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MULTIPLE SOLUTIONS FOR ELLIPTIC PROBLEMS INVOLVING THE $p(x)$-LAPLACIAN

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Multiplicity of solutions for $p(x)$-Laplacian Dirichlet problems is investigated. The approach is based on the critical point theory. The ordinary case is pointed out.

1. Introduction

The aim of this paper is to investigate the following Dirichlet problem involving the $p(x)$-laplacian

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(x,u) \text{ in } \Omega \\ \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, $p \in C(\bar{\Omega})$, $\Delta_{p(x)} u := div(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)-$Laplace operator, λ is a positive real parameter and f is a Carathéodory function.

The existence of three solutions for problem (P_λ) has been established in [1, Theorem 3.2]. However, in that case, one of the solutions, in absence of small perturbations of the nonlinear term, is the trivial solution. The aim of this paper is to point out the existence of three non-trivial solutions for (P_λ),

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which is a problem without small perturbations of the nonlinear term, whenever $p(x)$ is greater than N. Our main result is Theorem 3.1, where the existence of three solutions to (P_λ) is established. Moreover, as special case, the existence of three non-trivial solutions to a two-point boundary value problem is obtained (see Theorem 3.3). The main tool to prove our results, is a recent three critical points theorem of G. Bonanno and S.A. Marano (see Theorem 2.2).

This note consists of three sections. Section 2 contains some basic properties of the space $W^{1,p(x)}(\Omega)$ and $p(x)-$Laplace operator, while main results and their proofs are given in Section 3.

2. Preliminaries

The theory of the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ presents results similar to those for $L^p(\Omega)$ and $W^{m,p}(\Omega)$ spaces but there are, also, some crucial differences. In this Section, we recall some basic definitions and properties, while we refer to [7], [8], [6], [4] and references therein, for more details.

Here and in the sequel, we suppose that $p \in C(\bar{\Omega})$ satisfies the following condition:

$$N < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty. \tag{1}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \rho_p(u) < +\infty\}$$

where

$$\rho_p(u) := \int_\Omega |u(x)|^{p(x)} dx$$

is called the "modular" of the space $L^{p(x)}(\Omega)$. Such a space is a particular case of Orlicz-Musielak spaces (see [8]). On $L^{p(x)}(\Omega)$ we consider the "Luxembury" norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

The generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} := \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}$$

while, by $W_0^{1,p(x)}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.
On $W_0^{1,p(x)}(\Omega)$ we consider the norm

$$\|u\| := \||\nabla u|\|_{L^{p(x)}(\Omega)}.$$

With such norms, $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

As pointed out in [7] and [6], $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and, since $p^- > N$, $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, $W^{1,p(x)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. So, in particular, there exists a positive constant $c_0 > 0$ such that

$$\|u\|_{C^0(\overline{\Omega})} \leq c_0\|u\| \tag{2}$$

for each $u \in W_0^{1,p(x)}(\Omega)$.
Put

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$. It is known that $\Phi \in C^1(W_0^{1,p(x)}(\Omega),\mathbb{R})$, and

$$\left\langle \Phi'(u), v \right\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx$$

for each $u, v \in W_0^{1,p(x)}(\Omega)$. Moreover, (see [5, Theorem 3.1]), Φ is convex, sequentially weakly lower semi-continuous and $\Phi' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is an homeomorphism.
From Theorem 1.3 of [6], we obtain the following proposition.

Proposition 2.1. *Let $u \in W_0^{1,p(x)}(\Omega)$; then*

(i) $\|u\| < 1\ (= 1; > 1) \Leftrightarrow \rho_p(|\nabla u|) < 1\ (= 1; > 1)$;

(ii) *if* $\|u\| > 1$, *then* $\dfrac{1}{p^+}\|u\|^{p^-} \leq \Phi(u) \leq \dfrac{1}{p^-}\|u\|^{p^+}$;

(iii) *if* $\|u\| < 1$, *then* $\dfrac{1}{p^+}\|u\|^{p^+} \leq \Phi(u) \leq \dfrac{1}{p^-}\|u\|^{p^-}$.

Thanks to Proposition 2.1, the functional Φ turns out to be coercive.

Our main tool is a recent result obtained by G. Bonanno and S.A. Marano in [2], that we recall in a convenient form.

