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RESONANCE AND LANDESMAN-LAZER CONDITIONS FOR FIRST ORDER SYSTEMS IN $\mathbb{R}^2$

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The first part of the paper surveys the concept of resonance for T-periodic nonlinear problems. In the second part, some new results about existence conditions for nonlinear planar systems are presented. In particular, the Landesman-Lazer conditions are generalized to systems in $\mathbb{R}^2$ where the nonlinearity interacts with two resonant Hamiltonians. Such results apply to second order equations, generalizing previous theorems by Fabry [4] (for the undamped case), and Frederickson-Lazer [9] (for the case with friction). The results have been obtained with A. Fonda, and have been published in [8].

1. Introduction

Resonance is a physical phenomenon which has been widely studied by means of mathematical tools. The meaning of linear resonance is well understood: intuitively, we can think to a vibrating system, for instance a spring, whose motion is forced by a time-dependent external force which, roughly speaking, "constructively interacts" with the natural oscillation of the spring. More precisely, it is generally assumed that the ratio between the natural frequency of the spring and the frequency of the forcing term is a rational number, so that,

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at each multiple of a suitable interval of time, the amplitude of the oscillations increases (for an interesting and exhaustive survey about resonance, see [13]). From a mathematical point of view, hence, resonance is "against" the existence of periodic solutions (in some cases as in the linear one, it is indeed equivalent to the unboundedness of all the solutions). It is well known that, for the forced linear oscillator with T-periodic boundary conditions, i.e.,

$$\begin{cases} x'' + \lambda x = e(t) \\ x(0) = x(T),\ x'(0) = x'(T), \end{cases}$$

resonance can occur only in correspondence of the eigenvalues of the considered boundary value problem (namely, if $\lambda = (\frac{2N\pi}{T})^2$ for some nonnegative $N \in \mathbb{N}$). Indeed, if λ is not an eigenvalue, roughly speaking, the differential operator $x \mapsto x'' + \lambda x$ (acting from $C^2_T(0,T)$ to $C^0_T(0,T)$) becomes invertible.
Another problem of interest is the asymmetric oscillator, which intuitively can be thought again as a spring, this time subject to a potential which varies according to the position with respect to a point fixed in advance (usually, the origin). This system can be described, for instance, by the equation

$$x'' + \mu x^+ - \nu x^- = 0, \quad (\mu, \nu \in \mathbb{R}), \tag{1}$$

where $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$. In this context, the set that before was made up by the eigenvalues is replaced by the so called *Dancer-Fučik spectrum* (see [3, 10]). This set, which we will denote by Σ, is composed by the couples of nonnegative real numbers (μ, ν) such that equation (1) has a nontrivial solution. In the plane (μ, ν), it can be seen that Σ is made up by the axes $\{\mu = 0\}$, $\{\nu = 0\}$ and by the union of disjoint curves Γ_N (N positive integer), each one described by the equation

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{T}{N\pi}.$$

Dealing with the second order nonlinear problem

$$\begin{cases} x'' + g(t,x) = 0 \\ x(0) = x(T),\ x'(0) = x'(T), \end{cases} \tag{2}$$

it is then natural to compare the behavior of $g(t,x)$ (with respect to the variable x) with the lines λx having slopes λ equal to the eigenvalues of the linear problem (or, according to the sign of x, to the coordinates of a point belonging to a Fučik curve). Such an interaction turns out to be crucial in affecting (or not) the existence of a solution to the considered problem. Indeed, if we are "near" the eigenvalues, in general there is no hope of proving existence results without

further hypotheses on the nonlinearity. For this reason, there have been many contributions in literature seeking suitable conditions in order to prevent this danger, avoiding thus the unboundedness of all the solutions.
We will focus on the so called *Landesman-Lazer condition*, introduced by Lazer and Leach [12] in 1969 in the case

$$g(t,x) = \lambda_N x + h(x) - e(t),$$

with $\lambda_N = (\frac{2N\pi}{T})^2$ and h bounded, and generalized to elliptic PDEs by Landesman and Lazer [11] one year later. In the settings of [11, 12], this condition ensures existence for problem (2). In the case when $g(t,x) = \lambda_N x + h(t,x)$, it can be written as follows:

