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RIEMANN SURFACES WITH A QUASI LARGE ABELIAN GROUP OF AUTOMORPHISMS

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In this work we classify all Riemann surfaces having a quasi large abelian group of automorphisms, i.e. having an abelian group of automorphisms of order strictly bigger than $2(g-1)$, where g denotes the genus of the Riemann surface.

1. Introduction

In 1890 Schwartz [18] proved that a compact Riemann surface of genus $g \geq 2$ has a finite number of automorphisms.
A fundamental tool for the study of the groups of automorphisms of Riemann surfaces is the so-called Riemann-Hurwitz formula proved in 1893 [8]. By applying the formula Hurwitz established that a compact Riemann surface of genus $g \geq 2$ has at most $84(g-1)$ automorphisms.
Better bounds have been found with more restrictive assumptions: Wiman [19] proved in 1879 the upper bound $4g+2$ for a cyclic group of automorphisms of Riemann surfaces. In [13] Nakajima showed that an abelian group of automorphisms can't have order larger than $4g+4$.
Given a compact Riemann surface C of genus $g \geq 2$ and G group of automorphisms of C, G is called *large* if $|G| > 4(g-1)$.

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Kulkarni proved in [9] by using the formula of Riemann-Hurwitz, that if G is a large group of automorphisms of C and $C' = C/G$, then C' has genus $g = 0$ and the projection map $C \to C'$ has 3 or 4 critical values.
Clelia Lomuto applied Pardini's Theorem ([10]) about Galois covering to the result of Kulkarni in order to obtain a complete list of Riemann surfaces with a large abelian group of automorphisms.
In this paper we will extend the results of Lomuto by weakening the hypothesis $|G| > 4(g-1)$ using an elementary method based on the classic Riemann existence theorem.
We prove that if G is an abelian group of order $G > \left|2g-2\right|$ then C/G is a sphere. This motivates us in introducing the definition of *quasi large*: we will say that a group of automorphisms is quasi large if $G > 2(g-1)$. Our main result is the classification of all the Riemann surfaces with a quasi-large group of automorphisms.

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2. Riemann surfaces having a large group of automorphisms

In this section we'll briefly recall some facts about group actions on Riemann surfaces. For further details, see [11].
Let G be a finite group acting holomorphically and effectively on a Riemann surface C, and fix a point $P \in C$. Then:

(a) the stabilizer subgroup $H_P := \left\{\sigma \in G \middle| \sigma P = P\right\}$ is cyclic

(b) there is a finite number of points in C having a nontrivial stabilizer;

(c) there is a unique structure of Riemann surface on the quotient C/G such that the quotient map $p\colon C \to C/G$ is holomorphic. The degree of p is $|G|$. Moreover, assume that H_P is nontrivial and let g be a generator of the stabilizer subgroup H_P. Then there is a local coordinate z on C centered at P such that $g(z) = \lambda z$, where λ is a primitive $|H_P|^{th}$ root of unity.

A map as above is called a Galois cover with Galois group G; if G is abelian, we say that p is an abelian cover. These maps are special $|G|$-sheeted branched covers of C/G.
Let $Q_1, \dots, Q_k$ be the critical values of p, and let us associate to each of these Q_i their ramification index $l_i = |H_P|$, where H_P is the stabilizer subgroup of any $P \in p^{-1}(Q_i)$.
Note that the ramification index of a critical value is well-defined since if P

and P' are points of C with $p(P) = p(P')$, then their stabilizers H_P and $H_{P'}$ are conjugated. In this special case the Riemann-Hurwitz formula can be written as follows:

$$2g_C - 2 = |G| \left(2g_{C/G} - 2 + \sum_{i=1}^{k} \left(1 - \frac{1}{l_i} \right) \right) \tag{1}$$

We assume G abelian. Then $H_P = H_{P'}$ and therefore we can define, for every Q_i, $H_{Q_i} := H_P$ for a randomly chosen $P \in p^{-1}(Q_i)$. Applying the formula (1) and Pardini's Theorem, Lomuto proved in [10]

Theorem 2.1. *Let C be a Riemann surface and G a large abelian group of automorphisms of C and $C' = C/G$. Then $g_{C'} = 0$ and the projection map $p\colon C \longrightarrow C'$ has 3 or 4 critical values. The possible triples (group, ramification indices, genus of the Riemann surfaces) are listed in table 1.*

