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REMARK ON THE NUMBER OF SOLUTIONS IN THE THERMISTOR PROBLEM

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In this paper we study a nonlocal thermistor problem which models two simple electrical circuits. We prove that, under certain assumptions on the electrical conductivity, multiple states of thermal and electrical equilibria (typically three) are possible. All mathematical methods used are elementary.

1. Introduction

The term thermistor refers to a three-dimensional body made up of substances conducting both heat and electricity (typically a mixture of semiconductors) for which the thermal and electrical conductivities depend sharply on the temperature [10] and [15]. We shall represent the body of the thermistor by Ω, an open and bounded subset of $\mathbb{R}^3$. The regular boundary of Ω consists of three surfaces Γ_1, Γ_2 and Γ_0 $(\Gamma_1 \cap \Gamma_2 = \emptyset)$. Γ_1 and Γ_2 represent the electrodes to which a difference of potential is applied. Γ_0 is the part of the boundary which is insulated thermally and electrically. In this paper we treat a nonlocal boundary value problem for a system of P.D.E. which models two different circuits of which the thermistor is a part. In the first circuit the thermistor is in series with an ordinary resistor R_S and with an ideal generator of a difference V of potential. R_S limits the maximum of the current which may flow and may also represent the internal

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resistance of the generator. In the second circuit the thermistor is in parallel with the ordinary resistor R_P and with an ideal generator of a constant current I. In this second case R_P limits the maximum of the difference of potential which may exist across the generator and the thermistor. In any practical situation we have $R_S \neq 0$ and $R_P \neq 0$. Let $\mathbf{J}$ be the electrical current density in Ω. The total current crossing the thermistor in the unit time is given by

$$i = \int_{\Gamma_2} \mathbf{J} \cdot \mathbf{n} \, d\Gamma,$$

where $\mathbf{n}$ is the unit vector normal to Γ_2 pointing outward with respect to Ω. In the first circuit we have

$$V = X + R_S \int_{\Gamma_2} \mathbf{J} \cdot \mathbf{n} \, d\Gamma, \tag{1.1}$$

where V is the fixed difference of potential given by the generator, whereas X is the unknown difference of potential existing between Γ_1 and Γ_2. In the second circuit we have, if I is the constant current provided by the generator,

$$I = \frac{X}{R_P} + \int_{\Gamma_2} \mathbf{J} \cdot \mathbf{n} \, d\Gamma,$$

or equivalently

$$R_P I = X + R_p \int_{\Gamma_2} \mathbf{J} \cdot \mathbf{n} \, d\Gamma. \tag{1.2}$$

By the laws of Ohm and Fourier we have

$$\mathbf{J} = -\sigma(u) \nabla \varphi \tag{1.3}$$

and

$$\mathbf{q} = -\kappa(u) \nabla u, \tag{1.4}$$

where σ is the electrical conductivity, κ the thermal conductivity (both given functions of the temperature u), $\varphi(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3) \in \bar{\Omega}$ the electric potential and $\mathbf{q}$ the density of the heat flow. Conservation of charge and energy imply

$$\nabla \cdot \mathbf{J} = 0 \tag{1.5}$$

and

$$\nabla \cdot \mathbf{q} = \mathbf{E} \cdot \mathbf{J} \quad (\mathbf{E} = -\nabla \varphi), \tag{1.6}$$

where $\mathbf{E}$ is the electric field, and the term in the right hand side of (1.6) reflects the Joule heating. Setting $A = V$ and $R = R_P$ in (1.1) and $A = R_P I$ in (1.2) we arrive, substituting (1.3) in (1.5) and (1.4) in (1.6), at the same mathematical model (P) for the determination of the temperature $u(\mathbf{x})$ and of the potential $\varphi(\mathbf{x})$ in Ω for both the circuits.