$$\int_{\{v>0\}} \liminf_{x\to+\infty} h(t,x)v(t)\,dt + \int_{\{v<0\}} \limsup_{x\to-\infty} h(t,x)v(t)\,dt > 0, \tag{3}$$

for every v solving the homogeneous equation $x'' + \lambda_N x = 0$. Just as an intuitive idea, one can qualitatively think that a suitable shape for $h(t,x)$ to satisfy such a condition requires that h is positive for $x \to +\infty$ and negative for $x \to -\infty$.
After the pioneering works [11, 12], there have been many generalizations of (3), for several kinds of problems dealing with more general situations (we only cite [1, 5, 6], for a quite rich bibliography about the subject see [8]). In particular, in [5, 6] Fabry and Fonda considered the situation when the nonlinearity can asymptotically interact with two consecutive eigenvalues λ_N and λ_{N+1}, that is,

$$g(t,x) = \gamma(t,x)x + r(t,x),$$

with $\lambda_N \le \gamma(t,x) \le \lambda_{N+1}$ and $r(t,x)$ a bounded function. This situation is usually referred to as *double resonance*. This setting was further on extended by Fabry [4] in the asymmetric framework, assuming that

$$g(t,x) = \gamma_1(t,x)x^+ - \gamma_2(t,x)x^- + r(t,x),$$

with γ_1 and γ_2 such that $a_+ \le \gamma_1(t,x) \le b_+$ and $a_- \le \gamma_2(t,x) \le b_-$, and $r(t,x)$ bounded, being $(a_-,a_+) \in \Gamma_N$ (the N-th Fučik curve) and $(b_-,b_+) \in \Gamma_{N+1}$, for a positive integer N. Both in [4, 5], to prove the existence results, the authors assumed a Landesman-Lazer condition for each side, i.e., with respect to each eigenvalue, considered separately (see Theorem 2.4). Intuitively, this is due to the fact that the nonlinearity has to be kept sufficiently far from resonance with respect to both λ_N and λ_{N+1}.
In this paper, we present some of the results obtained in [8] for planar systems, generalizing most of the previous quoted ones. In particular, it is considered the problem

$$\begin{cases} Ju' = F(t,u) \\ u(0) = u(T), \end{cases} \tag{4}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix. This time, the double resonance assumption can be formulated following the approach by Fonda [7], in terms of interaction with gradients of positively homogeneous Hamiltonians, as it will be better explained in Section 2. Assumption 3) in Theorem 2.1 is the generalization of the Landesman-Lazer conditions (in particular, Theorem 2.1 contains [4, Theorem 1], as it is proved in [8, Corollary 3.3]). A few remarks will follow, focusing on some related questions and on another result in [8].

2. Main result

As already mentioned in the Introduction, we will consider problem (4), where, for the sake of simplicity, we will assume the function $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ to be continuous (the statement in [8] is proved in the more general Carathéodory setting). We look for classical solutions of (4).
A possible definition of resonance for first order systems was introduced, in a general framework, in [7]. The role of the term $\lambda_N x$ (or $a_+x^+ - a_-x^-$) in second-order equations is now played by gradients of positively homogeneous Hamiltonians, that is, gradients of C^1-functions $H : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$0 < H(\lambda u) = \lambda^2 H(u), \quad \lambda > 0,\, u \in \mathbb{R}^2.$$

We will denote this class of functions by $\mathscr{P}$. The remarkable point about such a class is that if $H \in \mathscr{P}$, then every solution of the Hamiltonian system

$$Ju' = \nabla H(u)$$

is periodic with the same minimal period τ (in this situation, the origin is usually said to be an isochronous center for the system). The fact that this minimal period τ is a submultiple of T affects the existence of a solution when the problem is forced. Precisely, in [7] it was proved that, for a T-periodic problem like

$$\begin{cases} Ju' = \nabla H(u) + f(t) \\ u(0) = u(T), \end{cases}$$

with $H \in \mathscr{P}$, if $\frac{T}{\tau} \notin \mathbb{N}$, then there is existence for every forcing term $f : [0,T] \to \mathbb{R}^2$. On the contrary, if $\frac{T}{\tau} \in \mathbb{N}$ (in this case, we will say that the Hamiltonian H is *resonant*), there exist forcing terms for which all the solutions are unbounded. A natural generalization of resonance is thus represented by this second case, which indeed extends the previously described resonant situations for scalar problems. The nonlinear extension of this concept, both for simple and double resonance, is based again on comparing the asymptotic behavior of the nonlinearity with the behavior of the gradient of a resonant Hamiltonian.

