LE MATEMATICHE
Vol. LXVI (2011) – Fasc. I, pp. 143–151
doi: 10.4418/2011.66.1.14

A NOTE ON SOME ELLIPTIC EQUATIONS OF ANISOTROPIC TYPE

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We prove the existence of weak solutions to some nonlinear elliptic equations governed by an anisotropic operator mapping an appropriate function space to its dual. A sign condition with no growth restrictions with respect to the variable solution is imposed to a perturbed nonlinear term to the operator. The data is considered to be close to L^1.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N, N \geq 1$. We denote $p_0, \ldots, p_N$ real numbers with $p_i > 1, i = 0, \ldots, N, \vec{p} = (p_1, \ldots, p_n)$. Let $X = W_0^{1,\vec{p},\varepsilon}(\Omega)$ be the anisotropic Sobolev space associated with the vector $\vec{p}$. Let A be the nonlinear operator from X into the dual X^* defined as

$$Au = -\operatorname{div}(a(x, u, \nabla u)),$$

where

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N, a(x, u, \xi) = \{a_i(x, u, \xi)\}, i = 1, \ldots, N,$$

is a Carathéodory vector-valued function, that is, measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$

Entrato in redazione: 14 giugno 2010

AMS 2010 Subject Classification: 35J60, 35J65, 35J70.
Keywords: Anisotropic nonlinear equations, Weak solutions, Sign condition.

for almost every x in Ω. Further, the vector field $a(x,u,\xi)$ is anisotropic, that is, each coordinate a_i behaves like $|\xi|^{p_i}$ for possibly different p_i.

More precisely, we assume that there exist two real positive constants α and β and a nonnegative function $k \in L^1(\Omega)$ such that

(A1)
$$|a_i(x,s,\xi)| \leq \beta \left[k(x) + |s|^{p_0} + \sum_{j=1}^{N} |\xi_j|^{p_j} \right]^{1-\frac{1}{p_i}},$$

for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and all $i = 1, \ldots .N$.

(A2)
$$\sum_{i=1}^{N} [a_i(x,s,\xi) - a_i(x,s,\xi^*)](\xi_i - \xi_i^*) > 0,$$

for a.e. $x \in \Omega$, for every $\xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$.

(A3)
$$a(x,s,\xi)\xi \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i}$$

for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Consider the following nonlinear Dirichlet problem

$$Au + g(x,u) = f \quad \text{in } \Omega, \tag{1.1}$$

where g is a nonlinear lower-order term having no growth conditions with respect to $|u|$ and verifying the following assumption

(G) $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function satisfying

$$\sup_{|u| \leq s} |g(x,u)| \leq h_s(x),$$

for a.e. $x \in \Omega$, all $s > 0$ and some function $h_s \in L^{\frac{1}{1-\varepsilon}}(\Omega), 0 < \varepsilon < 1$. We assume also the "sign condition" $g(x,u)u \geq 0$, for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$.

Let us mention that many results in the isotropic case have published for problems of the form (1.1) involving operators of type A in the variational case (i.e., where f belongs to the dual) and in the L^1 case, we restrict ourselves to papers dealing with L^1-data since our problem is close to this case and we cite the papers among others [4] and [7]. In the anisotropic case, it would be interesting to refer the reader to the works [3], [6] and to the recent works [2] and [5], where the authors proved the existence of solutions of some anisotropic elliptic equations for a general class of operators of higher order.

The purpose in this paper is to establish existence of weak solution to some anisotropic elliptic equations. In particular, based on techniques related to that of Webb [16], we give some estimates which help for the study of the problem of the form (1.1) taking zero boundary data on Ω.

2. Preliminaries

Anisotropic Sobolev spaces. We start by recalling that the notion of anisotropic Sobolev spaces were introduced and studied by Nikolskiĭ [9], Slobodeckiĭ [13], and Troisi [14], and later by Trudinger [15] in the framework of Orlicz spaces. Let Ω be a bounded open subset of $\mathbb{R}^N$, $(N \geq 1)$ and let $\vec{p} = (p_1, ..., p_N)$ be vector of real numbers, with $1 < p_i < \infty, i = 1, \ldots, N$. We denote by $W^{1,\vec{p}}(\Omega)$, called anisotropic Sobolev space, the space of all real-valued functions $u \in L^{p_0}(\Omega), p_0 > 1$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega) \text{ for all } i = 1, \ldots, N.$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,\vec{p},p_0} = \left(\int_\Omega |u(x)|^{p_0}\, dx\right)^{\frac{1}{p_0}} + \sum_{i=1}^{N} \left(\int_\Omega \left|\frac{\partial u(x)}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}}. \tag{2.1}$$