	Abelian group	Ramification indices	Genus of the curve C: g
		Four critical values	
1	$\mathbb{Z}_6$	$\{2,2,3,3\}$	2
		Three critical values	
2	$\mathbb{Z}_{4g+2}$	$\{2,2g+1,4g+2\}$	$g \geq 2$
3	$\mathbb{Z}_{4g}$	$\{2,4g,4g\}$	$g \geq 2$
4	$\mathbb{Z}_{12}$	$\{3,4,12\}$	3
5	$\mathbb{Z}_{15}$	$\{3,5,15\}$	4
6	$\mathbb{Z}_6$	$\{3,6,6\}$	2
7	$\mathbb{Z}_{21}$	$\{3,7,21\}$	6
8	$\mathbb{Z}_9$	$\{3,9,9\}$	3
9	$\mathbb{Z}_5$	$\{5,5,5\}$	2
10	$\mathbb{Z}_2 \times \mathbb{Z}_{2g+2}$	$\{2,2g+2,2g+2\}$	$g \geq 2$
11	$\mathbb{Z}_3 \times \mathbb{Z}_9$	$\{3,9,9\}$	7
12	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\{3,6,6\}$	4
13	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\{4,4,4\}$	3
14	$\mathbb{Z}_5 \times \mathbb{Z}_5$	$\{5,5,5\}$	6

Table 1: The table shows a list of abelian covers of $\mathbb{P}^1$. The second column represents the large abelian group G, the third the ramification indices and the fourth the genus of the Riemann surfaces C.

3. The method

In this section we quickly describe the method, based on the classical Riemann Existence Theorem, we use to describe and study finite group actions on Riemann surfaces. The method has been described in all details by the second author in [17], see also [7], [1], [2], [16], [5], [12], [3], [15], [4].

Let G be a finite subgroup of automorphisms acting on the Riemann surface C with k critical values $Q_1,\dots,Q_k$ and $p\colon C \longrightarrow C' := C/G$ the corresponding projection map. G is also a group of automorphisms of the unbranched cover $\overline{p}\colon \overline{C} \to \overline{C}'$ ($(\overline{C},\overline{p})$ for short) where $\overline{C}' := C' \setminus \{Q_1,\dots,Q_k\}$, $\overline{C} = p^{-1}(\overline{C}')$ and $\overline{p} := p\big|_{C\setminus p^{-1}(\{Q_1,\dots,Q_k\})}$.
We fix a point $\widetilde{x}_0 \in \overline{C}$ and define $x_0 := p(\widetilde{x}_0)$. The cover $(\overline{C},\overline{p})$ induces a monomorphism among the fundamental groups

$$\overline{p}_*\colon \pi(\overline{C},\widetilde{x}_0) \longrightarrow \pi(\overline{C}',x_0).$$

We define $H := \overline{p}_*(\pi(\overline{C},\widetilde{x}_0)) < \pi(\overline{C}',x_0)$. Then $H \triangleleft \pi(\overline{C}',x_0)$ and

$$\mathrm{Aut}(C,p) \cong G \cong \frac{\pi(\overline{C}',x_0)}{H}.$$

The last isomorphism is equivalent to the existence of a surjective homomorphism

$$\psi\colon \pi(\overline{C}',x_0) \longrightarrow G, \text{ with } \ker(\psi) = H.$$

Recall that the abelianization of the fundamental group of $\overline{C}'$ is the first homology group $H_1\left(\overline{C}',\mathbb{Z}\right)$. If G is abelian, then ψ factors uniquely through an homomorphisms $\psi'\colon H_1\left(\overline{C}',\mathbb{Z}\right) \to G$.

We have associated to each abelian cover(C,G) the following data: the Riemann surface $C' = C/G$, points $Q_1,\dots,Q_k \in C'$, and the surjection ψ. These data determine the cover as follows.

Theorem 3.1. *For all finite abelian group G, $\forall k \in \mathbb{N}, \forall l_1,\dots,l_k \in \mathbb{N}$ with $l_i \geq 2$, there exists a bijection between*

$$\left\{\begin{array}{c} \textit{Galois cover } (C,G) \\ \textit{where } C \textit{ is any Riemann surface} \\ \textit{with } k \textit{ critical values} \\ \textit{and ramification indices} \\ l_1,\dots,l_k. \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \textit{Riemann surfaces } C' \\ \textit{points } Q_1,\dots,Q_k \in C' \\ \textit{ and a surjection} \\ \psi'\colon H_1\left(\overline{C}',\mathbb{Z}\right) \to G \textit{ s. t.} \\ \forall i \quad \psi'(\gamma_i) \textit{ has order } l_i. \end{array}\right\}$$

Here γ_i is the class of a small circle positively oriented around Q_i.

