EXISTENCE RESULTS FOR A QUASI-LINEAR DIFFERENTIAL PROBLEM

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The aim of this paper is to establish the existence of at least one non-trivial solution for Neumann quasi-linear problems. Our approach is based on variational methods.

1. Introduction

The aim of this paper is to ensure the existence of at least one non-trivial solution for the following Neumann boundary value problem

\[
\begin{aligned}
-u'' + uh(u') &= \lambda \alpha(x)f(u)h(u') \\
u'(a) = u'(b) &= 0,
\end{aligned}
\]

\( (N_\lambda) \)

where \( \alpha : [a, b] \to \mathbb{R} \) is a positive continuous function, \( f : \mathbb{R} \to \mathbb{R} \) and \( h : \mathbb{R} \to \mathbb{R} \) are continuous functions and \( \lambda \) is a positive real parameter.

Existence and multiplicity of solutions for Neumann boundary value problems have been investigated by several authors and, for an overview on this subject, we refer to [1], [3] - [6], [8], [9], [11] - [14].

The main result of this paper is Theorem 3.1, which generalizes [6, Theorem 3.1] to the case where the nonlinear term is not constant with respect to \( u' \). Two relevant consequences of Theorem 3.1 (that is, Corollary 3.2 and Theorem
are also pointed out. Here, as an example, we presented a special case of our main result.

**Theorem 1.1.** Let \( \alpha : [a, b] \to \mathbb{R} \) be a nonnegative continuous function and \( f, h : \mathbb{R} \to \mathbb{R} \) be continuous functions. Suppose that \( h \) is bounded and strictly positive, and that
\[
\lim_{\xi \to 0^+} \frac{f(\xi)}{\xi} = +\infty.
\]
Then, there exists \( \lambda^* \) such that, for each \( \lambda \in ]0, \lambda^*[ \), the problem \((N_\lambda)\) admits at least one positive classical solution.

Our approach is based on a critical point theorem obtained in [2] (see Theorem 2.1).

The paper is arranged as follows: in Section 2, we recall some basic definitions and our main tool, while Section 3 is devoted to our main results.

### 2. Preliminaries and basic notations

Our main tool is the Ricceri variational principle [10, Theorem 2.5] as given in [2, Theorem 5.1] which is below recalled (see also [2, Proposition 2.1] and [7, Theorem 2.1]). First, given \( \Phi, \Psi : X \to \mathbb{R} \), put
\[
\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}([r_1, r_2])} \frac{\sup_{u \in \Phi^{-1}([r_1, r_2])} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},
\]
and
\[
\rho_2(r_1, r_2) = \sup_{v \in \Phi^{-1}([r_1, r_2])} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}([\infty, r_1])} \Psi(u)}{\Phi(v) - r_1},
\]
for all \( r_1, r_2 \in \mathbb{R} \), with \( r_1 < r_2 \).