Problem (P). Given $\sigma(u)$, $\kappa(u) \in C^0([0,\infty))$ such that

$$\sigma(u) > 0,\ \kappa(u) > 0 \tag{1.7}$$

and the positive number A, find $u(\mathbf{x}) \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $\varphi(\mathbf{x}) \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and the constant X such that

$$\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \ \text{ in } \Omega \tag{1.8}$$

$$\varphi = 0 \ \text{ on } \Gamma_1,\ \varphi = X \ \text{ on } \Gamma_2 \tag{1.9}$$

$$\frac{\partial\varphi}{\partial n} = 0 \ \text{ on } \Gamma_0 \tag{1.10}$$

$$-\nabla \cdot (\kappa(u)\nabla u) = \sigma(u)|\nabla\varphi|^2 \ \text{ in } \Omega \tag{1.11}$$

$$u = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \tag{1.12}$$

$$\frac{\partial u}{\partial n} = 0 \ \text{ on } \Gamma_0 \tag{1.13}$$

$$A = X + R\int_{\Gamma_2} \mathbf{J}\cdot\mathbf{n}\, d\Gamma. \tag{1.14}$$

At first sight problem (P) appears to involve a quadratic nonlinearity in the gradient. However, for classical solutions of problem (P) we have, taking into account (1.8),

$$\sigma(u)|\nabla\varphi|^2 = \nabla \cdot (\varphi\sigma(u)\nabla\varphi).$$

Thus equation (1.11) can be rewritten in the following full divergence form

$$\nabla \cdot (\kappa(u)\nabla u + \varphi\sigma(u)\nabla\varphi) = 0. \tag{1.15}$$

Hereafter we take in problem (P) equation (1.15) instead of (1.11). We also consider the following related problem (PO)

$$\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \ \text{ in } \Omega$$

$$\varphi = 0 \ \text{ on } \Gamma_1,\ \varphi = X \ \text{ on } \Gamma_2$$

$$\frac{\partial\varphi}{\partial n} = 0 \ \text{ on } \Gamma_0 \tag{1.16}$$

$$\nabla \cdot (\kappa(u)\nabla u + \varphi\sigma(u)\nabla\varphi) = 0$$

$$u = 0 \text{ on } \Gamma_1 \cup \Gamma_2$$

$$\frac{\partial u}{\partial n} = 0 \ \text{ on } \Gamma_0, \tag{1.17}$$

where now X is a given constant, expressing a directly applied potential to the thermistor. To problem (PO) the results of the papers [4] and [5] are applicable. They are based on the following definition of solution.

Definition. If $(u(\mathbf{x}), \varphi(\mathbf{x}))$ is a classical solution of problem (PO) and there exists a $C^2([0,x])$ function $\mathscr{U}(\varphi)$ such that

$$u(\mathbf{x}) = \mathscr{U}(\varphi(\mathbf{x})) \tag{1.18}$$

we say that $(u(\mathbf{x}), \varphi(\mathbf{x}))$ is a *functional solution* of problem (P).

Thus functional solutions are special classical solutions. This definition limits the class of problems for which the search of this kind of solutions is meaningful. For example, if in problem (P) we take, instead of (1.17), the condition

$$\kappa \frac{\partial u}{\partial n} = h(u_b - u) \ \text{ on } \Gamma_0, \ u_b \text{ and } h \text{ positive constants} \tag{1.19}$$

this new problem would be intractable with the present method in view of (1.16) and of the functional relation (1.18). Moreover, from the physical point of view, the condition (1.18) implies that the current density $\mathbf{J}$ and the density of the heat flow $\mathbf{q}$ are parallel in every point of Ω. On the positive side, the consideration of functional solutions is justified since their search is possible with a two-point problem for an ordinary differential equation depending on a parameter (see (2.10) below) and also by the following theorem [5]. Functional solutions are also useful in other problems of mathematical physics [6], [7].

