2](f*\Psi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} + 1\right) \\ \times\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)}\right), \end{cases} \tag{14}$$

then

$$\frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function p by

$$p(z) := \frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \quad (z \in \mathbb{U}). \tag{15}$$

We observe (in view of (6)) that the function $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Differentiating (15) with respect to z, and making use of (9), we obtain after elementary calculations the following:

$$\gamma_1 + \gamma_2 \left(\frac{\Theta[\alpha_1](f * \Phi)(z)}{\Theta[\alpha_1 + 1](f * \Psi)(z)} \right)^2 + \gamma_3 \frac{\Theta[\alpha_1](f * \Phi)(z)}{\Theta[\alpha_1 + 1](f * \Psi)(z)}$$
$$+ \frac{\gamma_4}{A_1} \left(\alpha_1 \frac{\Theta[\alpha_1 + 1](f * \Phi)(z)}{\Theta[\alpha_1](f * \Phi)(z)} - (\alpha_1 + 1) \frac{\Theta[\alpha_1 + 2](f * \Psi)(z)}{\Theta[\alpha_1 + 1](f * \Psi)(z)} + 1 \right)$$
$$\times \left(\frac{\Theta[\alpha_1](f * \Phi)(z)}{\Theta[\alpha_1 + 1](f * \Psi)(z)} \right) = \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z). \tag{16}$$

Using (16) in (13), we get

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z), \tag{17}$$

and by setting $\theta(w) := \gamma_1 + \gamma_2 w^2 + \gamma_3 w$ and $\phi(w) := \gamma_4 \ \ (w = q(z))$, it can be easily observed that both the functions $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$. Also, we notice that

$$Q(z) := zq'(z)\phi(q(z)) = \gamma_4 z q'(z)$$

and

$$h(z) := \theta(q(z)) + Q(z) = \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z).$$

It is clear that $Q(z)$ is starlike univalent in $\mathbb{U}$ and

$$\operatorname{Re}\left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re}\left\{ \frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(z) + \left(1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0.$$

In view of the hypothesis of Theorem 2.3, the result now follows by an application of Lemma 2.1. □

By choosing $A_t = 1$ $(t = 1, \ldots, l)$ and $B_t = 1$ $(t = 1, \ldots, m)$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.3, we state the following corollary.

Corollary 2.4. *Let* $\Phi, \Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, *q be convex univalent with* $q(0) = 1$, *and suppose that* (12) *holds true. If* $f \in \mathscr{A}$ *satisfies*

$$\gamma_1 + \gamma_2 \left(\frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \right)^2$$
$$+ \gamma_4 \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \left[\frac{z(f * \Phi)'(z)}{(f * \Phi)(z)} - \frac{z(f * \Psi)''(z)}{(f * \Psi)'(z)} + \frac{\gamma_3}{\gamma_4} - 1 \right]$$
$$\prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z),$$

then $\frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \prec q(z)$ *and q is the best dominant.*

Corollary 2.4 in the special case when $\Psi(z) = \Phi(z) = \frac{z}{1-z}$, yields the following interesting result.

Corollary 2.5. *Let $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, q be convex univalent with $q(0) = 1$, and suppose that* (12) *holds true. If $f \in \mathscr{A}$ satisfies*

$$\gamma_1 + \gamma_2 \left(\frac{f(z)}{zf'(z)}\right)^2 + \gamma_4 \frac{f(z)}{zf'(z)} \left[\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} + \frac{\gamma_3}{\gamma_4} - 1\right] \prec$$
$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z),$$

then $\frac{f(z)}{zf'(z)} \prec q(z)$ and q is the best dominant.

Remark 2.6. For $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = 1$, Corollary 2.5 yields the known result due to Shanmugam *et.al [22].*

If we choose $q(z)$ as a bilinear function, viz. $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.3, we get the following:

Corollary 2.7. *Let $\Phi, \Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, q be convex univalent with $q(0) = 1$, assume that* (12) *holds true. If $f \in \mathscr{A}$ and*

$$\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) \prec \gamma_1 + \gamma_2 (\frac{1+Az}{1+Bz})^2 + \gamma_3 \frac{1+Az}{1+Bz} + \gamma_4 \frac{(A-B)z}{(1+Bz)^2},$$

then

$$\frac{\Theta[\alpha_1](f * \Phi)(z)}{\Theta[\alpha_1 + 1](f * \Psi)(z)} \prec \frac{1+Az}{1+Bz},$$

where $\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi)$ is given by (14)*, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Next, by applying Lemma 2.2, we prove the following.