**Theorem 2.1.** ([2, Theorem 5.1]) Let \( X \) be a reflexive real Banach space; \( \Phi : X \to \mathbb{R} \) be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on \( X^* \); \( \Psi : X \to \mathbb{R} \) be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Put \( I_\lambda = \Phi - \lambda \Psi \) and assume that there are \( r_1, r_2 \in \mathbb{R} \), with \( r_1 < r_2 \), such that
\[
\beta(r_1, r_2) < \rho_2(r_1, r_2),
\]
where \( \beta \) and \( \rho_2 \) are given by (1) and (2). Then, for each \( \lambda \in \left[ \frac{1}{\rho_2(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)} \right] \) there is \( u_{0, \lambda} \in \Phi^{-1}([r_1, r_2]) \) such that \( I_\lambda(u_{0, \lambda}) \leq I_\lambda(u) \) for all \( u \in \Phi^{-1}([r_1, r_2]) \) and \( I'_\lambda(u_{0, \lambda}) = 0 \).
Let $X$ be the Sobolev space $W^{1,2}([a, b])$ endowed with the norm
\[
\|u\| := \left( \int_a^b |u'(x)|^2 \, dx + \int_a^b |u(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]
Throughout the sequel, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $h : \mathbb{R} \to \mathbb{R}$ is a positive continuous function, $\alpha : [a, b] \to \mathbb{R}$ is a sommable function and $\lambda$ is a positive real parameter. Put
\[
F(t) = \int_0^t f(\xi) \, d\xi, \quad \text{for all } t \in \mathbb{R},
\]
and, put
\[
F_1(x, t) = \int_0^t \alpha(x) f(\xi) \, d\xi = \alpha(x) F(t), \quad \text{for all } (x, t) \in [a, b] \times \mathbb{R},
\]
and, put
\[
H(y) = \int_y^\sigma \left( \int_0^\tau \frac{1}{h(\tau)} \, d\tau \right) d\sigma, \quad \text{for all } y \in \mathbb{R}.
\]
We recall that $u : [a, b] \to \mathbb{R}$ is called weak solution of Problem $(N_{\lambda})$ if $u \in W^{1,2}([a, b])$ and
\[
\int_a^b H'(u'(x)) v'(x) \, dx + \int_a^b u(x) v(x) \, dx = \lambda \int_a^b \alpha(x) f(u(x)) v(x) \, dx,
\]
for all $v \in W^{1,2}([a, b])$.
We also recall that a weak solution is a generalized solution, that is, $u \in C^1([a, b])$, $u' \in AC([a, b])$, $-u''(x) + u(x) h(u'(x)) = \lambda \alpha(x) f(u(x)) h(u'(x))$, for a.e. $x \in [a, b]$, and $u'(a) = u'(b) = 0$.
Moreover, if $\alpha$ is continuous, each weak solution is a classical solution, that is, $u \in C^2([a, b])$, $-u''(x) + u(x) h(u'(x)) = \lambda \alpha(x) f(u(x)) h(u'(x))$ for all $x \in [a, b]$, and $u'(a) = u'(b) = 0$. Finally, put
\[
\gamma = \left( \max \left\{ 2(b - a); \frac{2}{b - a} \right\} \right)^{\frac{1}{2}},
\]
we recall the following inequality which we use in the sequel
\[
\max_{x \in [a, b]} |u(x)| \leq \gamma \|u\|, \quad (4)
\]
for all $u \in X$ and for all $x \in [a, b]$. 
3. Main Results

In this Section, we establish existence results for the Neumann boundary value problem \((N_\lambda)\).

Given two positive constants \(m, M\), with \(m \leq M\), put

\[
\delta_1 = \left( \min \left\{ \frac{1}{M(b-a)}; \frac{1}{b-a} \right\} \right)^{\frac{1}{2}}, \quad \delta_2 = \left( \max \left\{ \frac{1}{m(b-a)}; \frac{1}{b-a} \right\} \right)^{\frac{1}{2}}.
\]

Moreover, given three nonnegative constants \(c_1, c_2, d\), with \(\delta_1 c_1 < \gamma d < \delta_2 c_2\), put

\[
a(c_2, d) := \frac{\max F(t) - F(d)}{\delta_2^2 c_2^2 - \gamma^2 d^2}
\]

and

\[
b(c_1, d) := \frac{F(d) - \max F(t)}{\gamma^2 d^2 - \delta_1^2 c_1^2}.
\]

We give our main result.

**Theorem 3.1.** Let \(\alpha : [a, b] \to \mathbb{R}\) be a nonnegative function and let \(f, h : \mathbb{R} \to \mathbb{R}\) be continuous functions. Assume that there exist two positive constants \(m, M\), such that

(i) \(m \leq h(y) \leq M\), for all \(y \in \mathbb{R}\),

and, assume that there exist three nonnegative constants \(c_1, c_2, d\), with \(\delta_1 c_1 < \gamma d < \delta_2 c_2\), such that

\[
a(c_2, d) < b(c_1, d).
\]

Then, for each \(\lambda \in \left[ \frac{b-a}{2\gamma^2 \| \alpha \|_1 b(c_1, d)}, \frac{b-a}{2\gamma^2 \| \alpha \|_1 a(c_2, d)} \right]\), the problem \((N_\lambda)\) admits at least one weak solution \(\bar{u}\), such that

\[
\frac{c_1}{\gamma} < \| \bar{u} \| < \frac{c_2}{\gamma}.
\]

**Proof.** Put

\[
\Phi(u) := \frac{1}{2} \int_a^b |u(x)|^2 \, dx + \int_a^b H(u'(x)) \, dx,
\]

\[
\Psi(u) := \int_a^b F_1(x, u(x)) \, dx,
\]

for all \(u \in X\).