Theorem 1.1. *All classical solutions of problem (PO) are functional solutions. Moreover, problem (PO) has a unique classical solution.*

We note that Γ_1 and Γ_2 are equipotential surfaces. Thus the boundary conditions (1.9) are the only physically meaningful. Moreover, the boundary condition (1.10) follows from the assumption that Γ_0 is electrically insulated, which means that

$$\mathbf{J} \cdot \mathbf{n} = 0 \ \text{ on } \Gamma_0.$$

The problem of the electrical heating of a conductor whose conductivities depend on the temperature is over one century old (see [9] and [12]). In the book of F. Llewellyn Jones [14] solutions in which $\mathbf{J}$ is parallel to $\mathbf{q}$ are considered in the perspective of the engineering applications. From the point of view of existence of weak solutions, problems like (PO), with general boundary conditions on the potential and the temperature, have been the object of many papers. We quote among others [2], [8], [11] and references therein. However the uniqueness of solutions is proved, for general boundary conditions, only for sufficiently small data [11].

Regarding problem (P) a result, valid only in a special case, on the number of solutions is presented in [3]. New in this paper is the formulation in terms of a two point problem (2.10) which greatly clarify the situation and Lemma 2.2 which, with formula (2.16), permits, in concrete cases, the calculation of the number of solutions. Also new it is the asymptotic formula (2.20). Moreover, the model studied here treats the case of a thermistor connected to a current generator. This situation is not covered by [3].

2. The basic equation.

For a classical solution of problem (P) the estimate

$$u(\mathbf{x}) \geq 0 \ \text{ in } \Omega \tag{2.1}$$

follows immediately from the maximum principle applied to equation (1.11). This is the reason why $\sigma(u)$ and κ need to be defined only for $u \geq 0$ as in (1.7). For later use we define the map

$$\mathscr{U} = \mathscr{F}(u) = \int_0^u \frac{\kappa(t)}{\sigma(t)}\, dt, \ \ u \geq 0$$

and assume

$$\int_0^\infty \frac{\kappa(t)}{\sigma(t)}\, dt = \infty. \tag{2.2}$$

$\mathscr{F}$ maps biunivocally $[0,\infty)$ onto $[0,\infty)$. Thus $\mathscr{U} = \mathscr{F}(u)$ gives a new scale on which we can measure the temperature. Let us consider the problem

$$\Delta z = 0 \ \text{ in } \Omega, \ z = 0 \ \text{ on } \Gamma_1, \ z = 1 \ \text{ on } \Gamma_2, \ \frac{\partial z}{\partial n} = 0 \ \text{ on } \Gamma_0 \tag{2.3}$$

$$k = \int_{\Gamma_2} \frac{\partial z}{\partial n}\, d\Gamma. \tag{2.4}$$

By the maximum principle in the form of Hopf [16] we have $k > 0$.

Theorem 2.1. *If $\sigma(u)$ and $\kappa(u)$ satisfy (1.7) and (2.2) all classical solutions of problem (P) are functional solutions and there exists at least one of such solutions. These solutions are in one-to-one correspondence with the solutions $X \in \mathbb{R}^1$ of the equation*

$$A = X + Rk\int_0^X \sigma\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2} - \frac{t^2}{2}\Big)\Big)\,dt.$$

If, in addition to (1.7) and (2.2), $\sigma(u) \in C^1([0,\infty))$, $\kappa(u) \in C^1([0,\infty))$ and

$$\sigma'(u) \geq 0, \tag{2.5}$$

the functional solution is also unique.

Proof. Let X be a fixed positive constant. By Theorem 1.1 all classical solutions of the problem

$$\nabla\cdot(\sigma(u)\nabla\varphi) = 0 \ \text{ in } \Omega \tag{2.6}$$

$$\varphi = 0 \ \text{ on } \Gamma_1,\ \varphi = X \ \text{ on } \Gamma_2,\ \frac{\partial\varphi}{\partial n} = 0 \ \text{ on } \Gamma_0 \tag{2.7}$$

$$\nabla\cdot(\kappa(u)\nabla u + \varphi\sigma(u)\nabla\varphi) = 0 \ \text{ in } \Omega \tag{2.8}$$

$$u = 0 \text{ on } \Gamma_1\cup\Gamma_2, \frac{\partial u}{\partial n} = 0 \ \text{ on } \Gamma_0 \tag{2.9}$$