Indeed, by standard algebraic topology, a triple $(C',\{Q_i\},\psi')$ as above determines uniquely an unbranched Galois cover of complex manifolds $\overline{p}\colon \overline{C} \to C' \setminus \{Q_i\}$ with Galois group G which, by the classical Riemann Existence Theorem [7, Theorem 4.10] determines uniquely a Riemann surface C and a Galois cover (C,G) yielding exactly those data.

4. Riemann surfaces having a quasi large group of automorphisms

We say that a subgroup G of $\mathrm{Aut}(C)$ is *quasi large* if:

$$|G| > 2\,(g_C - 1)\,.$$

Proposition 4.1. *Let C be a Riemann surface of genus at least 2, G a quasi large abelian group of automorphisms of C and $C' = C/G$. Then $g_{C'} = 0$.*

Proof. By the Riemann-Hurwitz formula (1), if G is quasi-large

$$g_{C'} + \frac{1}{2}\sum_{j=1}^{k}\left(1 - \frac{1}{l_j}\right) < \frac{3}{2}. \tag{2}$$

It follows that $g_{C'} \leq 1$. If $g_{C'} = 1$ then $k \neq 0$ (else $g_C = 1$) and therefore by (2) $k = 1$. In this case γ_1 is trivial in $H_1(\overline{C}')$, contradicting the fact that, by Theorem 3.1, $\psi'(\gamma_1)$ has order l_1. Therefore $g_{C'} = 0$. □

The statement is sharp in the sense that the inequality $|G| > 2\,(g-1)$ can't be substituted by any inequality of the form $|G| > ag + b$ with $a < 2$. We proved it in [17] by constructing infinitely many examples with elliptic C/G.

The following theorem includes the statement of Proposition 4.1: it's the main result of this article.

Theorem 4.2. *Let C be a Riemann surface of genus g and G a quasi large abelian group of automorphisms of C and $C' = C/G$. Then $g_{C'} = 0$ and the projection map $p : C \longrightarrow C'$ has 3, 4 or 5 branch points. The possible pairs (abelian group, ramification indices) are exactly those listed in Table 2.*

In the case $*$, $f(\alpha,\beta,\delta_1,\delta_2,\delta_3) := \frac{\alpha}{2}\{\alpha\beta\delta_1\delta_2\delta_3 - \delta_1 - \delta_2 - \delta_3\}$ and $(\alpha, \beta, \delta_1, \delta_2, \delta_3)$ is any quintuple of positive integers such that

- $\forall i \neq j$, $\gcd(\delta_i,\delta_j) = 1$*;*
- $\delta_1\delta_2\delta_3\,(\beta+1)$ *is even and*
- $f(\alpha,\beta,\delta_1,\delta_2,\delta_3) > 0$.

By means of Proposition 4.1 in order to classify the Riemann Surfaces with an abelian quasi-large group of automorphisms we have to consider only the case C/G rational. In this case the loops γ_i considered in Theorem 3.1 have the property that $\sum \gamma_i = 0$, and therefore their images g_i form a set of generators of G which is **spherical**, i.e. $\sum g_i = 0 \in G$. By Hurwitz formula (1), since we are only interested in the case $g_C > 1$, the cardinality of this set is at least 3.

Fix integers $l_1, \ldots, l_m$ such that $l_i \geq 2$ and denote by $G_{l_1 \ldots l_m}$ the quotient group $(\mathbb{Z}_{l_1} \times \ldots \times \mathbb{Z}_{l_m})\,/ < 1, \ldots, 1 >$. Let $e_1, \ldots, e_m$ be the standard generators

