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SUBHARMONIC SOLUTIONS OF PLANAR HAMILTONIAN SYSTEMS VIA THE POINCARÉ-BIRKHOFF THEOREM

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We revisit some recent results obtained in [1] about the existence of subharmonic solutions for a class of (nonautonomous) planar Hamiltonian systems, and we compare them with the existing literature. New applications to undamped second order equations are discussed, as well.

1. Introduction

The aim of this brief note is to present some recent results obtained in [1], via the Poincaré-Birkhoff fixed point theorem, about the existence of subharmonic solutions for a class of planar Hamiltonian system

$$Jz' = \nabla_z H(t,z), \tag{1}$$

being $z = (x,y) \in \mathbb{R}^2$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the standard symplectic matrix and $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ a function which is T-periodic in the first variable. Throughout the paper, by *subharmonic solution of order k* of system (1) (with $k \in \mathbb{N}_0$) we will mean a kT-periodic solution which is not lT-periodic for every integer $l = 1,\dots,k-1$.

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Just to start our discussion, we first consider the particular case of a scalar second order equation without damping

$$u'' + g(t,u) = 0, \tag{2}$$

with $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a function which is T-periodic in the first variable, and we focus on the case in which $g(t,x)$ satisfies a (possibly one-sided) sublinearity condition at infinity, that is to say

$$\lim_{x\to+\infty} \frac{g(t,x)}{x} = 0 \qquad \text{uniformly in } t \in [0,T]. \tag{3}$$

With such an assumption (3), the problem of the existence of subharmonic solutions for (2) has been tackled, independently and in the very same years, by Fonda-Ramos [4] and Ding-Zanolin [3]. Precisely, pairing the sublinearity condition (3) with the Landesman-Lazer condition

$$\int_0^T \limsup_{x\to-\infty} g(t,x)\,dt < 0 < \int_0^T \liminf_{x\to+\infty} g(t,x)\,dt \tag{4}$$

and using critical point theory, Fonda and Ramos proved the existence of a sequence $u_k(t)$ of kT-periodic solutions, with minimal periods and amplitudes (i.e. $\max_{[0,kT]} u_k - \min_{[0,kT]} u_k$) going to infinity with k. On the other hand, when (4) is replaced by the more restrictive sign condition

$$\liminf_{|x|\to+\infty} g(t,x)\mathrm{sgn}(x) > 0, \tag{5}$$

Ding and Zanolin showed, via the Poincaré-Birkhoff fixed point theorem, the existence, for every k large enough and for every integer $j \in [1,m_k]$ with $m_k \to +\infty$, of a kT-periodic solution $u_{j,k}(t)$ with $2j$ zeros in the interval $[0,kT[$, and with

$$\lim_{k\to+\infty} \min_{t\in[0,kT]} (u_{j,k}(t)^2 + u'_{j,k}(t)^2) = +\infty. \tag{6}$$

We point out that, as a quite general fact, the Poincaré-Birkhoff theorem seems to be sometimes more demanding about the assumptions on the nonlinear term with respect to variational tools but, on the other hand, it permits to greatly improve the result from the point of view of the multiplicity of periodic solutions. We refer the reader to [5] for more comments about this fact. Anyway, as it is clear, the results in [3, 4] are strictly related and show the existence of subharmonics which are larger and larger as their order k increases. From a dynamical point of view, conditions (4) or (5) imply that, fixed a time kT, solutions which are large enough wind the origin, while, due to the sublinearity assumption (3), much larger solutions do not: as a consequence, one gets the existence of subharmonics satisfying (6).

The aim of the paper [1] is twofold. On one hand, keeping (a weaker version of) the sublinearity condition (3), we look for another assumption implying the existence of subharmonic solutions with the same nodal characterization as in [3]. Precisely, we suppose to deal with the "unforced" case $g(t,0) \equiv 0$ [1] and we assume a condition implying a positive twist for small solutions. In this framework, our result is the following.

