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PERIODIC SOLUTIONS FOR SECOND ORDER HAMILTONIAN SYSTEMS

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In this paper we present some recent multiplicity results for a class of second order Hamiltonian systems. Exploiting the variational structure of the problem, it will be shown how the existence of multiple, even infinitely many, periodic solutions can be assured.

1. Introduction

The study of the existence of periodic solutions of the following Hamiltonian system

$$\begin{cases} u'' = \nabla F(t,u) \quad \text{a.e. in } [0,T] \\ u(T) - u(0) = u'(T) - u'(0) = 0, \end{cases} \tag{1}$$

where $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ and ∇F is the gradient of F with respect to u, over the last thirty or forty years, has been the subject of several monographs and papers. For example, we refer to the books of Mawhin and Willem [14] or Ekeland [12], as well as to the article of Brézis and Nirenberg [11] and the next results related to it [17–20].

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The natural variational structure of problem (1) allows to research the solutions as the critical points of the following functional

$$\varphi(u) := \frac{1}{2}\int_0^T |u'(t)|^2 dt + \int_0^T F(t,u(t))dt, \tag{2}$$

defined on the Sobolev space

$$H_T^1 := \left\{u \in L^2([0,T],\mathbb{R}^N) \text{ having weak derivative } u' \text{ in } L^2([0,T],\mathbb{R}^N)\right\}. \tag{3}$$

In particular, in [11], the Authors apply a well known multiple critical point theorem due to themselves, for a class of C^1 functionals which are bounded from below and satisfy a suitable local linking condition at zero, in order to obtain the existence of at least two nontrivial periodic solutions to problem (1), under a set of assumptions which, among the other, implies that $F(t,\cdot)$ has a particular behavior at zero, in addition to be coercive, namely:

there exist $r > 0$ *and an integer* $k \geq 0$ *such that*

$$-\frac{1}{2}(k+1)^2\omega^2|\xi|^2 \leq F(t,\xi) - F(t,0) \leq -\frac{1}{2}k^2\omega^2|\xi|^2, \tag{4}$$

for all $|\xi| \leq r$, *a. e.* $t \in [0,T]$, *where* $\omega = 2\pi/T$;

$$\lim_{|\xi|\to+\infty} F(t,\xi) = +\infty \text{ uniformly in } t. \tag{5}$$

In this note we consider the case when

$$F(t,\xi) = \frac{1}{2}A(t)\xi\cdot\xi - \lambda b(t)G(\xi), \tag{6}$$

where $A : [0,T] \to \mathbb{R}^{N\times N}$ is a suitable symmetric matrix-valued function, $A = (a_{ij})$, satisfying

$$A(t)\xi\cdot\xi \geq \mu|\xi|^2, \text{ a.e. in } [0,T], \ \forall \xi \in \mathbb{R}^N, \tag{7}$$

with $a_{ij} \in L^\infty([0,T])$, $\mu > 0$, $G \in C^1(\mathbb{R}^N)$, $b \in L^1([0,T])$ a. e. nonnegative and $\lambda > 0$. Hence, (1) reduces to

$$\begin{cases} u''(t) = A(t)u(t) - \lambda b(t)\nabla G(u(t)) & \text{a.e. in [0,T]} \\ u(T) - u(0) = u'(T) - u'(0) = 0. \end{cases} \tag{8}$$

More precisely, our aim is to present an overview of some recent results devoted to the study of problem (8) and that look at two possible aspects:

i) to assure the existence of at least two nontrivial periodic solutions of the considered problem, under assumptions on G that possibly avoid any coercivity of F, so that even the case when φ is indefinite can occur [1, 8, 9];

ii) to assure the existence of an unbounded sequence of periodic solutions of problem (8).