	Abelian group	Ramification indices	Genus of C: $g \geq 2$
Five critical values			
(a)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{2,2,2,2,2\}$	2
(b)	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\{2,2,2,2,2\}$	3
(c)	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\{2,2,2,2,2\}$	5
(d)	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\{2,2,2,3,3\}$	6
Four critical values			
(a)	$\mathbb{Z}_{2g}$	$\{2,2,2g,2g\}$	g
(b)	$\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$	$\{2,2,g+1,g+1\}$	g
(c)	$\mathbb{Z}_2 \times \mathbb{Z}_{4t+2}$	$\{2,2,2t+1,4t+2\}$	$4t, \quad t \in \mathbb{Z}, t \geq 1$
(d)	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2t}$	$\{2,2,2t,2t\}$	$4t-3, \quad t \in \mathbb{Z}, t \geq 2$
(e)	$\mathbb{Z}_6$	$\{2,3,3,6\}$	3
(f)	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\{2,3,3,6\}$	7
(g)	$\mathbb{Z}_{12}$	$\{2,3,4,12\}$	6
(h)	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	$\{2,3,4,12\}$	11
(j)	$\mathbb{Z}_{30}$	$\{2,3,5,30\}$	15
(j)	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\{2,3,6,6\}$	6
(k)	$\mathbb{Z}_6 \times \mathbb{Z}_6$	$\{2,3,6,6\}$	16
(l)	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\{2,4,4,4\}$	7
(m)	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	$\{2,4,4,4\}$	13
(n)	$\mathbb{Z}_3$	$\{3,3,3,3\}$	2
(o)	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\{3,3,3,3\}$	4
(p)	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\{3,3,3,3\}$	10
(q)	$\mathbb{Z}_{12}$	$\{3,3,4,4\}$	6
(r)	$\mathbb{Z}_{15}$	$\{3,3,5,5\}$	8
Three critical values			
$*$	$\mathbb{Z}_\alpha \times \mathbb{Z}_{\alpha\beta\delta_1\delta_2\delta_3}$	$\{\alpha\beta\delta_2\delta_3, \alpha\beta\delta_1\delta_3, \alpha\beta\delta_1\delta_2\}$	$1+f(\alpha,\beta,\delta_1,\delta_2,\delta_3)$

Table 2: The table shows all abelian covers with quasi large group G. The second column represents G, the third the ramification indices and the fourth the genus of the Riemann surfaces C.

of $\mathbb{Z}_{l_1} \times \ldots \times \mathbb{Z}_{l_m}$ and let $g_i \in G_{l_1 \ldots l_m}$ be the image of e_i, $i = 1, \ldots, m$. Of course we have $\sum g_i = 0$, i. e., the g_i are a set of spherical generators, and $\operatorname{ord}(g_i) | l_i$ for $i = 1, \ldots, m$. It is easy to check that $\operatorname{ord}(g_i) = l_i$ for every i iff the following condition holds:

$$\forall = 1, \ldots, m, \qquad l_i \big| \operatorname{lcm}_{j \neq i}(l_j). \tag{3}$$

So, by the Riemann Existence Theorem, if $l_1, \ldots, l_m$ satisfy (3), then there exists a $G_{l_1 \ldots l_m}$-cover $C \to \mathbb{P}^1$ with branching indices $l_1, \ldots, l_m$.

Conversely let G be a finite abelian group with spherical generators $h_1, \ldots,$ h_m and set $l_i := \operatorname{ord}(h_i)$. There is a surjective homomorphism $f \colon G_{l_1 \ldots l_m} \to G$ such that $f(g_i) = h_i$. Since $\operatorname{ord}(h_i) = l_i$, we have $\operatorname{ord}(g_i) = l_i$ for $i = 1, \ldots, m$ and the l_i satisfy (3). This remark proves the following:

Lemma 4.3. *There exists an abelian cover of* $\mathbb{P}^1$ *with branching indices* $l_1, \ldots, l_m$ *iff* $l_1, \ldots, l_m$ *satisfy (3).*

Remark 4.4. Denote by K the kernel of the homomorphism $f\colon G_{l_1 \ldots l_m} \to G$ above. Since $\operatorname{ord}(h_i) = l_i = \operatorname{ord}(g_i)$, we have $K \cap < g_i >= \{0\}$ for every $i = 1, \ldots, m$. Hence if $C \to \mathbb{P}^1$ is a $G_{l_1 \ldots l_m}-$cover as above, namely such that the non trivial stabilizers of points of C are precisely the subgroups $< g_i >$ and points in different orbits have different stabilizer, then K acts freely on C and $C \to C/K$ is unbranched. Note that $G_{l_1 \ldots l_m}$ is a quasi large group of automorphisms for C if and only if G is quasi large for C/K.

Let us consider first the case of three critical values.We need the following lemma.

Lemma 4.5. *Let* $l, \delta_1, \delta_2, \delta_3$ *be positive integers such that* $\gcd(\delta_i, \delta_j) = 1$ *for* $i \neq j$. *Then*

$$G_{l\delta_2\delta_3, l\delta_1\delta_3, l\delta_1\delta_2} \cong \mathbb{Z}_l \times \mathbb{Z}_{l\delta_1\delta_2\delta_3}.$$