Theorem 2.2. *[[2], Theorem 3.6] Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{x \in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that:

(a_1) $\dfrac{\sup\limits_{\Phi(x) \leq r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(a_2) *for each $\lambda \in \Lambda_r := \left]\dfrac{\Phi(\bar{x})}{\Psi(\bar{x})}, \dfrac{r}{\sup_{\Phi(x) \leq r} \Psi(x)}\right[$ the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

3. Main results

Consider the problem (P_λ) and assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following condition:

(f_1) there exist $s \in [1, p^-[$ and a positive constant c such that

$$|f(x,t)| \leq c(1 + |t|^{s-1})$$

for each $(x,t) \in \Omega \times \mathbb{R}$.

Put

$$F(x,\xi) := \int_0^\xi f(x,t)\, dt$$

for each $(x,\xi) \in \Omega \times \mathbb{R}$, (f_1) guarantees that the functional Ψ defined by

$$\Psi(u) = \int_\Omega F(x,u(x))dx$$

for each $u \in W_0^{1,p(x)}(\Omega)$, is in $C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, its derivative Ψ' is compact and

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx$$

for each $u, v \in W_0^{1,p(x)}(\Omega)$. Now, let us introduce the energy functional I_λ related to the problem (P_λ):

$$I_\lambda(\cdot) := \Phi(\cdot) - \lambda\Psi(\cdot)$$

and we observe that, for each $\lambda > 0$, the critical points u of I_λ are the weak solutions of (P_λ) i.e.

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \lambda \int_\Omega f(x,u(x))v(x)dx$$

for each $v \in W_0^{1,p(x)}(\Omega)$. Before introducing our result we observe that, putting

$$\delta(x) = \sup\{\delta > 0 : B(x,\delta) \subseteq \Omega\}$$

for all $x \in \Omega$, one can prove that there exists $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$, where

$$D = \sup_{x\in\Omega} \delta(x). \tag{3}$$

Finally, for each $r > 0$, put

$$\gamma_r := \max\left\{(p^+ r)^{\frac{1}{p^-}}, (p^+ r)^{\frac{1}{p^+}}\right\},$$

and

$$m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})},$$

where Γ is the Eulero function.

Theorem 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying (f_1) and such that $essinf_{x\in\Omega} F(x,\xi) \geq 0$ for all $\xi \in \mathbb{R}$. Assume also that there exist two positive constants r and h, with $r < \frac{1}{p^+}\min\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\}\, m\, D^N \frac{2^N-1}{2^N}$, such that*

$$\alpha_r := \frac{1}{r}\int_\Omega \sup_{|\xi|\leq c_0\gamma_r} F(x,\xi)\, dx < \frac{p^- essinf_{x\in\Omega} F(x,h)}{\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}(2^N-1)} := \beta_h. \tag{4}$$

Then, for each $\lambda \in]\frac{1}{\beta_h}, \frac{1}{\alpha_r}[$, the problem (P_λ) admits at least three weak solutions.

Proof. Our aim is to apply Theorem 2.2. To this end, take $X := W_0^{1,p(x)}(\Omega)$ with the usual norm and

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx,$$

$$\Psi(u) = \int_\Omega F(x,u(x))dx,$$

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$$

for all $u \in X$.

As seen before, the functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.2.

Now, let $\bar{v} \in X$ be defined by

$$\bar{v}(x) = \begin{cases} 0 & x \in \Omega \setminus B(x_0, D) \\ h & x \in B(x_0, \frac{D}{2}) \\ \frac{2h}{D}(D - |x - x_0|) & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}), \end{cases} \tag{5}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^N$. We have

$$\frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} m\left(D^N - \left(\frac{D}{2}\right)^N\right) \leq$$

$$\leq \Phi(\bar{v}) \leq \frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} m\left(D^N - \left(\frac{D}{2}\right)^N\right)$$

and

$$\Psi(\bar{v}) \geq \int_{B(x_0, \frac{D}{2})} F(x, \bar{v}(x))dx \geq essinf_{x\in\Omega} F(x,h)\, m\left(\frac{D}{2}\right)^N \cdot$$

From $r < \frac{1}{p^+}\min\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\}\, m\, D^N \frac{2^N - 1}{2^N}$, one has $r < \Phi(\bar{v})$. Moreover, thanks to the embedding $X \hookrightarrow C^0(\bar{\Omega})$, for each $u \in X$ with $\Phi(u) \leq r$, the following relation holds:

$$\max_{x\in\Omega}|u(x)| \leq c_0 \max\left\{(p^+ r)^{\frac{1}{p^-}}, (p^+ r)^{\frac{1}{p^+}}\right\} = c_0\gamma_r$$

and so

$$\sup_{\Phi(u)\leq r} \Psi(u) \leq \int_\Omega \sup_{|\xi|\leq c_0\gamma_r} F(x,\xi)dx \cdot$$