Corollary 1.1. *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, T-periodic in the first variable and such that $g(t,0) \equiv 0$ and assume that the uniqueness and the global continuability for the solutions to the Cauchy problems associated to* (2) *are guaranteed. Let us suppose:*

(g_0) *there exists $q_0 \in L^1([0,T])$ with $\int_0^T q_0(t)\,dt > 0$ such that*

$$\liminf_{x\to 0} \frac{g(t,x)}{x} \geq q_0(t) \qquad \text{uniformly for a.e. } t \in [0,T],$$

(g_∞)

$$\limsup_{x\to +\infty} \frac{g(t,x)}{x} \leq 0 \qquad \text{uniformly for a.e. } t \in [0,T].$$

Then there exists $k^* \in \mathbb{N}_0$ such that, for every integer $k \geq k^*$, there exists an integer m_k such that, for every integer $j \in [1, m_k]$ with j and k coprime, equation (2) has at least two subharmonic solutions $u^1_{j,k}(t), u^2_{j,k}(t)$ of order k, not belonging to the same periodicity class [2], with exactly $2j$ zeros in the interval $[0,kT[$. In particular, if hypothesis (g_0) is satisfied with $q_0 \in L^\infty([0,T])$, then [3]

$$m_k = \left\lfloor k \frac{\int_0^T q_0(t)\,dt}{2\pi(\operatorname{esssup}_{[0,T]} q_0(t))^{1/2}} \right\rfloor. \tag{7}$$

We point out that the subharmonics produced can be quite small and possibly they do not satisfy relation (6). We also notice that assumption (g_0) avoids the use of a linearized equation at zero (contrary to [7]), while the estimate (7), which is not explicitly emphasized in [1], is sharp in the autonomous case $q_0(t) \equiv q_0$ and can be useful in some concrete applications. We will check this claim by applying the result to the Sitnikov problem (see Proposition 4.2).

[1] Of course, if (2) has a T-periodic solution, we can enter into this setting with a change of variables which does not modify the asymptotic properties of $g(t,x)$.

[2] i.e., $u^1_{j,k}(t) \not\equiv u^2_{j,k}(t+lT)$ for every $l = 1,\ldots,k-1$.

[3] Here, and throughout the paper, for $a \in \mathbb{R}^+$ we denote by $\lfloor a \rfloor$ the greatest integer strictly less than a.

On the other hand, the second (and major) task of [1] is to generalize Corollary 1.1 to a planar Hamiltonian system (1), with $\nabla_z H(t,0) \equiv 0$. The main idea in order to achieve such a result is to compare the dynamical properties of (1) to those of suitably chosen positively homogeneous planar Hamiltonian systems. In particular, on the line of [8], a *modified rotation number* is introduced and extensively used. We point out that a technical difficulty here is given by the fact that a (not even two-sided) sublinearity condition at infinity for $g(t,x)$ does not imply any well-recognized condition for the associated Hamiltonian $H(t,x,y) = \frac{1}{2}y^2 + \int_0^x g(t,\xi)\,d\xi$.

2. The Poincaré-Birkhoff fixed point theorem and the modified rotation number

The Poincaré-Birkhoff fixed point theorem is a classical result of planar topology, ensuring the existence of two fixed points for an area-preserving "twist" homeomorphism of the annulus. For a very nice survey on the theorem and its application to ODE's, we refer to [2]. Here, we just state, for the reader's convenience, a consequence of the Poincaré-Birkhoff fixed point theorem when applied to equation (1) in the unforced case. To this aim, we first recall that, for a (at least absolutely continuous) path $z = (x,y) : [s_1,s_2] \to \mathbb{R}^2$ such that $z(t) \neq 0$ for every t, the quantity

$$\mathrm{Rot}(z;[s_1,s_2]) := \frac{1}{2\pi}\int_{s_1}^{s_2} \frac{y(t)x'(t) - x(t)y'(t)}{x(t)^2 + y(t)^2}\,dt \tag{8}$$

is defined as the *rotation number* of $z(t)$ on $[s_1,s_2]$ and, as well known, it represents an algebric count of the *clockwise* windings around the origin of the path $z(t)$ in the time interval $[s_1,s_2]$. Assuming the uniqueness and the global continuability for the solutions to the Cauchy problems associated to (1), we will denote by $z(\cdot;z_0)$ the solution with $z(0;z_0) = z_0$.