Proof. We take

$$\begin{array}{lll} \alpha_1, \alpha_2 & \text{with} & \alpha_1\delta_2 - \alpha_2\delta_1 = 1 \\ \beta_1, \beta_2 & \text{with} & \beta_1\delta_3 - \beta_2\delta_1\delta_2 = 1 \end{array}$$

and the map $DA\colon \mathbb{Z}_{l\delta_2\delta_3} \times \mathbb{Z}_{l\delta_1\delta_3} \times \mathbb{Z}_{l\delta_1\delta_2} \longrightarrow \mathbb{Z}_l \times \mathbb{Z}_{l\delta_1\delta_2\delta_3}$, with given matrices

$$A = \begin{pmatrix} \beta_1 & -\beta_1 & -\beta_2 \\ -\alpha_2\delta_1 & \alpha_1\delta_2 & 0 \\ -\delta_1\delta_2 & \delta_1\delta_2 & \delta_3 \end{pmatrix} \qquad D := \begin{pmatrix} 1 & \beta_2 & 0 \\ 0 & -\delta_3 & 1 \end{pmatrix}.$$

A tedious but straightforward computation shows that the map is surjective and its kernel is the cyclic subgroup generated by $(1,1,1)$.

An alternative proof is the following. It is easy to compute that the subgroup generated by $(1,0,1)$ and $(1,1,0)$ in $\mathbb{Z}_{l\delta_2\delta_3} \times \mathbb{Z}_{l\delta_1\delta_3} \times \mathbb{Z}_{l\delta_1\delta_2}$ is isomorphic to $\mathbb{Z}^2_{l\delta_1\delta_2\delta_3}$. Since the maximal order in $\mathbb{Z}_{l\delta_2\delta_3} \times \mathbb{Z}_{l\delta_1\delta_3} \times \mathbb{Z}_{l\delta_1\delta_2}$ is $l\delta_1\delta_2\delta_3$, then the group is isomorphic to $\mathbb{Z}_l \times \mathbb{Z}^2_{l\delta_1\delta_2\delta_3}$. The statement follows because $(1,1,1)$ has order $l\delta_1\delta_2\delta_3$. □

Proposition 4.6. *Let* G *be an abelian group and let* $g_1, g_2, g_3 \in G$ *be spherical generators of orders* l_1, l_2, l_3 *respectively. Then there are* $\alpha, \beta, \delta_1, \delta_2, \delta_3 \in \mathbb{N}$ *such that*

- $\gcd(\delta_i, \delta_j) = 1$ *for* $i \neq j$,
- *the generators have the orders* $(\alpha\beta\delta_2\delta_3, \alpha\beta\delta_1\delta_3, \alpha\beta\delta_1\delta_2)$,

- $G \cong \mathbb{Z}_\alpha \times \mathbb{Z}_{\alpha\beta\delta_1\delta_2\delta_3}$,

- *the product* $\delta_1\delta_2\delta_3(\beta+1)$ *is even.*

Vice versa, for all $\alpha, \beta, \delta_1, \delta_2, \delta_3 \in \mathbb{N}$ *with* $\gcd(\delta_i, \delta_j) = 1$ *for* $i \neq j$ *and even product* $\delta_1\delta_2\delta_3(\beta+1)$*, the group* $G \cong \mathbb{Z}_\alpha \times \mathbb{Z}_{\alpha\beta\delta_1\delta_2\delta_3}$ *has a set of three spherical generators of orders* $(\alpha\beta\delta_2\delta_3, \alpha\beta\delta_1\delta_3, \alpha\beta\delta_1\delta_2)$.

Proof. ($\Rightarrow$) By assumption there is a short exact sequence

$$1 \longrightarrow K \longrightarrow \mathbb{Z}_{l_1} \times \mathbb{Z}_{l_2} \times \mathbb{Z}_{l_3} \longrightarrow G \longrightarrow 1. \tag{4}$$

The images of $(1,0,0)$, $(0,1,0)$ and of $(0,0,1)$ in G have respectively the orders l_1, l_2 e l_3. As $(1,1,1) \in K$ it follows that

$$(\mathrm{lcm}(l_2,l_3), \mathrm{lcm}(l_2,l_3), \mathrm{lcm}(l_2,l_3)) = (\mathrm{lcm}(l_2,l_3),0,0) \in K \Rightarrow l_1 \big| \mathrm{lcm}(l_2,l_3).$$

Similarly we get $l_2 \big| \mathrm{lcm}(l_1,l_3)$ e $l_3 \big| \mathrm{lcm}(l_1,l_2)$. In particular we define

$$\begin{array}{rcl} L & := & \mathrm{lcm}(l_1,l_2,l_3) = \mathrm{lcm}(l_i,l_j) \quad \forall i \neq j \\ l & := & \gcd(l_1,l_2,l_3) \\ \delta_i & := & \gcd\left(\frac{l_j}{l}, \frac{l_k}{l}\right) \text{ con } \{i,j,k\} = \{1,2,3\}. \end{array}$$