are functional solutions and the solution is unique. Moreover, the function $\mathscr{U}(\varphi)$ entering in the definition (1.18) can be obtained as solution of the two-point problem

$$\kappa(\mathscr{U})\frac{d\mathscr{U}}{d\varphi} + \varphi\sigma(\mathscr{U}) = \gamma\sigma(\mathscr{U}),\ \mathscr{U}(0) = 0,\ \mathscr{U}(X) = 0, \tag{2.10}$$

where γ is a constant to be determined in order to satisfy both the boundary conditions. Problem (2.10) can be easily solved by separation of variables since $\frac{1}{\sigma(u)}$ is an integrating factor for it. We find as solution of problem (2.10)

$$\mathscr{F}(\mathscr{U}) = \frac{\varphi X}{2} - \frac{\varphi^2}{2},$$

where

$$\mathscr{F}(\mathscr{U}) = \int_0^{\mathscr{U}} \frac{\kappa(t)}{\sigma(t)}\,dt.$$

Hence

$$\mathscr{U}(\varphi) = \mathscr{F}^{-1}\left(\frac{X\varphi}{2} - \frac{\varphi^2}{2}\right).$$

From (1.8) we have, for the determination of $\varphi(\mathbf{x})$, the problem

$$\nabla \cdot \Big(\sigma\Big(\mathscr{F}^{-1}\Big(\frac{X\varphi}{2}-\frac{\varphi^2}{2}\Big)\Big)\nabla\varphi\Big)=0 \text{ in } \Omega \tag{2.11}$$

$$\varphi=0 \text{ on } \Gamma_1,\ \varphi=X \text{ on } \Gamma_2,\ \frac{\partial\varphi}{\partial n}=0 \text{ on } \Gamma_0. \tag{2.12}$$

To this problem we can apply the transformation

$$w=G(\varphi)=\int_0^{\varphi}\sigma\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)\Big)\,dt$$

which gives

$$\nabla w=\sigma\Big(\mathscr{F}^{-1}\Big(\frac{X\varphi}{2}-\frac{\varphi^2}{2}\Big)\Big)\nabla\varphi. \tag{2.13}$$

By (2.11) and (2.12) we have

$$\Delta w=0 \text{ in } \Omega$$

and

$$w=0 \text{ on } \Gamma_1,\ w=G(X)=\int_0^{X}\sigma\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)\Big)\,dt,\ \ \frac{\partial w}{\partial n}=0 \text{ on } \Gamma_0.$$

If $z(\mathbf{x})$ is the solution of (2.3) we obtain

$$w(\mathbf{x})=z(\mathbf{x})G(X)=z(\mathbf{x})\int_0^{X}\sigma\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)\Big)\,dt. \tag{2.14}$$

We now rewrite the nonlocal condition (1.14). On Γ_0 we have, from (2.13) and (2.14),

$$\mathbf{J}\cdot\mathbf{n}=\frac{\partial w}{\partial n}=\int_0^{X}\sigma\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)\Big)\,dt\ \frac{\partial z}{\partial n}.$$

Therefore (1.14) becomes, by (2.4),

$$A=X+Rk\int_0^{X}\sigma\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)\Big)\,dt. \tag{2.15}$$

Thus the solutions of problem (P) are in a one-to-one correspondence with the solutions of equation (2.15). On the other hand, if we define

$$f(X)=X+Rk\int_0^{X}\sigma\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)\Big)\,dt-A,$$

we have $f(0) = -A < 0$ and $f(X) > 0$ for all $X \geq A$. Hence a solution to problem (P) always exists. Besides, if $\sigma'(u) \geq 0$ the solution is also unique. For, we have

$$f'(X) = 1 + Rk\sigma(0) + Rk\int_0^X \frac{\sigma'\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)\Big)\sigma\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)t}{2\kappa\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)}\,dt > 0.$$

□

To treat cases in which $\sigma'(u)$ does not satisfy (2.5) it is useful to rewrite (2.15) in a more convenient form using the following