With these preliminaries, we will then focus on the case when double resonance can occur, so that $F(t,u)$ interacts asymptotically with the gradients of two resonant Hamiltonians $H_1, H_2 \in \mathscr{P}$. We will assume

$$H_1(u) \leq H_2(u), \qquad \text{for every } u \in \mathbb{R}^2. \tag{5}$$

The main result in [8] is the following.

Theorem 2.1. *Assume (5) and the following hypotheses:*

1) *there exist a continuous functions $\gamma : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, such that $0 \leq \gamma(t,u) \leq 1$, and a bounded continuous function $r : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$, such that*

$$F(t,u) = (1-\gamma(t,u))\nabla H_1(u) + \gamma(t,u)\nabla H_2(u) + r(t,u), \tag{6}$$

for every $t \in \mathbb{R}$ and every $u \in \mathbb{R}^2$;

2) *if φ and ψ satisfy $J\varphi' = \nabla H_1(\varphi)$, and $J\psi' = \nabla H_2(\psi)$, and τ_φ, τ_ψ are their minimal periods, then there exists a positive $N \in \mathbb{N}$ such that*

$$\frac{T}{N+1} \leq \tau_\psi < \tau_\varphi \leq \frac{T}{N}; \tag{7}$$

3) *for every $\theta \in [0,T]$, the following relations are satisfied:*

$$\Gamma_1(\theta) := \int_0^T \liminf_{\substack{\lambda \to +\infty \\ \omega \to \theta}} [\langle F(t, \lambda\varphi(t+\omega)) | \varphi(t+\omega)\rangle - 2\lambda H_1(\varphi(t))]\, dt > 0, \tag{8}$$

$$\Gamma_2(\theta) := \int_0^T \liminf_{\substack{\lambda \to +\infty \\ \omega \to \theta}} [2\lambda H_2(\psi(t)) - \langle F(t, \lambda\psi(t+\omega)) | \psi(t+\omega)\rangle]\, dt > 0. \tag{9}$$

Then problem (4) has a solution.

The proof relies on accurate estimates of an angular-type coordinate, in order to perform suitable a priori estimates on the solutions (to this aim, (8) and (9) are essential). The conclusion follows from [2, Theorem 2], showing that the coincidence degree associated to the problem is 1. We refer to [8] for the details, now focusing instead on a few complementary comments.

Remark 2.2. Notice that, in the statement, N is assumed to be positive. With particular attention to the scalar case, this means that the situation when resonance occurs with the first eigenvalue (or with the axes of the Fučik spectrum) is not included in the framework of the theorem. The generalization of this situation to first order systems, from the point of view of positively homogeneous Hamiltonians, seems indeed more delicate.

Remark 2.3. Let us recall the main result by Fabry [4] for the scalar second order problem

$$\begin{cases} x'' + g(t,x) = 0 \\ x(0) = x(T), x'(0) = x'(T), \end{cases} \tag{10}$$

in order to carefully analyze its relationships with Theorem 2.1.

Theorem 2.4 (Fabry 1995)**.** *Let a_-, a_+, b_-, b_+ be positive numbers such that*

$$\frac{1}{\sqrt{a_+}} + \frac{1}{\sqrt{a_-}} = \frac{T}{N\pi}, \qquad \frac{1}{\sqrt{b_+}} + \frac{1}{\sqrt{b_-}} = \frac{T}{(N+1)\pi}, \tag{11}$$

for some positive integer N. Let $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$g(t,x) = \gamma_1(t,x)x^+ - \gamma_2(t,x)x^- + r(t,x),$$

with $a_+ \leq \gamma_1(t,x) \leq b_+$ and $a_- \leq \gamma_2(t,x) \leq b_-$, and r bounded. Moreover, assume that for every ϕ, ξ satisfying

$$\phi'' + a_+\phi^+ - a_-\phi^- = 0, \qquad \xi'' + b_+\xi^+ - b_-\xi^- = 0,$$

respectively, the following conditions are satisfied:

$$\int_{\{\phi>0\}} \liminf_{x\to+\infty}(g(t,x) - a_+x)\,\phi(t)\,dt + \int_{\{\phi<0\}} \limsup_{x\to-\infty}(g(t,x) - a_-x)\,\phi(t)\,dt > 0, \tag{12}$$

and

$$\int_{\{\xi>0\}} \limsup_{x\to+\infty}(g(t,x) - b_+x)\,\xi(t)\,dt + \int_{\{\xi<0\}} \liminf_{x\to-\infty}(g(t,x) - b_-x)\,\xi(t)\,dt < 0. \tag{13}$$

Then problem (10) has a solution.