From this definition $\gcd(\delta_i, \delta_j) = 1$. We observe that

$$\begin{array}{rcl} \delta_2\delta_3 = \mathrm{lcm}(\delta_2,\delta_3) \big| \frac{l_1}{l} \Rightarrow l\delta_2\delta_3 \big| l_1 & \Rightarrow & \exists \mu_1 \in \mathbb{N}: \ l_1 = \mu_1 l \delta_2\delta_3 \\ & \Rightarrow & \exists \mu_2 \in \mathbb{N}: \ l_2 = \mu_2 l \delta_1\delta_3 \\ & \Rightarrow & \exists \mu_3 \in \mathbb{N}: \ l_3 = \mu_3 l \delta_1\delta_2 \end{array}$$

$\gcd\left(\frac{l_1}{l}, \frac{l_2}{l}\right) = \delta_3 \Rightarrow \gcd(\mu_1\delta_2, \mu_2\delta_1) = 1$, similarly $\gcd(\mu_i\delta_j, \mu_j\delta_i) = 1$. Furthermore

$$\begin{array}{rcl} l_1 \big| \mathrm{lcm}(l_2,l_3) = L = l\delta_1 \mathrm{lcm}(\mu_2\delta_3, \mu_3\delta_2) & \Rightarrow & \mu_1\delta_2\delta_3 \big| \delta_1 \mathrm{lcm}(\mu_2\delta_3, \mu_3\delta_2) \\ & \Rightarrow & \mu_1\delta_2\delta_3 \big| \delta_1\delta_2\delta_3\mu_2\mu_3 \Rightarrow \mu_1 \big| \delta_1\mu_2\mu_3 \end{array}$$

But $\gcd(\mu_1, \delta_1) = \gcd(\mu_1, \mu_2) = \gcd(\mu_1, \mu_3) = 1$. It follows $\mu_1 = 1$. Similarly $\mu_2 = \mu_3 = 1$. We conclude $l_1 = l\delta_2\delta_3$, $l_2 = l\delta_1\delta_3$, $l_3 = l\delta_1\delta_2$.

By lemma 4.5 and the exact sequence (4) $G \cong \frac{\mathbb{Z}_l \times \mathbb{Z}_{l\delta_1\delta_2\delta_3}}{H}$ for a subgroup $H < \mathbb{Z}_l \times \mathbb{Z}_{l\delta_1\delta_2\delta_3}$.

From the classification Theorem of abelian groups follows that there exist α and β such that

$$G \cong \mathbb{Z}_\alpha \times \mathbb{Z}_{\alpha\beta\delta_1\delta_2\delta_3}.$$

Finally we have to prove that $\delta_1\delta_2\delta_3(\beta+1)$ is even. Let $\{g_1,g_2,g_3\}$ be the set of spherical generators. We have

$$\begin{array}{lcl} \operatorname{ord}(g_1)=\alpha\beta\delta_2\delta_3 & \Rightarrow & g_1=(a_1,k_1\delta_1)\in\mathbb{Z}_\alpha\times\mathbb{Z}_{\alpha\beta\delta_1\delta_2\delta_3} \\ \operatorname{ord}(g_2)=\alpha\beta\delta_1\delta_3 & \Rightarrow & g_2=(a_2,k_2\delta_2) \\ \operatorname{ord}(g_3)=\alpha\beta\delta_1\delta_2 & \Rightarrow & g_3=(a_3,k_3\delta_3). \end{array}$$

If, by reductio ad absurdum, $\delta_1\delta_2\delta_3(\beta+1)$ is odd, then β is even

$$\begin{array}{ll} \Rightarrow & \alpha\beta\delta_1\delta_2\delta_3 \text{ is even} \\ \Rightarrow & k_1\delta_1+k_2\delta_2+k_3\delta_3 \text{ is even} \\ \Rightarrow & \exists i\in\{1,2,3\} \text{ such that } k_i\delta_i \text{ is even.} \end{array}$$

Since $\delta_1\delta_2\delta_3(\beta+1)$ is odd, then $\forall j\in\{1,2,3\}$, δ_j is odd. This implies, that there exists an $i\in\{1,2,3\}$ such that k_i is even: $k_i=2h_i$.
We choose j,k with $\{i,j,k\}=\{1,2,3\}$. It follows that

$$\alpha\cdot\frac{\beta}{2}\cdot\delta_j\delta_k\cdot g_i=\left(\alpha\left[\frac{\beta}{2}\delta_j\delta_k a_i\right],\alpha\beta\delta_1\delta_2\delta_3\,[h_i]\right)\equiv 0 \text{ in } \mathbb{Z}_\alpha\times\mathbb{Z}_{\alpha\beta\delta_1\delta_2\delta_3}.$$

$\Rightarrow \operatorname{ord}(g_i)\big|\alpha\frac{\beta}{2}\delta_j\delta_k$, this contradicts $\operatorname{ord}(g_i)=l_i=\alpha\beta\delta_j\delta_k$.