It is well known that \(\Phi\) and \(\Psi\) satisfy all regularity assumptions requested in
Theorem 2.1 and that the critical points in $X$ of the functional $\Phi - \lambda \Psi$ are exactly the weak solutions of the problem $(N_\lambda)$. By using $(i)$, one has
\[
\min \left\{ \frac{1}{2M^2}; \frac{1}{2} \right\} \|u\|^2 \leq \Phi(u) \leq \max \left\{ \frac{1}{2m}; \frac{1}{2} \right\} \|u\|^2,
\]
for every $u \in X$. Our aim is to apply Theorem 2.1. To this end, put
\[
r_1 = \frac{b - a}{2} \frac{\delta_1^2}{\gamma^2} c_1^2, \quad r_2 = \frac{b - a}{2} \frac{\delta_2^2}{\gamma^2} c_2^2
\]
and
\[
u_0(x) = d, \quad \text{for all } x \in [a, b].
\]
Clearly, $u_0 \in X$ and one has
\[
\Phi(u_0) = \frac{1}{2} \int_a^b |u_0|^2 dx + \int_a^b F(u_0) dx = \frac{1}{2} d^2 (b - a),
\]
\[
\Psi(u_0) = \int_a^b F_1(x, u_0(x)) dx = \|\alpha\|_1 F(d),
\]
where
\[
\|\alpha\|_1 := \int_a^b |\alpha(x)| dx.
\]
From $\delta_1 c_1 < \gamma d < \delta_2 c_2$, one has $r_1 < \Phi(u_0) < r_2$. Moreover, for all $u \in X$ such that $\Phi(u) < r_2$, taking $(4)$ into account, one has
\[
|u(x)| < c_2, \quad \text{for all } x \in [a, b],
\]
and
\[
\int_a^b F_1(x, u(x)) dx \leq \int_a^b \max_{|t| \leq c_2} F_1(x, t) dx = \|\alpha\|_1 \max_{|t| \leq c_2} F(t).
\]
Therefore
\[
\sup_{u \in \Phi^{-1}([-\infty, r_2])} \Psi(u) \leq \|\alpha\|_1 \max_{|t| \leq c_2} F(t).
\]
Arguing as before, we obtain
\[
\sup_{u \in \Phi^{-1}([-\infty, r_1])} \Psi(u) \leq \|\alpha\|_1 \max_{|t| \leq c_1} F(t).
\]
Therefore, one has
\[
\beta(r_1, r_2) \leq \sup_{u \in \Phi^{-1}([-\infty, r_2])} \Psi(u) - \Psi(u_0)
\]
\[
\leq \frac{2\gamma^2 \|\alpha\|_1}{b - a} \max_{|t| \leq c_2} F(t) - F(d)
\]
\[
\leq \frac{2\gamma^2 \|\alpha\|_1}{b - a} \frac{\delta_2^2 c_2^2 - \gamma^2 d^2}{\delta_2^2 c_2^2 - \gamma^2 d^2} = \frac{2\gamma^2 \|\alpha\|_1}{b - a} a(c_2, d). \quad (6)
\]
On the other hand, one has
\[
\rho_2(r_1, r_2) \geq \frac{\Psi(u_0) - \sup_{u \in \Phi^{-1}(\mathbb{R})} \Psi(u)}{\Phi(u_0) - r_1} \geq \frac{2\gamma^2 \|\alpha\|_1}{b - a} \frac{F(d) - \max_{|t| \leq c_1} F(t)}{\gamma^2 d^2 - \delta_1^2 c_1^2} = \frac{2\gamma^2 \|\alpha\|_1}{b - a} b(c_1, d). \tag{7}
\]

Hence, from (5) one has
\[
\beta(r_1, r_2) < \rho_2(r_1, r_2).
\]

Therefore, owing to Theorem 2.1, for each
\[
\lambda \in \left[\frac{b - a}{2\gamma^2 \|\alpha\|_1 b(c_1, d)}, \frac{b - a}{2\gamma^2 \|\alpha\|_1 a(c_2, d)}\right],
\]
\[
\Phi - \lambda \Psi \text{ admits at least one critical point } \bar{u} \text{ such that }
\]
\[
r_1 < \Phi(\bar{u}) < r_2,
\]
that is
\[
\frac{c_1}{\gamma} < \|\bar{u}\| < \frac{c_2}{\gamma}.
\]

Hence, the proof is complete. \(\square\)

Now, we point out the following consequence of Theorem 3.1.