The space $W_0^{1,\vec{p}}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{1,\vec{p}}$. The theory of such anisotropic spaces was developed in [10],[11], [12], [14]. It was proved that $C_0^\infty(\Omega)$ is dense in $W_0^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$ is a reflexive Banach space for any $\vec{p} = (p_1, ..., p_N)$, with $1 < p_i < \infty, i = 1, \ldots, N$. We recall that the dual space of the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ is equivalent to $W^{-1,\vec{p}'}(\Omega)$, where $\vec{p}'$ is the conjugate of $\vec{p}$, i. e., $p_i' = \frac{p_i}{p_i - 1}, i = 1, \ldots, N$.
In the following, we will use the anisotropic Sobolev space given by

$$W^{1,\vec{p},\varepsilon}(\Omega) = \left\{ u \in L^{1+\frac{1}{\varepsilon}}(\Omega), \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, \cdots, N \right\},$$

under the norm

$$\|u\|_{1,\vec{p},\varepsilon} = \|u\|_{L^{1+\frac{1}{\varepsilon}}(\Omega)} + \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}.$$

Let $W_0^{1,\vec{p},\varepsilon}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\vec{p},\varepsilon}(\Omega)}$ endowed with the norm

$$\|u\| = \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}.$$

The dual of $W_0^{1,\vec{p},\varepsilon}(\Omega)$ is denoted by $W^{-1,\vec{p}',\varepsilon}(\Omega)$, where $\vec{p}' = \{p_i', i = 1,\dots,N\}$, $p_i' = \frac{p_i}{p_i-1}$ and $p_0 = 1 + \frac{1}{\varepsilon}$, $p_i > 1$ for $i = 1,\dots,N$. Here ε is a positive number satisfying $0 < \varepsilon < 1$. We prove the existence of distributional solutions in an appropriate function space for nonlinear elliptic equation

$$(P_\varepsilon)\begin{cases} u \in W_0^{1,\vec{p},\varepsilon}(\Omega), \\ Au + g(x,u) = f \text{ in } \Omega, \end{cases}$$

where $f \in L^{1+\varepsilon}(\Omega)$. The operator A and the function g are assumed to satisfy the conditions $(A1),(A2),(A3)$ and (G) respectively.

We state our main result as follows.

Theorem 2.1. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and $0 < \varepsilon < 1$. Assume $(A1),(A2),(A3)$ and (G) hold. Then for all $f \in L^{1+\varepsilon}(\Omega)$, the problem (P_ε) has at least one nontrivial solution, i.e., there exists $u \in W_0^{1,\vec{p},\varepsilon}(\Omega)$ such that*

$$\langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle \ \forall v \in W_0^{1,\vec{p},\varepsilon}(\Omega) \cap L^\infty(\Omega).$$

Remark 2.2. Remark that $\langle f, v\rangle$ is well defined since v is in $W_0^{1,\vec{p},\varepsilon}(\Omega)$, thus $v \in L^{1+\frac{1}{\varepsilon}}(\Omega)$ and $f \in L^{1+\varepsilon}(\Omega)$. This is also true in the case where $p_i = 2$ for all $i \geq 1$ and ε approaches 1, u belongs to $H_0^1(\Omega)$ and f belongs to $L^2(\Omega)$.

Remark 2.3. The conclusion of Theorem 2.1 remains true if we assume, instead of the condition $(A1)$, the following

$$(A1)' \qquad |a_i(x,s,\xi)| \leq \beta \left[k(x) + |s|^{\frac{p_0}{p_i'}} + \left(\sum_{j=1}^N |\xi_j|^{p_j} \right)^{\frac{1}{p_i'}} \right],$$

for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, all $i = 1,\dots,N$ and some function $k(x) \in L^{p_i'}(\Omega)$.

Remark 2.4.

1. As example of such an operator satisfying $(A1),(A2)$ and $(A3)$, we consider

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right),$$

where the exponents $p_i > 1$ for $i = 1,\dots,N$.

2. Note that the growth condition upon g with respect to u (Condition (G)) can generically have the following form

$$|g(x,u)| \leq F(u)h(x),$$

with $h \in L^{\frac{1}{1-\varepsilon}}(\Omega)$ and F is locally bounded and nondecreasing.

Proof of Theorem 2.1.