Theorem 2.8. *Let $\Phi, \Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, q be convex univalent with $q(0) = 1$, and assume that*

$$Re \left\{ \frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(z) \right\} \geq 0. \tag{18}$$

Suppose that $f \in \mathscr{A}$, $\frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \in \mathbb{H}[q(0), 1] \cap Q$, and let $\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi)$ be univalent in $\mathbb{U}$ and

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi), \tag{19}$$

where $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ *is given by* (14)*, then*

$$q(z) \prec \frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)}$$

and q is the best subordinant.

Proof. Let the function $p(z)$ be defined by (15)(which is analytic in the unit disk $\mathbb{U}$ with $p(0)=1$). Following the same steps as in proof of Theorem 2.3, and using (19), we infer that

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z) \prec \gamma_1+\gamma_2 p^2(z)+\gamma_3 p(z)+\gamma_4 zp'(z).$$

By setting $\vartheta(w)=\gamma_1+\gamma_2 w^2+\gamma_3 w$ and $\phi(w)=\gamma_4$, it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\phi(w)$ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0$. In view of the result [20, p.159, Theorem 6.2], if we let

$$L(z,t)=\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 tzq'(z)=a_1(t)z+\ldots, \tag{20}$$

then on differentiating (20) with respect to z and t, we get

$$\frac{\partial L(z,t)}{\partial z}=2\gamma_2 q(z)q'(z)+\gamma_3 q'(z)+t\gamma_4 zq''(z)+t\gamma_4 q'(z)=a_1(t)+\ldots$$

and $\frac{\partial L(z,t)}{\partial t}=\gamma_4 zq'(z)$. Also, $\frac{\partial L(0,t)}{\partial z}=\gamma_4 q'(0)\left[\frac{\gamma_3}{\gamma_4}+\frac{2\gamma_2}{\gamma_4}q(0)+t\right]$.
From the univalence of q we have $q'(0)\neq 0$ and $q(0)=1$, it follows that $a_1(t)\neq 0$ for $t\geq 0$ and $\lim_{t\to\infty}|a_1(t)|=+\infty$, and simple computation now yields

$$\operatorname{Re}\left\{z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\}=\operatorname{Re}\left\{\frac{\gamma_3}{\gamma_4}+\frac{2\gamma_2}{\gamma_4}q(z)+t\left(1+\frac{zq''(z)}{q'(z)}\right)\right\}.$$

Using the fact that q is a convex univalent function in $\mathbb{U}$ and $\gamma_4\neq 0$, we conclude that

$$\operatorname{Re}\left\{z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\}>0 \text{ if } Re\left\{\frac{\gamma_3}{\gamma_4}+\frac{2\gamma_2}{\gamma_4}q(z)\right\}>0 \quad (z\in\mathbb{U},\ t\geq 0)$$

and Theorem 2.8, follows by applying Lemma 2.2. □

Remark 2.9. We mention here that a recent known result [18, p.90, Theorem 2.12] evidently follows from the above Theorem 2.8 by reducing the Dziok-Raina operator (8) to the corresponding Dziok-Srivastava operator (defined in [7]) in view of the parameteric substitutions stated in (4). It may also be noted that Theorem 2.8 gives a *corrected form of the proof derivation* of two known recent results [17, p.123, Theorem 3.1] and [18, p.90, Theorem 2.12].

When $A_t = 1$ $(t = 1,\ldots,l)$ and $B_t = 1$ $(t = 1,\ldots,m)$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.8, we derive the following corollary.

Corollary 2.10. *Let $\Phi,\Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1,\ldots,4)(\gamma_4 \neq 0)$, q be convex univalent with $q(0) = 1$, and assume that* (18) *holds true.*

If $f \in \mathscr{A}$, $\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \in \mathbb{H}[q(0),1] \cap Q$ let $\gamma_1 + \gamma_2 \left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\right)^2 + \gamma_4 \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}$ $\left[\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)} + \frac{\gamma_3}{\gamma_4} - 1\right]$ be univalent in $\mathbb{U}$ and

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \gamma_1 + \gamma_2 \left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\right)^2 + \gamma_4 \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \left[\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)} + \frac{\gamma_3}{\gamma_4} - 1\right].$$

Then $q(z) \prec \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}$ and q is the best subordinant.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Corollary 2.10, we obtain the following corollary.

Corollary 2.11. *Let $\gamma_i \in \mathbb{C}$ $(i = 1,\ldots,4)(\gamma_4 \neq 0)$, q be convex univalent with $q(0) = 1$, and* (18) *holds true. If $f \in \mathscr{A}$, $\frac{f(z)}{zf'(z)} \in \mathbb{H}[q(0),1] \cap Q$. Let $\gamma_1 + \gamma_2 \left(\frac{f(z)}{zf'(z)}\right)^2 + \gamma_4 \frac{f(z)}{zf'(z)} \left[\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} + \frac{\gamma_3}{\gamma_4} - 1\right]$ be univalent in $\mathbb{U}$ and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \gamma_1 + \gamma_2 \left(\frac{f(z)}{zf'(z)}\right)^2 + \gamma_4 \frac{f(z)}{zf'(z)} \left[\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} + \frac{\gamma_3}{\gamma_4} - 1\right],$$

then $q(z) \prec \frac{f(z)}{zf'(z)}$ and q is the best subordinant.