Application of the Poincaré-Birkhoff fixed point theorem
Let $k \in \mathbb{N}_0$ be fixed. Assume that there exist two circumferences $\Gamma_i = r\mathbb{S}^1$ and $\Gamma_o = R\mathbb{S}^1$, with $0 < r < R$, and two numbers $l_r, l_R \in \mathbb{N}_0$ with $l_R \leq l_r$ such that:

- $\mathrm{Rot}(z(t;z_0);[0,kT]) > l_r$ *for every* $z_0 \in \Gamma_i$*;*
- $\mathrm{Rot}(z(t;z_0);[0,kT]) < l_R$ *for every* $z_0 \in \Gamma_o$*.*

Then, denoting with $\mathscr{A}$ the closed annulus having as inner and outer boundaries the circumferences Γ_i and Γ_o respectively, for every integer $j \in [l_R,l_r]$ equation (1) *has at least two kT-periodic solutions $z^1_{j,k}(t)$, $z^2_{j,k}(t)$ with $z^1_{j,k}(0), z^2_{j,k}(0) \in \mathring{\mathscr{A}}$ and such that*

$$\mathrm{Rot}(z^1_{j,k};[0,kT]) = \mathrm{Rot}(z^2_{j,k};[0,kT]) = j.$$

Now we pass to the description of the modified rotation number used in [1]. To this aim, we introduce the class $\mathscr{P}$ made up by all the C^1 functions $V : \mathbb{R}^2 \to \mathbb{R}$ which are positively homogeneous of degree 2 (i.e., $V(\lambda z) = \lambda^2 V(z)$ for every $\lambda > 0$ and for every $z \in \mathbb{R}^2$) and such that $V(z) > 0$ for every $z \neq 0$. For every $V \in \mathscr{P}$ and for a (at least absolutely continuous) path $z = (x,y) : [s_1,s_2] \to \mathbb{R}^2$ such that $z(t) \neq 0$ for every t, we set $A_V := \int_{\{V\leq 1\}} dx\,dy$ and we define the quantity

$$\mathrm{Rot}_V(z;[s_1,s_2]) := \frac{1}{2A_V}\int_{s_1}^{s_2} \frac{y(t)x'(t) - x(t)y'(t)}{V(x(t),y(t))}\,dt \tag{9}$$

as the *modified V-rotation number* of $z(t)$ on $[s_1,s_2]$. Clearly, the classical rotation number (8) is just the particular case of (9) with $V(x,y) = x^2 + y^2$. All we need to know, for the application of the Poincaré-Birkhoff fixed point theorem, is the following fact, which is proved in [1, Proposition 2.2].

Lemma 2.1. *Let $z : [s_1,s_2] \to \mathbb{R}^2$ be an absolutely continuous path, such that $z(t) \neq 0$ for every $t \in [s_1,s_2]$, and $j \in \mathbb{Z}$. Then, for every $V \in \mathscr{P}$:*

$$\mathrm{Rot}_V(z;[s_1,s_2]) \gtreqless j \Longleftrightarrow \mathrm{Rot}(z;[s_1,s_2]) \gtreqless j.$$

3. The planar Hamiltonian system

We are now ready to state our main result, which is a slight improvement of [1, Theorems 3.1-4.1]. Throughout the section, we assume that $H : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is a function measurable and T-periodic in the t-variable, of class C^1 in the z-variable and such that for every $r > 0$ there exists $\zeta_r \in L^1([0,T],\mathbb{R}^+)$ such that $|\nabla_z H(t,z)| \leq \zeta_r(t)$ for a.e. $t \in [0,T]$ and for every $z \in \mathbb{R}^2$ with $|z| \leq r$.
Recall that, for $V \in \mathscr{P}$, $A_V = \int_{\{V\leq 1\}} dx\,dy$.