The point i) is treated in Section 2 exploiting some critical points theorems due to Bonanno [2, 3] and Bonanno-Candito [4]. While point ii) is discussed in Section 3 through a suitable application of a variational principle due to Ricceri [16] and its generalization obtained by Bonanno and Molica Bisci [10]. Indeed, the generality of the abstract results contained in [4, 10, 16] allows us to approach several other differential problems like a class of Sturm-Liouville boundary value problems [5] or a type of Neumann problems for elliptic equations involving the p-Laplacian [6, 7].

2. Multiple solutions

Before beginning to show the main results, it is worth noticing that the Sobolev space H^1_T defined in (3), equipped with the usual norm defined by $\|u\|_{H^1_T} = \left(\int_0^T |u'(t)|^2dt + \int_0^T |u(t)|^2dt\right)^{\frac{1}{2}}$, is compactly embedded in $C^0([0,T],\mathbb{R}^N)$. Moreover, if, as announced in the Introduction, we assume that, in addition to (7), the function $A:[0,T]\to\mathbb{R}^{N\times N}$ satisfies the following condition

$$A(\cdot) = (a_{ij}(\cdot)) \text{ is a symmetric matrix with } a_{ij}\in L^\infty([0,T]), \tag{9}$$

it is possible to introduce on H^1_T an equivalent norm by putting

$$\|u\|^2 := \int_0^T |u'(t)|^2dt + \int_0^T A(t)u(t)\cdot u(t)dt \quad \forall u \in H^1_T,$$

so that $(H^1_T,\|\cdot\|)$ is still compactly embedded in $C^0([0,T],\mathbb{R}^N)$ with constant of embedding $\bar{c}$ that can be estimated as follows

$$\bar{c} \le c := \sqrt{\frac{2}{m}}\max\left\{\sqrt{T}, \frac{1}{\sqrt{T}}\right\}, \tag{10}$$

where $m := \min\{1,\mu\}$.

The first multiple result that we can state has been proved in [1] and reads as follows.

Theorem 2.1. *Assume $G(0)\ge 0$ and let $\gamma>0$, $\bar{\xi}\in\mathbb{R}^N$ be such that*

$$(\mathrm{i}_1)\ |\bar{\xi}| > \frac{\gamma}{c\sqrt{\mu T}}, \qquad (\mathrm{i}_2)\ \frac{\max_{|\xi|\le\gamma} G(\xi)}{\gamma^2} < L\frac{G(\bar{\xi})}{|\bar{\xi}|^2},$$

being $L = \dfrac{1}{c^2T\sum_{i,j=1}^N \|a_{ij}\|_\infty}$, and c as in (10);

(i$_3$) $\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^2} < \frac{1}{2c^2\lambda^*}$,

where

$$\lambda^* := \frac{p\gamma^2}{c^2\left(L\gamma^2\frac{G(\bar{\xi})}{|\bar{\xi}|^2} - \max_{|\xi|\le\gamma} G(\xi)\right)}, \quad p > 1/2.$$

Then, for every function $b \in L^1([0,T]) \setminus \{0\}$ that is a.e. nonnegative, there exist an open interval $\Lambda \subseteq \left[0, \frac{\lambda^*}{\|b\|_1}\right]$ and $\rho > 0$ such that for every $\lambda \in \Lambda$ problem (8) admits at least three solutions, whose norms are less than ρ.

The main tool in the proof of the above result is the critical points theorem obtained by Bonanno in [2] as a refinement of a previous similar three critical point theorem due to Ricceri [15].
Although Theorem 2.1 represents a first step in finding at least two nontrivial periodic solutions to problem (8), it should be noted that, analyzing its conclusion, the estimation of the number ρ as well as a more precise determination of the interval Λ, which in this context is only localized, are two natural questions that deserve further studies. In fact, exploiting a subsequent abstract critical point theorem, again due to Bonanno [3], an answer to the previous issues has been pointed out in [8] thanks to the following result

Theorem 2.2. *Assume that $G(0) = 0$ and let $\gamma_1, \gamma_2 > 0$, $\bar{\xi} \in \mathbb{R}^N$ be such that*