($\Leftarrow$) We can suppose $\delta_3(\beta+1)$ even and $\delta_1\delta_2$ odd up to permutations of δ_i. We choose

$$\begin{array}{lll} \widetilde{\alpha}_1,\widetilde{\alpha}_2\in\mathbb{N} & \text{con} & \widetilde{\alpha}_1\delta_2-\widetilde{\alpha}_2\delta_1=1 \\ \widetilde{\beta}_1,\widetilde{\beta}_2\in\mathbb{N} & \text{con} & \widetilde{\beta}_1\delta_3-\widetilde{\beta}_2\delta_1\delta_2=1. \end{array}$$

We note that they exist because $\gcd(\delta_i,\delta_j)=1$. For every $\eta,\theta\in\mathbb{Z}$ we write

$$\begin{array}{lclclcl} \alpha_1 & = & \widetilde{\alpha}_1+\eta\delta_1 & \quad & \beta_1 & = & \widetilde{\beta}_1+\theta\delta_1\delta_2 \\ \alpha_2 & = & \widetilde{\alpha}_2+\eta\delta_2 & & \beta_2 & = & \widetilde{\beta}_2+\theta\delta_3. \end{array}$$

and still $\alpha_1\delta_2-\alpha_2\delta_1=\beta_1\delta_3-\beta_2\delta_1\delta_2=1$.
We consider then the maps D and A in the proof of lemma 4.5 for $l=\alpha\beta$, and let E be the composition of the surjection DA with the natural projection

$$\mathbb{Z}_{\alpha\beta}\times\mathbb{Z}_{\alpha\beta\delta_1\delta_2\delta_3}\longrightarrow\mathbb{Z}_\alpha\times\mathbb{Z}_{\alpha\beta\delta_1\delta_2\delta_3}.$$

Set $g_1:=E(1,0,0)$, $g_2:=E(0,1,0)$ and $g_3:=E(0,0,1)$. Then

(a) $\{g_1,g_2,g_3\}$ is a set of generators, because it is the image of a set of generators of $\mathbb{Z}_{l\delta_2\delta_3}\times\mathbb{Z}_{l\delta_1\delta_3}\times\mathbb{Z}_{l\delta_1\delta_2}$ through a surjective map.

(b) $\{g_1,g_2,g_3\}$ is a set of spherical generators, since

$$g_1+g_2+g_3=E(1,1,1)=(0,0).$$

(c) We leave to the reader the long but standard verification that one can choose η and θ so that the order of the g_i are exactly as prescribed. □

Proof of 4.2. From the inequality (2) we deduce that at most the following possibilities for the branching indices can occur:

(I) Five critical values:

- (a) One infinite family: $\{2,2,2,2,n\}$, $2 \leq n$.
- (b) Three exceptional cases: $\{2,2,2,3,a\}$, $3 \leq a \leq 5$.

(II) Four critical values:

- (a) One infinite family depending on 2 parameters: $\{2,2,m,n\}$, $2 \leq m \leq n$; and if $m = 2 \Rightarrow 3 \leq n$.
- (b) Four infinite families: $\{2,3,a,n\}$, $3 \leq a \leq 6$ e $a \leq n$.
- (c) 35 cases: $\{2,3,7,a\}$, $7 \leq a \leq 41$.
- (d) 16 cases: $\{2,3,8,a\}$, $8 \leq a \leq 23$.
- (e) 9 cases: $\{2,3,9,a\}$, $9 \leq a \leq 17$.
- (f) 5 cases: $\{2,3,10,a\}$, $10 \leq a \leq 14$.
- (g) 3 cases: $\{2,3,11,a\}$, $11 \leq a \leq 13$.
- (h) One infinite family: $\{2,4,4,n\}$, $4 \leq n$.
- (i) 15 cases: $\{2,4,5,a\}$, $5 \leq a \leq 19$.
- (j) 6 cases: $\{2,4,6,a\}$, $6 \leq a \leq 11$.
- (k) 3 cases: $\{2,4,7,a\}$, $7 \leq a \leq 9$.
- (l) 5 cases: $\{2,5,5,a\}$, $5 \leq a \leq 9$.
- (m) 2 cases: $\{2,5,6,a\}$, $6 \leq a \leq 7$.
- (n) One infinite family: $\{3,3,3,n\}$, $3 \leq n$.
- (o) 8 cases: $\{3,3,4,a\}$, $4 \leq a \leq 11$.
- (p) 3 cases: $\{3,3,5,a\}$, $5 \leq a \leq 7$.
- (q) 2 cases: $\{3,4,4,a\}$, $4 \leq a \leq 5$.

