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EXISTENCE RESULTS FOR AN ELLIPTIC DIRICHLET PROBLEM

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The main purpose of this paper is to present recent existence results for an elliptic eigenvalue Dirichlet problem. Precisely, our method ensures the existence of an exactly determined open interval (possibly unbounded) of positive parameters for which the problem admits infinitely many weak solutions.

1. Introduction

This note concerns existence of infinitely many solutions to the Dirichlet problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \qquad (D_\lambda^{q,f})$$

where Ω is a bounded open subset of the Euclidean space $(\mathbb{R}^N, |\cdot|)$, $N \geq 1$, with boundary $\partial\Omega$ of class C^1 and Lebesgue measure "meas(Ω)". Moreover $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > N$, is the usual Laplace operator, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $q \in L^\infty(\Omega)$ and λ is a positive real parameter.

The existence of infinitely many solutions for problem $(D_\lambda^{0,f})$ has been widely investigated. The most classical results in this topic are essentially based on the Ljusternik-Schnirelman theory.

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In them, a key role is played by the oddness of the nonlinearity. Moreover, conditions which do not allow an oscillating behaviour of the nonlinearity f are necessary in order to check the Palais-Smale condition. Multiplicity results with an oscillating behaviour of f are more rare.

In this direction, very recently, in [2], under some hypotheses on the behavior of the potential of the nonlinear term at infinity, the existence of an interval Λ such that, for each $\lambda \in \Lambda$, problem $(D_\lambda^{q,f})$ admits a sequence of pairwise distinct weak solutions is proved (see Theorem 3.1). Moreover, replacing the conditions at infinity of the potential by a similar at zero, the same results hold and, in addition, the sequence of pairwise distinct solutions uniformly converges to zero (see Remark 1.4).

In order to recall our result, let us introduce some constants associated to the geometry of the set Ω. Define

$$\sigma(N,p) := \inf_{\mu \in]0,1[} \frac{1-\mu^N}{\mu^N(1-\mu)^p},$$

and consider $\overline{\mu} \in]0,1[$ such that $\sigma(N,p) = \dfrac{1-\overline{\mu}^N}{\overline{\mu}^N(1-\overline{\mu})^p}$.

Further, let $\tau := \sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)$ and

$$\kappa := \frac{\tau^p \overline{\mu}^N}{\operatorname{meas}(\Omega) m^p (\overline{\mu}^N \sigma(N,p) + \|q\|_\infty \tau^p g_{\overline{\mu}}(p,N))}, \tag{1}$$

where $\|q\|_\infty := \operatorname{ess\,sup}_{x\in\Omega} q(x)$,

$$m := \frac{N^{-\frac{1}{p}}}{\sqrt{\pi}} \left[\Gamma\left(1+\frac{N}{2}\right)\right]^{\frac{1}{N}} \left(\frac{p-1}{p-N}\right)^{1-\frac{1}{p}} \operatorname{meas}(\Omega)^{\frac{1}{N}-\frac{1}{p}},$$

and

$$g_{\overline{\mu}}(p,N) := \overline{\mu}^N + \frac{1}{(1-\overline{\mu})^p} N B_{(\overline{\mu},1)}(N,p+1).$$

Here, $B_{(\overline{\mu},1)}(N,p+1)$ denotes the generalized incomplete beta function defined as follows

$$B_{(\overline{\mu},1)}(N,p+1) := \int_{\overline{\mu}}^1 t^{N-1}(1-t)^{(p+1)-1} dt,$$

and Γ is the Gamma function. Throughout the sequel, ω_τ denotes the measure of the N-dimensional ball of radius τ,

$$F(\xi) := \int_0^\xi f(t)dt,$$

for each $\xi \in \mathbb{R}$ and

$$B := \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p}.$$

Our existence result, obtained in [2], can be stated as follows

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(i) $F(\xi) \geq 0$ *for every* $\xi \geq 0$;