Theorem 3.1. *Assume that $\nabla_z H(t,0) \equiv 0$ and that the uniqueness and the global continuability for the solutions to the Cauchy problems associated to* (1) *are guaranteed. Let us suppose:*

(H_0) *there exist $V_0 \in \mathscr{P}$, $a_0 \in L^1([0,T])$ with $\int_0^T a_0(t)\,dt > 0$ such that*

$$\liminf_{z\to 0} \frac{\nabla_z H(t,z)\cdot z}{V_0(z)} \geq a_0(t) \qquad \textit{uniformly for a.e. } t \in [0,T],$$

(H_∞) *there exist $\theta^1_\infty, \theta^2_\infty \in [0,+\infty)$ with $0 < \theta^2_\infty - \theta^1_\infty \leq 2\pi$ and two sequences $(V^n_\infty)_n \subset \mathscr{P}$, $(a^n_\infty)_n \subset L^1([0,T],\mathbb{R}^+)$ such that:*

i) $\inf_n \frac{1}{A_{V^n_\infty}} \int_{\theta^1_\infty}^{\theta^2_\infty} \frac{ds}{V^n_\infty(\cos s, -\sin s)} > 0$;

ii) $\inf_n \frac{\int_0^T a_\infty^n(t)\,dt}{A_{V_\infty^n}} = 0$,

iii) *for every $n \in \mathbb{N}_0$ and for every $\varepsilon > 0$ there exist $R_{n,\varepsilon} > 0$ and $b_{n,\varepsilon} \in L^1([0,T],\mathbb{R}^+)$ with $\int_0^T b_{n,\varepsilon}(t)\,dt \leq \varepsilon$ such that*

$$\frac{\nabla_z H(t,z)\cdot z}{V_\infty^n(z)} \leq a_\infty^n(t) + b_{n,\varepsilon}(t)$$

for a.e. $t \in [0,T]$ and for every $z = \rho e^{-i\theta} \in \mathbb{R}^2$ with $\theta_\infty^1 \leq \theta \leq \theta_\infty^2$ and $\rho \geq R_{n,\varepsilon}$.

Set

$$m_k := \left\lfloor k \frac{\int_0^T a_0(t)\,dt}{2A_{V_0}} \right\rfloor.$$

Then there exists $k^* \in \mathbb{N}_0$ such that, for every integer $k \geq k^*$, there exists an integer m_k such that, for every integer $j \in [1, m_k]$ with j and k coprime, equation (1) *has at least two subharmonic solutions $z_{j,k}^1(t), z_{j,k}^2(t)$ of order k, not belonging to the same periodicity class, with*

$$\mathrm{Rot}(z_{j,k}^1;[0,kT]) = \mathrm{Rot}(z_{j,k}^2;[0,kT]) = j.$$

Sketch of the proof. For a fixed $k \in \mathbb{N}_0$, it is possible to see that there exist $0 < r < R$ such that

- $\mathrm{Rot}_{V_0}(z(t;z_0);[0,kT]) > m_k$ for every $|z_0| = r$;
- for at least one n, $\mathrm{Rot}_{V_\infty^n}(z(t;z_0);[0,kT]) < 1$ for every $|z_0| = R$.

The conclusion follows in a standard way from the Poincaré-Birkhoff fixed point theorem, together with Lemma 2.1. $\square$

Remark 3.2. A more general version of assumption (H_0), which allows a different behavior for $H(t,z)$ in different angular regions, is presented in [1, Theorem 4.1].