We proceed by steps to prove our result.

`Step (1) Existence of the approximate problem.`
Set for a.e. $x \in \Omega$

$$g_k(x,u) = T_k g(x,u),$$

and

$$b_k(u,v) = \int_\Omega g_k(x,u) v\, dx \quad \text{for all } u,v \in W_0^{1,\vec{p},\varepsilon}(\Omega),$$

where T_k is the usual truncation given by

$$T_k \eta = \begin{cases} \eta & \text{if } |\eta| \le k \\ \dfrac{k\eta}{|\eta|} & \text{if } |\eta| > k. \end{cases}$$

Note that $b_k(u,v)$ is well defined. Since $L^{1+\frac{1}{\varepsilon}}(\Omega) \subset L^{\frac{1}{\varepsilon}}(\Omega)$ and $(\frac{1}{\varepsilon})' = \frac{1}{1-\varepsilon}$, it is easy to see that

$$\begin{aligned} G_k : W_0^{1,\vec{p},\varepsilon}(\Omega) &\longrightarrow W^{-1,\vec{p}',\varepsilon}(\Omega) \\ u &\longrightarrow G_k u \end{aligned}$$

is also well defined, where the operator $G_k u$ is given by

$$\begin{aligned} G_k u : W_0^{1,\vec{p},\varepsilon}(\Omega) &\longrightarrow \mathbb{R} \\ v &\longrightarrow \int_\Omega g_k(x,u) v\, dx. \end{aligned}$$

On the other hand, notice that under the assumptions $(A1),(A2),(A3)$ and (G), the operator $A + G_k$ is coercive, monotone, hemicontinuous and bounded. Precisely, note that g_k satisfies also the sign condition. Indeed, the coercivity follows easily from $(A3)$ and the monotonicity follows immediately from $(A2)$ and the sign condition of the function g stated in (G).
The continuity of the map $\lambda \in \mathbb{R} \mapsto \langle (A+G_k)(u+\lambda v), w\rangle$ is an easy consequence of the assumptions that $a_i, i = 1,\ldots,N$ and g are Carathéodory functions and the growth condition $(A1)$. The boundedness follows

$$|\langle (A+G_k)u, v\rangle| = \langle Au, v\rangle + \int_\Omega g_k(x,u) v\, dx \le$$

$$\le \sum_{i=1}^N \left[\beta \left(\int_\Omega \left(k(x) + |u|^{1+\frac{1}{\varepsilon}} + \sum_{j=1}^N \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \right) \right)^{\frac{1}{p_i'}} \left(\int_\Omega \left| \frac{\partial v}{\partial x_i} \right|^{p_i} \right)^{\frac{1}{p_i}} \right] + c_1 \|v\| \le$$

$\leq c_2\|v\|(c_3+\|u\|)^{\gamma}$

where c_1 and c_2 are positive constants and γ is a positive real number. This implies the boundedness of $A+G_k$.
Therefore, thanks to Theorem 2.1, page 171 of [8], there exists $u_k \in W_0^{1,\vec{p},\varepsilon}(\Omega)$ solution of the problem

$$Au_k + g_k(x,u_k) = f,$$

i.e.

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_k,\nabla u_k)\nabla v\,dx + \int_{\Omega} g_k(x,u_k)v\,dx = \langle f,v\rangle \tag{2.2}$$

for all $v \in W_0^{1,\vec{p},\varepsilon}(\Omega)$.

`Step (2) A priori estimates.`
Substituting $v = u_k$ in (2.2), using (A3) and (G), the result is

$$\|u_k\| \leq C, \tag{2.3}$$

$$\int_{\Omega} g_k(x,u_k)u_k\,dx \leq C \tag{2.4}$$

for some constant $C > 0$ independent of k. By the similar arguments as above, we can prove that A is a bounded operator, thus we get

$$\|Au_k\|_{-1,\vec{p}',\varepsilon} \leq C', \tag{2.5}$$

for some constant $C' > 0$ independent of k.