From (4) one has

$$\frac{1}{r}\sup_{\Phi(u)\leq r}\Psi(u) \leq \frac{1}{r}\int_\Omega \sup_{|\xi|\leq c_0\gamma_r} F(x,\xi)dx = \alpha_r <$$

$$< \beta_h = \frac{essinf_{x\in\Omega} F(x,h) m\left(\frac{D}{2}\right)^N}{\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} m\left(D^N - \left(\frac{D}{2}\right)^N\right)} \leq \frac{\Psi(\bar{v})}{\Phi(\bar{v})}$$

and so condition (a_1) of Theorem 2.2 is verified. Finally, we prove that, for each $\lambda > 0$, the functional I_λ is coercive. For each $u \in X$, from (f_1) one has

$$\Psi(u) = \int_\Omega F(x,u(x))\, dx \leq \int_\Omega c(|u(x)| + \frac{1}{s}|u(x)|^s)\, dx \leq$$

$$c\|u\|_{C^0(\bar{\Omega})}|\Omega| + \frac{c}{s}\|u\|^s_{C^0(\bar{\Omega})}|\Omega| \leq cc_0\|u\||\Omega| + \frac{c}{s}(c_0\|u\|)^s|\Omega|\cdot$$

If $\|u\| \geq 1$, then this leads to

$$I_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \lambda cc_0\|u\||\Omega| - \frac{\lambda c}{s}(c_0\|u\|)^s|\Omega|$$

and, since $s < p^-$, coercivity of I_λ is obtained. Taking into account that

$$\bar{\Lambda} :=]\frac{1}{\beta_h}, \frac{1}{\alpha_r}[\subseteq]\frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}[,$$

Theorem 2.2 ensures that, for each $\lambda \in \bar{\Lambda}$, the functional I_λ admits at least three critical points in X that are weak solutions of the problem (P_λ). □

Remark 3.2. Actually, in Theorem 3.1, it is enough to require

$$essinf_{x\in\Omega}F(x,\xi) \geq 0$$

for all $\xi \in [0,h]$.

Now, we point out the following special case of Theorem 3.1. To this end, let $\alpha : [0,1] \to \mathbb{R}$ be a positive, bounded, and measurable function. Put $\alpha_0 = essinf_{x\in[0,1]}\alpha(x)$ and $\|\alpha\|_1 = \|\alpha\|_{L^1([0,1])}$. Moreover, put

$$k = \frac{p^-}{p^+}\frac{1}{4^{p^+}}\frac{\alpha_0}{\|\alpha\|_1}.$$

Theorem 3.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\lim_{|t|\to+\infty}\frac{g(t)}{|t|^v} = 0$$

for some $0 \leq v < p^- - 1$, and $g(0) \neq 0$. Put $G(\xi) = \int_0^\xi g(t)dt$ for all $\xi \in \mathbb{R}$ and assume that there exist two positive constants l and h, with $l \leq 1 < \left(\frac{1}{2}\right)^{\frac{1}{p^+}}(4h)^{\frac{p^-}{p^+}}$ such that

$$\frac{G(l)}{l^{p^+}} < k\frac{G(h)}{h^{p^+}}$$

Then, for each $\lambda \in \left] \frac{4^{p^+}}{p^-\alpha_0} \frac{h^{p^+}}{G(h)}, \frac{1}{p^+\|\alpha\|_1} \frac{l^{p^+}}{G(l)} \right[$, *the problem*

$$\begin{cases} -(|u'|^{p(x)-2}u')' = \lambda \alpha(x) g(u) \text{ in }]0,1[\\ \\ u(0) = u(1) = 0 \end{cases} \tag{D_λ}$$

admits at least three non-trivial weak solutions.

Proof. From [3, Proposition 2.4] one has

$$\int_0^1 |u'(t)|dt \leq 2\|u'\|_{L^{p(x)}([0,1])} \|1\|_{L^{q(x)}([0,1])}$$

for all $u \in W_0^{1,p(x)}([0,1])$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Since $\|1\|_{L^{q(x)}([0,1])} \leq 1$, one has

$$|u(t)| \leq \|u\|$$

for all $t \in [0,1]$ and for all $u \in W_0^{1,p(x)}([0,1])$. Hence, taking (2) into account, one has $c_0 \leq 1$. Now, by choosing $r = \frac{l^{p^+}}{p^+}$, simple computations show that all assumptions of Theorem 3.1 are verified and the conclusion is achieved. $\square$

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