Lemma 2.2. *Define*

$$\tilde{\sigma}(t) = \sigma(\mathscr{F}^{-1}(t)).$$

We have

$$\int_0^X \sigma\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)\Big)\,dt = \sqrt{2}\int_0^{\frac{X^2}{8}} \frac{\tilde{\sigma}(z)}{\sqrt{\frac{X^2}{8}-z}}\,dz. \tag{2.16}$$

Proof. With the change of variable of integration $\zeta = t - \frac{X}{2}$ in the left hand side of (2.16) we have

$$\int_0^X \sigma\Big(\mathscr{F}^{-1}\Big(\frac{Xt}{2}-\frac{t^2}{2}\Big)\Big)\,dt = 2\int_0^{\frac{X}{2}} \sigma\Big(\mathscr{F}^{-1}\Big(\frac{X^2}{8}-\frac{\zeta^2}{2}\Big)\Big)\,d\zeta. \tag{2.17}$$

With the further change $z = \frac{X^2}{8} - \frac{\zeta^2}{2}$ in (2.17) we obtain

$$2\int_0^{\frac{X}{2}} \sigma\Big(\mathscr{F}^{-1}\Big(\frac{X^2}{8}-\frac{\zeta^2}{2}\Big)\Big)\,d\zeta = \sqrt{2}\int_0^{\frac{X^2}{8}} \frac{\tilde{\sigma}(z)}{\sqrt{\frac{X^2}{8}-z}}\,dz.$$

Hence (2.16) follows. □

By Theorem 2.1 and Lemma 2.2 the solutions of problem (P) are in a one-to-one correspondence with the solutions X of the equation

$$A = X + RkF(X), \tag{2.18}$$

where

$$F(X) = \sqrt{2} \int_0^{\frac{X^2}{8}} \frac{\tilde{\sigma}(z)}{\sqrt{\frac{X^2}{8} - z}}\, dz. \tag{2.19}$$

We note that $F(X)$ is well-defined for $X > 0$ since

$$F(X) \leq X\, max\Big\{\tilde{\sigma}(z),\ z \in \Big[0, \frac{X^2}{8}\Big]\Big\}.$$

The following development of $F(X)$ in powers of X could be useful in applying equation (2.18). Assume $\tilde{\sigma}(z) \in C^n([0,\infty))$. With n integration by parts in the right hand side of (2.19) we have

$$F(X) = \tilde{\sigma}(0) + \frac{1}{2^{2\cdot 1}} \frac{\tilde{\sigma}^{(1)}(0)}{1\cdot 3} X^3 + \frac{1}{2^{2\cdot 2}} \frac{\tilde{\sigma}^{(2)}(0)}{1\cdot 3\cdot 5} X^5 + \tag{2.20}$$

$$\frac{1}{2^{2\cdot 3}} \frac{\tilde{\sigma}^{(3)}(0)}{1\cdot 3\cdot 5\cdot 7} X^7 + \ldots + \frac{1}{2^{2\cdot n}} \frac{\tilde{\sigma}^{(n)}(0)}{1\cdot 3\cdot 5\cdot 7 \ldots \cdot (2n-1)(2n+1)} X^{2n+1} + R_n(X),$$

where

$$R_n(X) = \frac{\sqrt{2}\, 2^{2n-1}}{1\cdot 3 \cdot \ldots \cdot (2n-1)(2n+1)} \int_0^{\frac{X^2}{8}} \Big(\frac{X^2}{8} - z\Big)^{\frac{2n+1}{2}} \tilde{\sigma}^{(n+1)}(z)\, dz.$$

If

$$|\tilde{\sigma}^{(n)}(z)| \leq L$$

we have

$$|R_n(X)| \leq \frac{L\sqrt{2}}{2^3(2n+3)(1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)(2n+3))} X^{2n+3}.$$

Thus the series converges as $n \to \infty$ in $[0,A]$.