(ii) *there exist two sequences* $\{a_n\}$ *and* $\{b_n\}$ *such that*

$$0 \leq a_n < \frac{1}{\overline{\mu}^{N/p} m \Big(\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty \frac{g_{\overline{\mu}}(p,N)}{\overline{\mu}^N} \Big)^{1/p} \omega_\tau^{1/p}} b_n,$$

for every $n \in \mathbb{N}$ *and* $\lim_{n \to +\infty} b_n = +\infty$ *such that*

$$A_1 < \mathrm{meas}(\Omega) \kappa B, \tag{2}$$

where κ *is given by* (1) *and*

$$A_1 := \lim_{n \to +\infty} \frac{\mathrm{meas}(\Omega) \max_{|t| \leq b_n} F(t) - \overline{\mu}^N \omega_\tau F(a_n)}{b_n^p - m^p a_n^p \omega_\tau \Big[\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty \frac{g_{\overline{\mu}}(p,N)}{\overline{\mu}^N} \Big] \overline{\mu}^N}.$$

Then, for every

$$\lambda \in \Lambda_{q,g} := \Big] \frac{1}{pB} \Big(\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty \frac{g_{\overline{\mu}}(p,N)}{\overline{\mu}^N} \Big), \frac{1}{m^p p A_1} \Big[,$$

the problem $(D_\lambda^{q,f})$ *admits a sequence of weak solutions which is unbounded in* $W_0^{1,p}(\Omega)$.

The main tool in order to obtain Theorem 1.1 is a refinement of the Variational Principle of Ricceri (see the quoted paper [10]) recently obtained in [1].

Remark 1.2. Condition (ii) in Theorem 1.1 is technical and could be replaced by the more simple and sufficient assumption

(ii′) $\liminf_{\xi \to +\infty} \frac{\max_{|t| \leq \xi} F(t)}{\xi^p} < \kappa \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p}.$

In this setting, if

$$A := \liminf_{\xi\to+\infty} \frac{\max_{|t|\le\xi} F(t)}{\xi^p},$$

for every

$$\lambda \in \Lambda'_{q,g} := \left]\frac{1}{pB}\Big(\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty \frac{g_{\overline{\mu}}(p,N)}{\overline{\mu}^N}\Big), \frac{1}{m^p pA}\right[,$$

the problem $(D_\lambda^{q,f})$ admits a sequence of weak solutions which is unbounded in $W_0^{1,p}(\Omega)$. If, in addition, the nonlinearity f is non-negative, hypothesis (ii$'$) can be written as follows

(ii$''$) $\displaystyle\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} < \kappa \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}.$

Remark 1.3. We observe that in the very interesting paper [8], the authors assume

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} = 0 \quad\text{and}\quad \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} = +\infty,$$

which are conditions that imply our key assumptions. Moreover, when $q \equiv 0$, our theorems and the results in [7] and [8] are mutually independent (see Theorem 1.1, Example 4.1 and Remark 4.1 in [2]).

Remark 1.4. Replacing the condition (ii) at infinity by the following one at zero

(jj) there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$0 \le a_n < \frac{1}{\overline{\mu}^{N/p} m\Big(\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty \frac{g_{\overline{\mu}}(p,N)}{\overline{\mu}^N}\Big)^{1/p} \omega_\tau^{1/p}} b_n,$$

for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} b_n = 0$ such that

$$A_1 < \kappa\,\mathrm{meas}(\Omega) \limsup_{\xi\to0^+}\frac{F(\xi)}{\xi^p},$$

a sequence of pairwise distinct solutions uniformly converging to zero is obtained.

Remark 1.5. We point out that the results contained in [4] are direct consequences of the main Theorem. On the other hand, we do not require as in [6, Corollary 3.1] that the function f is definitively non-positive.

2. Main Results

Taking Remark 1.2 into account, a particular case of Theorem 1.1 reads as follows.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative function and assume that*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} < \frac{\tau^p}{m^p \operatorname{meas}(\Omega)\sigma(N,p)} \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p}.$$

Then, for each $\lambda \in \left] \frac{\sigma(N,p)}{p\tau^p \limsup\limits_{\xi \to +\infty} \frac{F(\xi)}{\xi^p}}, \frac{1}{pm^p \operatorname{meas}(\Omega) \liminf\limits_{\xi \to +\infty} \frac{F(\xi)}{\xi^p}} \right[$, *the problem*

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{$D_\lambda^{0,f}$}$$

admits a sequence of pairwise distinct positive weak solutions in $W_0^{1,p}(\Omega)$.