**Corollary 3.2.** Let \(\alpha : [a, b] \to \mathbb{R}\) be a nonnegative function, \(h : \mathbb{R} \to [0, +\infty[\) and \(f : \mathbb{R} \to [0, +\infty[\) be continuous functions. Assume that (i) holds and that there exist two positive constants \(c, d\), with \(c > \frac{\gamma}{\delta_2} d\), such that
\[
\frac{F(c)}{c^2} < \left(\frac{\delta_2}{\gamma}\right)^2 \frac{F(d)}{d^2}. \tag{8}
\]
Then, for each \(\lambda \in \left[\frac{b - a}{2\|\alpha\|_1 F(d)}, \frac{b - a}{2\|\alpha\|_1} \left(\frac{\delta_2}{\gamma}\right)^2 \frac{c^2}{F(c)}\right]\), the problem \((N_\lambda)\) admits at least one nontrivial weak solution \(\bar{u}\) such that \(\|\bar{u}\| < \frac{c}{\gamma}\).

**Proof.** Our aim is to apply Theorem 3.1. To this end, we pick \(c_1 = 0\) and \(c_2 = c\). From (8) one has
\[
a(c_2, d) = \max_{|t| \leq c} \frac{F(t) - F(d)}{\delta_2^2 c^2 - \gamma^2 d^2} \leq \frac{F(c) - \left(\frac{\gamma^2 d^2}{\delta_2^2 c^2} F(c)\right)}{\delta_2^2 c^2 - \gamma^2 d^2} = \frac{F(c)}{\delta_2^2 c^2}.
\]
On the other hand, one has $b(c_1, d) = \frac{F(d)}{\gamma^2 d^2}$. Hence, owing to (8), Theorem 3.1 ensures the conclusion.

Now, we point out the following relevant consequence of Corollary 3.2.

**Theorem 3.3.** Let $\alpha : [a, b] \to \mathbb{R}$ be a nonnegative function and $f, h : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that (i) holds. Assume that
\[
\lim_{\xi \to 0^+} \frac{f(\xi)}{\xi} = +\infty,
\]
and put $\lambda^* = \frac{b - a}{2\|\alpha\|_1} \left( \frac{\delta_2}{\gamma} \right)^2 \sup_{c > 0} \frac{c^2}{F(c)}$. Then, for each $\lambda \in ]0, \lambda^*[,$ the problem $(N_\lambda)$ admits at least one positive weak solution.

**Proof.** Fix $\lambda \in ]0, \lambda^*[.$ Then, there is $c > 0$ such that $\lambda < \frac{b - a}{2\|\alpha\|_1} \left( \frac{\delta_2}{\gamma} \right)^2 \frac{c^2}{F(c)}.$ From (9) there is $d < \frac{\delta_2}{\gamma} c$ such that $\frac{2\|\alpha\|_1 F(d)}{b - a} > \frac{1}{\lambda}$. Hence, Corollary 3.2 ensures the conclusion.

**Remark 3.4.** Taking (9) into account, fix $\rho > 0$ such that $f(\xi) > 0$ for all $\xi \in ]0, \rho[.$ Then, put $\tilde{\lambda} = \frac{b - a}{2\|\alpha\|_1} \left( \frac{\delta_2}{\gamma} \right)^2 \sup_{c \in ]0, \rho[} \frac{c^2}{F(c)}.$ Clearly, $\tilde{\lambda} \leq \lambda^*.$ Now, fixed $\lambda \in ]0, \tilde{\lambda}[,$ and arguing as in the proof of Theorem 3.3, there are $c \in ]0, \rho[,$ and $d < \frac{\delta_2}{\gamma} c$ such that $\frac{b - a}{2\|\alpha\|_1 F(d)} < \lambda < \frac{b - a}{2\|\alpha\|_1} \left( \frac{\delta_2}{\gamma} \right)^2 \frac{c^2}{F(c)}.$ Hence, Corollary 3.2 ensures that, for each $\lambda \in ]0, \tilde{\lambda}[,$ the problem $(N_\lambda)$ admits at least one positive weak solution $\bar{u}_\lambda$ such that
\[
|\bar{u}_\lambda(x)| < \frac{\rho}{\gamma},
\]
for all $x \in [a, b]$. 
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REFERENCES


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