(j$_1$) $\gamma_1 < |\bar{\xi}| < \sqrt{L}\gamma_2$,

(j$_2$) $\max\left\{\frac{\max_{|\xi|\le\gamma_1} G(\xi)}{\gamma_1^2}, \frac{\max_{|\xi|\le\gamma_2} G(\xi)}{\gamma_2^2}\right\} < R\frac{G(\bar{\xi})}{|\bar{\xi}|^2}$,

where $R = \frac{L}{1+L}$.
Then, for every $b \in L^1([0,T]) \setminus \{0\}$ that is a.e. nonnegative and for every

$$\lambda \in \left]\frac{1}{2R\|b\|_1 c^2}\frac{|\bar{\xi}|^2}{G(\bar{\xi})}, \frac{1}{2\|b\|_1 c^2}\min\left\{\frac{\gamma_1^2}{\max_{|\xi|\le\gamma_1} G(\xi)}, \frac{\gamma_2^2}{\max_{|\xi|\le\gamma_2} G(\xi)}\right\}\right[$$

problem (8) admits at least two solutions u_1, u_2 such that $\|u_1\|_{C^0} \le \gamma_1$ and $\|u_2\|_{C^0} \le \gamma_2$.

In order to better understand the meaning of assumptions (j$_1$) $-$ (j$_2$), we point out that Theorem 2.2 can be applied whenever there exist three concentric balls in $\mathbb{R}^N$, centered at zero, with radius γ_1, $|\bar{\xi}|$ and γ_2 respectively, such that G has a strong superquadratic growth up to the intermedia ball and a subquadratic growth in the annulus $B_{\gamma_2} \setminus B_{|\bar{\xi}|}$.

It is also meaningful to observe that, since no growth condition at infinity is required on G, the functional F defined in (6) could be not coercive, namely (5) fails, as well as φ could be unbounded from below.

The conclusion of Theorem 2.2 can be refined under a set of assumptions that is slightly less general. In particular, making use of a general critical points result due to Bonanno and Candito [4], in [9] the following has been proved

Theorem 2.3. *Assume that $G(\xi) \geq G(0) = 0$ for every $\xi \in \mathbb{R}^N$ and let $\gamma_1, \gamma_2 > 0$, $\bar{\xi} \in \mathbb{R}^N$ be such that*

(k₁) $\gamma_1 < |\bar{\xi}| < \sqrt{\frac{L}{2}}\gamma_2$,

(k₂) $\dfrac{\max_{|\xi| \leq \gamma_1} G(\xi)}{\gamma_1^2} < R\dfrac{G(\bar{\xi})}{|\bar{\xi}|^2}, \qquad \dfrac{\max_{|\xi| \leq \gamma_2} G(\xi)}{\gamma_2^2} < \dfrac{R}{2}\dfrac{G(\bar{\xi})}{|\bar{\xi}|^2},$

being $R = \dfrac{L}{1+L}$.
Then, for every $b \in L^1([0,T]) \setminus \{0\}$ a.e. nonnegative and for every $\lambda \in \Lambda_{\gamma_1,\gamma_2} :=$

$$\left]\frac{1}{2R\|b\|_1 c^2}\frac{|\bar{\xi}|^2}{G(\bar{\xi})}, \frac{1}{2\|b\|_1 c^2}\min\left\{\frac{\gamma_1^2}{\max_{|\xi|\leq\gamma_1} G(\xi)}, \frac{\gamma_2^2}{2\max_{|\xi|\leq\gamma_2} G(\xi)}\right\}\right[$$

problem (8) admits at least two ***non trivial*** *solutions u_1, u_2 such that $\|u_i\|_{C^0} \leq \gamma_2$, $i = 1,2$.*