`Step (3) Convergence of` u_k.
Observe that $W_0^{1,\vec{p},\varepsilon}(\Omega)$ is reflexive (recall that $p_i > 1$ for all $i = 0,1,\dots,N$), we deduce from (2.3) and (2.5)

$$u_k \rightharpoonup u \text{ weakly in } W_0^{1,\vec{p},\varepsilon}(\Omega),$$

$$Au_k \rightharpoonup \chi \text{ weakly in } W^{-1,\vec{p}',\varepsilon}(\Omega).$$

Hence we can extract a subsequence still denoted by u_k such that

$$u_k \to u \text{ a.e. in } \Omega \quad \text{and} \quad g_k(x,u_k) \longrightarrow g(x,u) \text{ a.e. in } \Omega.$$

Now let $\delta > 0$, since $|g_k(x,t)|\delta \leq |g_k(x,t)t|$ for $|t| \geq \delta$, and then $|g_k(x,t)| \leq \delta^{-1}|g_k(x,t)t|$ for $|t| \geq \delta$, we have

$$\begin{aligned} |g_k(x,u_k)| &\leq \sup_{|t|\leq\delta}|g_k(x,t)| + \delta^{-1}|g_k(x,u_k)u_k| \\ &\leq h_\delta(x) + \delta^{-1}|g_k(x,u_k)u_k|. \end{aligned}$$

This implies

$$\int_E |g_k(x,u_k)|\,dx \leq \int_E h_\delta(x)\,dx + \delta^{-1}C,$$

where E is a measurable subset of Ω and C is the constant of (2.4) which is independent of k.

For $|E|$ sufficiently small and $\delta = \frac{2C}{\varepsilon_1}$ with $\varepsilon_1 > 0$ we obtain

$$\int_E |g_k(x,u_k)|\,dx \leq \varepsilon_1.$$

Then by Vitali's theorem we get

$$g_k(x,u_k) \to g(x,u) \text{ strongly in } L^1(\Omega).$$

Passing to the limit, we obtain

$$\langle \chi, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle \tag{2.6}$$

for all $v \in W_0^{1,\vec{p},\varepsilon}(\Omega) \cap L^\infty(\Omega)$.
It remains to show that $Au = \chi$. For this purpose, note that since A is bounded, hemicontinuous and monotone, then A is pseudo-monotone (see Proposition 2.5, page 179 of [8]).
Put $v = T_k u$ in (2.6) where $T_k u$ is the truncation of u ($T_k u \in W_0^{1,\vec{p},\varepsilon}(\Omega) \cap L^\infty(\Omega)$).
On one hand we have

$$\langle \chi - f, T_k u\rangle \to \langle \chi - f, u\rangle.$$

On the other hand, using Lebesgue's dominated convergence theorem, since

$$|g(x,u)T_k u| \leq |g(x,u)||u| \in L^1(\Omega)$$

and

$$g(x,u)T_k u \to g(x,u)u \text{ a.e. in } \Omega,$$

we deduce that

$$g(x,u)T_k u \to g(x,u)u \text{ in } L^1(\Omega).$$

Therefore, we obtain

$$\langle \chi, u\rangle + \int_\Omega g(x,u)u\,dx = \langle f, u\rangle.$$

Now, by substituting $v = u_k$ in (2.2), then in view of Fatou's lemma we get

$$\limsup_{k\to+\infty} \langle Au_k, u_k\rangle \leq \langle f, u\rangle - \int_\Omega g(x,u)u\,dx.$$

This implies

$$\limsup_{k\to+\infty}\langle Au_k,u_k\rangle \le \langle \chi,u\rangle.$$

Since A is a pseudo-monotone operator, then $\chi = Au$.
Finally, we conclude that

$$\langle Au,v\rangle + \int_\Omega g(x,u)v\,dx = \langle f,v\rangle,$$

for all $v \in W_0^{1,\vec{p},\varepsilon}(\Omega)\cap L^\infty(\Omega)$. This completes the proof.

3. Concluding remarks

Remark 3.1. Note that $\left(L^{1+\frac{1}{\varepsilon}}(\Omega)\right)' = L^{1+\varepsilon}(\Omega)$. So that, f is considered in a dual space close to $L^1(\Omega)$ for ε small enough.

Remark 3.2. Observe that when $\varepsilon \longrightarrow 0$, then $f \in L^1(\Omega)$ and the solution u will reach the maximal regularity, i.e., $u \in L^\infty(\Omega)$. Hence the duality pairing between $L^1(\Omega)$ and $L^\infty(\Omega)$, $\langle f,u\rangle$, is still well defined.

Remark 3.3. Let us point out that in the work [5], the authors have studied the existence of solutions for a general class of anisotropic equations of order m with L^1-data under the condition $m\underline{p} > N, m \ge 1$, with $\underline{p} = \min\{p_i, i = 1,\dots,N\}$. Here with specific conditions on the operator A and the function g, we prove the existence result when the right hand side term becomes close to $L^1(\Omega)$.

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