3. The “one-or-three” solutions case.

Three states of thermal and electrical equilibria with the circuits studied here occur in reality with industrially made thermistors (see [15]). It is therefore of interest to exhibit specific cases of conductivity laws which give the one-or-three solutions situation on the ground of the present model. To this end we have the following

Theorem 3.1. *Let $\tilde{\sigma}(z) \in C^1([0,\infty))$ be such that the function $F(X)$ defined by (2.19) satisfies*

$$\lim_{X\to\infty} F(X) = 0. \tag{3.1}$$

If there exists a unique $X_M \in (0,\infty)$ such that

$$F'(X_M) = 0,\ F'(X) > 0 \text{ in } [0,X_M),\ F'(X) < 0 \text{ in } (X_M,\infty)$$

and

$$\lim_{X\to\infty} F'(X) = 0, \tag{3.2}$$

then

$$0 > \inf\{F'(X),\ X \in [0,\infty)\} > -\infty.$$

Let $\mu^2 = -\inf\{F'(X),\ X \in [0,\infty)\}$. If

$$Rk\mu^2 < 1, \tag{3.3}$$

problem (P) has one and only one solution. If

$$Rk\mu^2 > 1, \tag{3.4}$$

then there exist two numbers A_1 and A_2 with $0 < A_2 < A_1$ such that, if

$$0 \le A < A_2 \ \text{ or } \ A > A_1,$$

problem (P) has one and only one solution. If

$$A_2 < A < A_1,$$

then problem (P) has exactly three solutions.

Proof. We have $F(0) = 0$ and $F(X) > 0$ for all $X > 0$. We claim that

$$\inf\{F'(X),\ X \in [0,\infty)\} > -\infty.$$

By contradiction, assume

$$\inf\{F'(X),\ X \in [0,\infty)\} = -\infty. \tag{3.5}$$

Let $\varepsilon > 0$. By (3.2) there exists $X_\varepsilon > 0$ such that $F'(X) > -\varepsilon$ for all $X > X_\varepsilon$. On the other hand, in $[X_M,X_\varepsilon]$ we have $\inf\{F'(X),\ X \in [0,\infty)\} > -\infty$. Hence (3.5) is not possible. By Theorem 2.1 the solutions of problem (P) are in a one-to-one correspondence with the solutions of the equation

$$X + Rk\, F(X) = 0.$$

If $L(X) = X + Rk\, F(X)$, by (3.1) we have

$$\lim_{X\to\infty} L(X) = \infty. \tag{3.6}$$

Moreover $L'(X) = 1 + RkF'(X)$. Thus, if (3.3) holds, we have $L'(X) > 0$. Since $L(0) = 0$ we conclude, by (3.6), that (P) has one and only one solution. If (3.4) holds there exist two numbers X_1 and X_2 with $0 < X_1 < X_2$ such that $RkF'(X) < -1$ in (X_1, X_2). Therefore $L'(X) > 0$ in $(0, X_1) \cup (X_2, \infty)$ and $L'(X) < 0$ in (X_1, X_2). Hence $L(X)$ is strictly increasing in $(0, X_1) \cup (X_2, \infty)$ and strictly decreasing in (X_1, X_2). Thus X_1 is a strict relative maximum and X_2 a strict relative minimum for $L(X)$. We conclude that problem (P) has exactly three solutions if $A_2 = L(X_2) < A < A_1 = L(X_1)$. $\square$

As an application of the theorem we take

$$\tilde{\sigma}(z) = e^{-z}. \tag{3.7}$$

With a direct calculation we find in this case

$$F(X) = 2\sqrt{2}\, e^{\frac{-X^2}{8}} \int_0^{\frac{X}{2\sqrt{2}}} e^{t^2}\, dt. \tag{3.8}$$

It is easy to verify that (3.8) satisfies all the assumptions of Theorem 3.1. Therefore, if (3.7) holds, which is compatible with the empirical conductivity laws reported in [15], the “one or three solutions” situation occurs.

Remark 3.2. It would be interesting to prove a similar “one or three solutions” result assuming (1.19) as boundary condition for the temperature.

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