The next result is a simpler but less general form of Theorem 1.1 in the ordinary case (see [5, Theorem 1.1]).

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative function. Assume that the following condition holds*

$f_2'')$ $\displaystyle \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} < \frac{1}{4} \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}.$

Then, for each

$$\lambda \in \Lambda := \left] \frac{8}{\limsup\limits_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}}, \frac{2}{\liminf\limits_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}} \right[,$$

for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$g_1')$ $\displaystyle G_\infty^\star := \lim_{\xi \to +\infty} \frac{\int_0^\xi g(t)\,dt}{\xi^2} < +\infty,$

and for every $\mu \in [0, \widehat{\mu}_{g,\lambda}[$, *where*

$$\widehat{\mu}_{g,\lambda} := \frac{1}{G_\infty^\star}\Big(2 - \lambda \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}\Big),$$

the following problem

$$\begin{cases} -u'' = \lambda f(u) + \mu g(u) & \text{in} \quad]0,1[\\ u(0) = u(1) = 0, \end{cases} \qquad (P_\lambda^{f,g})$$

admits a sequence of pairwise distinct positive classical solutions.

Example 2.3. For instance, for each $(\lambda, \mu) \in \Lambda \times [0, +\infty[$, the following Dirichlet problem

$$\begin{cases} -u'' = \lambda f(u) + \mu \sqrt{|u|} & \text{in} \quad]0,1[\\ u(0) = u(1) = 0, \end{cases}$$

where

$$f(u) := \begin{cases} u \cos^2(\ln(u)) & \text{if } u > 0 \\ 0 & \text{if } u \leq 0. \end{cases}$$

possesses a sequence of pairwise distinct positive classical solutions.

Here, we present a simple and direct consequence of Theorem 1.1 that improves Proposition 1.1 in [4] (see also Remark 2.5).

Proposition 2.4. *Let* $\{a_n\}$, $\{b_n\}$ *be two sequences in* $]0, +\infty[$, $a_n < b_n < a_{n+1}$ $\forall\, n \geq n_0$, *for some* $n_0 \in \mathbb{N}$, $\lim_{n \to +\infty} b_n = +\infty$ *and* $\lim_{n \to +\infty} \frac{b_n}{a_n} = +\infty$. *Moreover, let* $\varphi \in C^1([0,1])$ *be a non-negative and non-zero function such that* $\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0$ *and let* $g : \mathbb{R} \to \mathbb{R}$ *be the function defined by*

$$g(t) := \begin{cases} \varphi\Big(\dfrac{t - b_n}{a_{n+1} - b_n}\Big) & \text{if } t \in \bigcup_{n \geq n_0} [b_n, a_{n+1}] \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

Then, for every

$$\lambda > \lambda_\varphi := \frac{1}{p \max_{s \in [0,1]} \varphi(s)} \Big(\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty \frac{g_{\overline{\mu}}(p,N)}{\overline{\mu}^N}\Big),$$

the problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda h(u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \qquad (D_\lambda^{q,h})$$

where

$$h(u) := u^{p-1}(pg(u) + ug'(u)),$$

admits an unbounded sequence of positive weak solutions in $W_0^{1,p}(\Omega)$.

Proof. Let $\{a_n\},\{b_n\}$ be two positive sequence satisfying our assumptions. We claim that all the hypotheses of Theorem 1.1 are verified. In fact, taking into account the definition of φ, one has that g is a $C^1(\mathbb{R})$-function and

$$F(\xi)=\int_0^{\xi} h(t)dt=\xi^p g(\xi),\ \ \forall\,\xi\in\mathbb{R}^+.$$

Hence, hypothesis (i) is verified. Moreover,

$$0<a_n<\frac{1}{\overline{\mu}^{N/p}m\Big(\frac{\sigma(N,p)}{\tau^p}+\|q\|_\infty\frac{g_{\overline{\mu}}(p,N)}{\overline{\mu}^N}\Big)^{1/p}\omega_\tau^{1/p}}b_n,$$

for every n sufficiently large. Then, condition (2) holds. Indeed, direct computations ensure that

$$\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}=\limsup_{\xi\to+\infty}g(\xi)=\max_{s\in[0,1]}\varphi(s),$$

and $A_1=0$. Hence, for every $\lambda>\lambda_\varphi$, Theorem 1.1 guarantees the existence of an unbounded sequence in $W_0^{1,p}(\Omega)$ of weak solutions of problem $(D_\lambda^{q,h})$. Finally, arguing as in [2, Remark 3.3], the Strong Maximum Principle (see [9, Theorem 11.1]) ensures that the obtained solutions are positive. □