If we choose $q(z) = (1+Az)/(1+Bz)$ $(-1 \le B < A \le 1)$ in Theorem 2.8, then we obtain the following corollary.

Corollary 2.12. *Let $\gamma_i \in \mathbb{C}$ $(i = 1,\ldots,4)(\gamma_4 \neq 0)$, q be convex univalent with $q(0) = 1$, and assume that* (18) *holds true. If $f \in \mathbb{A}$, $\frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \in \mathbb{H}[q(0),1]$ $\cap Q$. Let $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ be univalent in $\mathbb{U}$ and*

$$\gamma_1 + \gamma_2 (\frac{1+Az}{1+Bz})^2 + \gamma_3 \frac{1+Az}{1+Bz} + \gamma_4 \frac{(A-B)z}{(1+Bz)^2} \prec \Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi),$$

then

$$\frac{1+Az}{1+Bz} \prec \frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

3. Composition Results

Combining the two results given by Theorem 2.3 (involving differential subordination) and Theorem 2.8 (involving differential superordination), we have the following sandwich theorem.

Theorem 3.1. *Let* q_1 *and* q_2 *be convex univalent in* $\mathbb{U}$, $\gamma_i \in \mathbb{C}$ $(i = 1,\ldots,4)$ $(\gamma_4 \neq 0)$, *and let* q_2 *satisfy* (12) *and* q_1 *satisfy* (18). *For* $f,\Phi,\Psi \in \mathscr{A}$, *let* $\frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \in \mathbb{H}[1,1]\cap Q$ *and* $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ *defined by* (14) *be univalent in* $\mathbb{U}$ *satisfying*

$$\begin{aligned}\gamma_1+\gamma_2 q_1^2(z)+\gamma_3 q_1(z)+\gamma_4 z q_1'(z) &\prec \Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)\\ &\prec \gamma_1+\gamma_2 q_2^2(z)+\gamma_3 q_2(z)+\gamma_4 z q_2'(z),\end{aligned}$$

then

$$q_1(z) \prec \frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are, respectively, the best subordinant and best dominant.*

On putting $q_1(z) = \frac{1+A_1z}{1+B_1z}$ $(-1 \leq B_1 < A_1 \leq 1)$ and $q_2(z) = \frac{1+A_2z}{1+B_2z}$ $(-1 \leq B_2 < A_2 \leq 1)$ in Theorem 3.1,we obtain the following result.

Corollary 3.2. *For* $f,\Phi,\Psi \in \mathscr{A}$, *let* $\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \in \mathscr{H}[1,1]\cap Q$ *and* $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ *defined by* (14) *be univalent in* $\mathbb{U}$ *satisfying*

$$\begin{aligned}\gamma_1+\gamma_2\Big(\frac{1+A_1z}{1+B_1z}\Big)^2+\gamma_3\frac{1+A_1z}{1+B_1z}+\gamma_4\frac{(A_1-B_1)z}{(1+B_1z)^2} &\prec \Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)\\ &\prec \gamma_1+\gamma_2\Big(\frac{1+A_2z}{1+B_2z}\Big)^2+\gamma_3\frac{1+A_2z}{1+B_2z}+\gamma_4\frac{(A_2-B_2)z}{(1+B_2z)^2},\end{aligned}$$

then

$$\frac{1+A_1z}{1+B_1z} \prec \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \prec \frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$, $\frac{1+A_2z}{1+B_2z}$ *are, respectively, the best subordinant and best dominant.*

Remark 3.3. We conclude this paper by remarking that the various results can be derived from Theorems 2.3, 2.8 and 3.1 for the different choices of the arbitraray functions $\Phi(z)$ and $\Psi(z)$. In fact, by appropriately selecting the arbitrary parameters A_t, B_t, l, m, $\alpha_1, \ldots, \alpha_l$, $\beta_1, \ldots, \beta_m$, involved in the operator (6), we observe that the generalized form of the linear operator of Dziok and Raina [8] reduced to well known operators like the Dziok-Srivastava linear operator, Hohlov linear operator, Saitoh generalized linear operator, Carlson-Shaffer linear operator, Ruscheweyh derivative operator (as well as its generalized versions), Bernardi-Libera-Livingston operator, Cho-Kwon-Srivastava operator, Choi-Saigo-Srivastava operator and Srivastava-Owa fractional derivative operator. These consequences are easy to deduce, and we make here no attempt to mention these obvious special cases of Theorems 2.3, 2.8 and 3.1.

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