In this setting, as it is proved in [8, Corollary 3.3], conditions (8), (9) for problem (10) are more general than (12), (13), respectively. Nevertheless, we now show that the converse implication - which is not proved in [8] - holds, as well. For $\theta \in [0,T]$ fixed, consider the explicit expression of $\Gamma_1(\theta)$ in (8), that is,

$$\int_{\{\phi(t+\theta)>0\}} w_1(t,\theta)\,dt + \int_{\{\phi(t+\theta)<0\}} w_2(t,\theta)\,dt, \tag{14}$$

where we set $w_1(t,\theta) := \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} (g(t,\lambda\phi(t+\omega)) - \lambda a_+\phi(t+\omega))\phi(t+\omega)$, and $w_2(t,\theta) := -\limsup_{\substack{\lambda\to+\infty \\ \omega\to\theta}} (g(t,\lambda\phi(t+\omega)) - \lambda a_-\phi(t+\omega))(-\phi(t+\omega))$. For the sake of brevity, we will focus only on the first term in (14), the reasoning for the other summand being the same. Since the second factor has limit,

$$w_1(t,\theta) = \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} (g(t,\lambda\phi(t+\omega)) - \lambda a_+\phi(t+\omega))\phi(t+\theta). \tag{15}$$

However, by writing the explicit expression of the right-hand side in (15), a standard argument implies now

$$w_1(t,\theta) \leq \liminf_{\lambda\to\infty}(g(t,\lambda\phi(t+\theta)) - \lambda a_+\phi(t+\theta))\phi(t+\theta) =: W_1(t,\theta),$$

since, for every $\delta,\gamma > 0$,

$$\inf_{\substack{\lambda\geq\gamma \\ |\theta-\omega|\leq\delta}} g(t,\lambda(\phi(t+\omega))) - \lambda a_+\phi(t+\omega) \leq \inf_{\lambda\geq\gamma} g(t,\lambda(\phi(t+\theta))) - \lambda a_+\phi(t+\theta).$$

It follows

$$\Gamma_1(\theta) \leq \int_{\{\phi(t+\theta)>0\}} W_1(t,\theta)\,dt + \int_{\{\phi(\theta)<0\}} W_2(t,\theta)\,dt,$$

where $W_2(t,\theta) := \limsup_{\lambda\to+\infty}(g(t,\lambda\phi(t+\theta)) - \lambda a_-\phi(t+\theta))\phi(t+\theta)$. It is now sufficient to notice that, at $\theta \in [0,T]$ fixed, if $t \in \{\phi(t+\theta) > 0\}$,

$$W_1(t,\theta) = \liminf_{x\to+\infty}(g(t,x) - a_+x)\phi(t+\theta),$$

while, if $t \in \{\phi(t+\theta) < 0\}$,

$$W_2(t,\theta) = \limsup_{x\to-\infty}(g(t,x) - a_-x)\phi(t+\theta).$$

Consequently, (8) implies (12). Similarly, (9) implies (13).

Remark 2.5. We only mention that another result in [8], imposing little further restrictions on the form of $F(t,u)$, improves the Landesman-Lazer conditions (8), (9). Precisely, in [8, Theorem 6.1], it is assumed

$$F(t,u) = \hat{\gamma}(t,u)\nabla H(u) + r(t,u),$$

with $0 < \alpha \leq \hat{\gamma}(t,u) \leq \beta$, being αH and $\beta H \in \mathscr{P}$ two "consecutive" resonant Hamiltonians (like H_1 and H_2 in Theorem 2.1). In this case, it is possible to take into account also the radial component of solutions, yielding more accurate existence conditions which give rise to a "multiple choice" situation for every $\theta \in [0,T]$. This result generalizes an interesting theorem proved by Frederickson and Lazer in [9].

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