Example 2.4. Let us define

$$G(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0 \\ e^{e^{\xi}} - e(\xi+1) & \text{if } 0 < \xi < 2 \\ (e^{e^2}e^2 - e)\xi - e^{e^2}e^2 - e & \text{if } 2 \leq \xi < e^5\sqrt{e} \\ \frac{e^{e^2}e^2 - e}{3e^5\sqrt{e}}\xi^3 + \rho & \text{if } \xi \geq e^5\sqrt{e}, \end{cases} \tag{11}$$

where $\rho = -\frac{e^{e^2}e^2-e}{3}e^{11} + (e^{e^2}e^2 - e)e^5\sqrt{e} - e^{e^2}e^2 - e$.
Then, for every $b \in L^1([0,1]) \setminus \{0\}$ that is a.e. nonnegative and for every

$$\lambda \in \left]\frac{3}{\|b\|_1(e^{e^2} - 3e)}, \frac{e^{11}}{8\|b\|_1(e^5\sqrt{e} - 1)(e^{e^2}e^2 - 1)}\right[$$

the problem

$$\begin{cases} u'' = u - \lambda b(t)G'(u) & \text{a.e. in } [0,1] \\ u(1) - u(0) = u'(1) - u'(0) = 0. \end{cases}$$

admits at least two non trivial solutions u_1, u_2 such that $\|u_i\|_{C^0} \leq e^5\sqrt{e}$ with $i = 1,2$. In fact, we can apply Theorem 2.3 where $N = 1$, $T = 1$, $A(t) = 1$, $c = \sqrt{2}$, $L = 1/2$, $\gamma_1 = 1$, $\gamma_2 = e^5\sqrt{e}$ and $\bar{\xi} = 2$.

The preceding example assures that Theorem 2.3 is independent from Theorem 7 of [11]. Indeed, if F is as defined in (6), where G is given in (11), it is easy to observe that (4) fails. Moreover, if, in addition $b \in C^0([0,1],\mathbb{R}^+)$, one has that

$$\lim_{\xi\to+\infty} F(t,\xi) = -\infty, \quad \lim_{\xi\to-\infty} F(t,\xi) = +\infty$$

uniformly with respect to t. Hence, the coercivity assumption required by Brézis-Nirenberg (see (5)) and weakened in [18], [20] does not hold.
Finally, a simple computation shows that the functional φ is unbounded from below.

A simple but meaningful consequence of Theorem 2.3 is the following

Corollary 2.5. *Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $g(x)x \geq 0$ for all $x \in \mathbb{R}$ and put $G(\xi) = \int_0^\xi g(x)dx$ for all $\xi \in \mathbb{R}$. Assume that there are three positive constants $\gamma_1, \gamma_2, \bar{\xi}$ such that*

(k$'_1$) $\gamma_1 < \bar{\xi} < \frac{\gamma_2}{2}$,

(k$'_2$) $\dfrac{G(\gamma_1)}{\gamma_1^2} < \dfrac{1}{3}\dfrac{G(\bar{\xi})}{\bar{\xi}^2}, \qquad \dfrac{G(\gamma_2)}{\gamma_2^2} < \dfrac{1}{6}\dfrac{G(\bar{\xi})}{\bar{\xi}^2}.$

Then, for every $b \in L^1([0,T],\mathbb{R}^N)\setminus\{0\}$ a.e. nonnegative and for every

$$\lambda \in \Lambda_{\gamma_1,\gamma_2} := \left]\frac{3}{4\|b\|_1}\frac{|\bar{\xi}|^2}{G(\bar{\xi})}, \frac{1}{4\|b\|_1}\min\left\{\frac{\gamma_1}{G(\gamma_1)}, \frac{\gamma_2}{2G(\gamma_2)}\right\}\right[$$

the problem

$$\begin{cases} u'' = u - \lambda b(t)g(u) & \text{a.e. in } [0,1] \\ u(1)-u(0) = u'(1)-u'(0) = 0. \end{cases}$$

admits at least two non trivial solutions u_1, u_2 such that $\|u_i\|_{C^0} \leq \gamma_2$ with $i = 1,2$.