Remark 2.5. We point out that if $\{a_n\}$, $\{b_n\}$ are two sequences in $]0,+\infty[$ such that $b_{n+1}<a_n<b_n$, $\forall n\geq n_0$, for some $n_0\in\mathbb{N}$, $\lim\limits_{n\to+\infty}b_n=0$, $\lim\limits_{n\to+\infty}\dfrac{b_n}{a_n}=+\infty$ and $g:\mathbb{R}\to\mathbb{R}$ is defined by

$$g(t):=\begin{cases}\varphi\Big(\dfrac{t-b_{n+1}}{a_n-b_{n+1}}\Big) & \text{if } t\in\bigcup\limits_{n\geq n_0}[b_{n+1},a_n]\\ 0 & \text{otherwise,}\end{cases}$$

then, for every $\lambda>\lambda_\varphi$, the problem $(D_\lambda^{q,h})$ admits a sequence of pairwise distinct positive weak solutions which strongly converges to zero in $W_0^{1,p}(\Omega)$. For instance, let

$$a_n:=\frac{1}{n!n}\quad\text{and}\quad b_n:=\frac{1}{n!},$$

for every $n\geq 2$ and $\varphi\in C^1([0,1])$ given by

$$\varphi(s):=\exp\left(\frac{1}{s(s-1)}\right),\quad\forall s\in[0,1],$$

and zero otherwise.

Hence, let g be the $C^1(\mathbb{R})$-function given by

$$g(t) := \exp\left(\frac{1}{n(n+1)!\Big(t-\frac{1}{(n+1)!}\Big)\Big(n\Big(t-\frac{1}{(n+1)!}\Big)(n+1)!-1\Big)}\right),$$

for every $t \in \bigcup_{n\geq 2}\left[\frac{1}{(n+1)!}, \frac{1}{n!n}\right]$ and zero otherwise.
Consider the problem

$$\begin{cases} -\Delta_p u + e|u|^{p-2}u = \lambda h(u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \qquad (D_\lambda^{e,h})$$

where

$$h(t) := u^{p-1}(pg(u) + ug'(u)).$$

Then, for every

$$\lambda > \frac{e^4}{p}\Big(\frac{\sigma(N,p)}{\tau^p} + e\frac{g_{\overline{\mu}}(p,N)}{\overline{\mu}^N}\Big),$$

problem $(D_\lambda^{e,h})$ possesses a sequence of pairwise distinct positive weak solutions which strongly converges to zero in $W_0^{1,p}(\Omega)$.

Finally, we just observe that, by using the same variational approach, in [3], the existence of infinitely many weak solutions for quasilinear elliptic systems has been widely investigated. A special case reads as follows; see [3, Theorem 1.2].

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^N$ be a non-empty bounded open set with boundary of class C^1. Let $f,g : \mathbb{R}^2 \to \mathbb{R}$ be two positive $C^0(\mathbb{R}^2)$-functions such that the differential 1-form $\omega := f(\xi,\eta)d\xi + g(\xi,\eta)d\eta$ is integrable and let F be a primitive of ω such that $F(0,0) = 0$. Fix $p,q > N$, with $p \leq q$, and assume that*

$$\liminf_{y\to+\infty}\frac{F(y,y)}{y^p} = 0 \quad \textit{and} \quad \limsup_{y\to+\infty}\frac{F(y,y)}{y^q} = +\infty.$$

Then, the problem

$$\begin{cases} -\Delta_p u = f(u,v) & \text{in} \quad \Omega \\ -\Delta_q v = g(u,v) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases}$$

admits an unbounded sequence $\{(u_n,v_n)\} \subset W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of positive weak solutions.

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