Remark 3.3. We notice that assumption (H_∞) is satisfied if one can choose $a_\infty^n(t) \equiv 0$, $V_\infty^n(z) = |z|^2$, that is when $\nabla_z H(t,z)$ satisfies a sublinearity-like condition (at infinifty) in the angular sector $\{\rho e^{-i\theta} \mid \theta_\infty^1 \leq \theta \leq \theta_\infty^2\}$. On the other hand, some interesting situations in which $a_\infty^n(t)$ and $V_\infty^n(z)$ actually depend on n are covered by the theorem. In this case, the condition *i)* in (H_∞) can be quite difficult to check, but we point out that, as shown in [1], it is always satisfied when $\{\rho e^{-i\theta} \mid \theta_\infty^1 \leq \theta \leq \theta_\infty^2\}$ is the whole plane or when it contains at least one of the four quadrants and $V_\infty^n(x,y) = c_\infty^n x^2 + d_\infty^n y^2$ for some $c_\infty^n, d_\infty^n > 0$.

4. The second order equation

The result which can be derived from Theorem 3.1, when dealing with the second order equation (2), is just Corollary 1.1 of the Introduction. We give a sketch of the proof.

Sketch of the proof of Corollary 1.1. Assumption (H_0) is satisfied for $a_0(t) := \min\{q_0(t), \frac{1}{\sigma}\} - \rho$ (with $\sigma, \rho > 0$ so small that $\int_0^T a_0(t)\,dt > 0$) and $V_0(x,y) := x^2 + \rho y^2$. Assumption (H_∞) is satisfied (see also Remark 3.3) for $[\theta_\infty^1, \theta_\infty^2] := [-\frac{\pi}{2}, \frac{\pi}{2}]$, $a_\infty^n(t) := \frac{1}{nT}$ and $V_\infty^n(x,y) := x^2 + 2nTy^2$. □

Remark 4.1. We point out that assumption (g_0) can be replaced by the following more general condition:

(g_0') *$g(t,x)$ is continuous and there exist $q_0^+, q_0^- \in L^1([0,T])$ with $\int_0^T q_0^\pm(t)\,dt >$ 0 such that*

$$\liminf_{x\to 0^\pm} \frac{g(t,x)}{x} \geq q_0^\pm(t) \qquad \textit{uniformly in } t \in [0,T].$$

Indeed, one can use the improved version of Theorem 3.1 recalled in Remark 3.2.

We end the paper by showing an application of Corollary 1.1 to the Sitnikov problem. For a detailed description of this classicl problem in Celestial Mechanics, as well as of some classical and recent results, we refer to [6]. Here we just recall that such a problem leads to the following undamped and unforced second order ODE

$$u'' + \frac{u}{(u^2 + r(t,e)^2)^{3/2}} = 0, \tag{10}$$

being $e \in [0,1[$ the eccentricity of the orbits described by the primaries and $r(\cdot,e)$ implicitly defined by

$$r(t,e) := \frac{1}{2}(1 - e\cos v(t)), \qquad v(t) - e\sin v(t) = t.$$

The possibility of achieving multiplicity results for periodic solutions of (10) via the Poincaré-Birkhoff theorem is suggested in [7, Example 1], but only a semi-abstract result in term of the weighted eigenvalues of $u'' + \lambda \frac{u}{r(t,e)^3} = 0$ is given there. Here, as a straight consequence of Corollary 1.1 together with some computations for the number (7), we can get the following result.

Proposition 4.2. *For every $k \in \mathbb{N}_0$ and for every integer*

$$j \in \left[1, k\left(\frac{2}{1+e}\right)^{\frac{3}{2}}\right[$$

prime with k, equation (10) *has at least two* $2k\pi$*-periodic solutions (not* $2l\pi$*-periodic for* $l = 1,\cdots,k-1$*) with* $2j$ *zeros in* $[0,2k\pi[$.

We point out that the conclusion is optimal for the circular Sitnikov problem (i.e., the autonomous case $e = 0$), while subharmonics with larger number of zeros seem to be lost for greater values of e. This fact seems to be strictly related to the question raised in [6, p. 731].

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