3. Infinitely many solutions

Throughout this section, A satisfies assumptions (9) and (7), while $G \in C^1(\mathbb{R}^N)$. The first main result that we recall is contained in [13].

Theorem 3.1. *Let $b \in L^1([0,T])\setminus\{0\}$ be an a.e. nonnegative function. Assume that there are sequences $\{r_n\}$ in $\mathbb{R}^+$, with $\lim_{n\to+\infty} r_n = +\infty$, and $\{\xi_n\}$ in $\mathbb{R}^N$ such that, for each integer n one has*

$$|\xi_n| < (2r_n/(T\sum_{i,j=1}^N \|a_{ij}\|_\infty))^{1/2}, \qquad G(\xi_n) = \max_{|\xi|\leq\bar{c}(2r_n)^{1/2}} G(\xi). \tag{12}$$

Moreover, assume that

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^2} > \frac{T}{2}\Big(\sum_{i,j=1}^{N} \|a_{ij}\|_\infty\Big)/\|b\|_1.$$

Then, problem

$$\begin{cases} u''(t) = A(t)u(t) - b(t)\nabla G(u(t)) & \text{a.e. in } [0,T] \\ u(T) - u(0) = u'(T) - u'(0) = 0. \end{cases}$$

admits an unbounded sequence of solutions.

The preceding result has been proved applying a general variational principle obtained by Ricceri [16]. A refinement of the cited Ricceri's theorem has been proved by Bonanno and Molica Bisci in [10]. It is the main tool in obtaining the following result dealing with the existence of infinitely many solutions to problem (8)

Theorem 3.2. *Put*

$$\alpha := \liminf_{\rho\to+\infty} \frac{\max_{|\xi|\le\rho} G(\xi)}{\rho^2}, \qquad \beta := \limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^2}$$

and assume that

$$\alpha < L\beta, \tag{13}$$

where L is defined in Theorem 2.1.
Then, for every $b \in L^1([0,T]) \setminus \{0\}$ *that is a.e. nonnegative and for every* $\lambda \in \Lambda := \left] \frac{1}{2c^2 L\beta\|b\|_1}, \frac{1}{2c^2\alpha\|b\|_1} \right[$ *problem (8) admits an unbounded sequence of solutions.*

The next result, which is a consequence of the preceding Theorem 3.2, shows that assumption (13) allows us to consider a class of nonlinearities G which is different from that one satisfying assumption (12).

Corollary 3.3. *Let* $G \in C^1(\mathbb{R})$ *be a non decreasing function. Assume that*

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} < \frac{1}{2}\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{\xi^2}.$$

Then, for every $b \in L^1([0,1]) \setminus \{0\}$ *that is a.e. nonnegative and for every*

$$\lambda \in \left] \frac{1}{2\|b\|_1 \limsup_{|\xi|\to+\infty} \frac{G(\xi)}{\xi^2}}, \frac{1}{4\|b\|_1 \liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}} \right[$$

the following problem

$$\begin{cases} u'' = u - \lambda b(t)G'(u) & a.e.\ in\ [0,1] \\ u(1) - u(0) = u'(1) - u'(0) = 0, \end{cases}$$

admits an unbounded sequence of solutions.

Example 3.4. For every $n \in \mathbb{N}$, put

$$a_n := \frac{2n!(n+2)! - 1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)! + 1}{4(n+1)!},$$

$D = \bigcup_{n\in\mathbb{N}} [a_n, b_n]$ and define the following continuous functions $g_1, g_2, g : \mathbb{R} \to \mathbb{R}$ by putting

$$g_1(x) := \pi(n+1)! \left[(n+1)!^2 - n!^2\right] \chi_D(x) \sin(2\pi(n+1)!(x - a_n)),$$

$$g_2(x) = \max\{0, x\}, \qquad g(x) = g_1(x) + g_2(x)$$

for every $x \in \mathbb{R}$. Then, applying Corollary 3.3 where $G(\xi) := \int_0^\xi g(x)dx$, we conclude that for every $b \in L^1([0,1]) \setminus \{0\}$ that is a.e. nonnegative and for every $\lambda \in \left]\frac{1}{5\|b\|_1}, \frac{1}{2\|b\|_1}\right[$, the problem

$$\begin{cases} u'' = u - \lambda b(t)g(u) & \text{a.e. in } [0,1] \\ u(1) - u(0) = u'(1) - u'(0) = 0, \end{cases}$$

admits an unbounded sequence of solutions.

We conclude observing that, in the case $N = 1$, Theorems 3.1 and 3.2 can be applied whenever G has a suitable oscillating behavior at infinity. To have a model of this kind of oscillations we can consider the function $G : \mathbb{R} \to \mathbb{R}$ defined by putting

$$G(x) = \begin{cases} x^2 \cos^2(\ln x) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

REFERENCES

[1] G. Barletta - R. Livrea, *Existence of three periodic solutions for a non autonomous second order system*, Le Matematiche 57 (2) (2002), 205–215.

[2] G. Bonanno, *Some remarks on a critical points theorem*, Nonlinear Anal. 54 (2003), 651–665.

[3] G. Bonanno, *Multiple critical points theorems without the Palais-Smale condition*, J. Math. Anal. Appl. 299 (2004), 600–614.

[4] G. Bonanno - P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations 244 (2008), 3031–3059.

[5] G. Bonanno - G. D'Aguì, *A Neumann boundary value problem for Sturm-Liouville equation*, Appl. Mat. Comput. 208 (2009), 318–327.

[6] G. Bonanno - G. D'Aguì, *On the Neumann problem for elliptic equations involving the p-Laplacian*, J. Math. Anal. Appl. 358 (2009), 223–228.

[7] G. Bonanno - G. D'Aguì, *Multiplicity Results for a Perturbed elliptic Neumann Problem*, Abstr. Appl. Anal. (2010), 10 pp.

[8] G. Bonanno - R. Livrea, *Periodic solutions for a class of second-order Hamiltonian systems*, Electron. J. Differential Equations 115 (2005), 1–13.

[9] G. Bonanno - R. Livrea, *Multiple periodic solutions for Hamiltonian systems with not coercive potential*, J. Math. Anal. Appl. 363 (2010), 627–638.

[10] G. Bonanno - G. Molica Bisci, *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl. (2009), 20 pp.

[11] H. Brézis - L. Nirenberg, *Remarks on finding critical points*, Comm. Pure. Appl. Math. 44 (1991), 939–963.

[12] I. Ekeland, *Convexity Methods in Hamiltonian Mechanics*, Springer-Verlag, Berlin, 1990.

[13] F. Faraci - R. Livrea, *Infinitely many periodic solutions for a second order nonautonomous system*, Nonlinear Anal. 54 (2003), 417–429.

[14] J. Mawhin - M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.

[15] B. Ricceri, *On a three critical points theorem*, Arch. Math. (Basel) 75 (2000) 220–226.

[16] B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math. 113 (2000) 401–410.

[17] C. L. Tang, *Existence and multiplicity periodic solutions of nonautonomous second order systems*, Nonlinear Anal. 32 (1998), 299–304.

[18] C. L. Tang - X. P. Wu, *Periodic solutions for second order systems with not uniformly coercive potential*, J. Math. Anal. Appl. 259 (2001), 386–397.

[19] C. L. Tang - X. P. Wu, *Notes on periodic solutions of subquadretic second order systems*, J. Math. Anal. Appl. 285 (2003), 8–16.

[20] Y-W. Ye - C. L. Tang, *Periodic solutions for some nonautonomous second order Hamiltonian systems*, J. Math. Anal. Appl. 344 (2007), 462–471.

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