(III) Three critical values:

- (a) One infinite family depending on 2 parameters: $\{2,m,n\}$, $3 \leq m \leq n$; if $m = 3$ then $n \geq 7$, if $m = 4$ then $n \geq 5$.
- (b) One infinite family depending on 2 parameters: $\{3,m,n\}$, $3 \leq m \leq n$; if $m = 3$ then $n \geq 4$.
- (c) One infinite family depending on 3 parameters: $\{l,m,n\}$, $4 \leq l \leq m \leq n$.

We start by considering the cases in the previous list with 5 critical values. By Theorem 3.1 there exist $m_1, m_2, m_3, m_4, m_5 \in G$ such that $\langle m_1, m_2, m_3, m_4, m_5 \rangle = G$, $\sum_{i=1}^{5} m_i = 0$ and $\operatorname{ord}(m_i) = l_i$, $\forall i = 1, \ldots, 5$.

(I,a) One infinite family: $\{2,2,2,2,n\}$, $2 \leq n$.
Here $l_1 = l_2 = l_3 = l_4 = 2$, $l_5 = n$. By Lemma 4.3 follows $n = 2$ and then G is a quotient of $G_{2,2,2,2,2} \cong \mathbb{Z}_2^4$. Then $G \cong \mathbb{Z}_2^r$, $r \leq 4$. By (1) $g_C = 1 + 2^{r-2}$. Since $g_C \in \mathbb{N}$, $r \geq 2$. We have then 3 cases, for $r = 2,3,4$.
The following examples show their existence.
- For $r = 2$, we can take $m_1 = (1,0)$, $m_2 = (0,1)$, $m_3 = (1,1)$, $m_4 = (1,0) \Rightarrow m_5 = -m_1 - m_2 - m_3 - m_4 = (1,0)$.
- For $r = 3$, we choose $m_1 = (1,0,0)$, $m_2 = (0,1,0)$, $m_3 = (0,0,1)$, $m_4 = (1,1,0) \Rightarrow m_5 = -m_1 - m_2 - m_3 - m_4 = (0,0,1)$.
- For $r = 4$, $m_1 = (1,0,0,0)$, $m_2 = (0,1,0,0)$, $m_3 = (0,0,1,0)$, $m_4 = (0,0,0,1) \Rightarrow m_5 = -m_1 - m_2 - m_3 - m_4 = (1,1,1,1)$.
In this case we have then the three possibilities

$$\begin{aligned} G &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ with } \{2,2,2,2,2\} \text{ and } g_C = 2, \\ G &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ with } \{2,2,2,2,2\} \text{ and } g_C = 3, \\ G &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ with } \{2,2,2,2,2\} \text{ and } g_C = 5. \end{aligned}$$

(I,b) Three exceptional cases: $\{2,2,2,3,a\}$, $3 \leq a \leq 5$.
By Lemma 4.3 follows $a = 3$ and then G is quotient of $G_{2,2,2,3,3} \cong \mathbb{Z}_2 \times \mathbb{Z}_6$. By (1), $g_C = 1 + 5\frac{|G|}{12}$: 12 divides $|G|$.
Then

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_6 \text{ with } \{2,2,2,3,3\} \text{ and } g_C = 6.$$

We show the existence by taking $m_1 = (1,0)$, $m_2 = (0,3)$, $m_3 = (1,3)$, $m_4 = (0,2) \Rightarrow m_5 = -m_1 - m_2 - m_3 - m_4 = (0,4)$.

We consider now the cases with 4 critical values.

(II,a) One infinite family depending on 2 parameters: $\{2,2,m,n\}$, $2 \leq m \leq n$; and if $m = 2 \Rightarrow 3 \leq n$.
If $m = 2$, $n \geq 3$ contradicts Lemma 4.3. Then $3 \leq m \leq n$ and , again by Lemma 4.3, either $n = m$ or $n = 2m$ and m is odd. Moreover G is a quotient of $\mathbb{Z}_2^2 \times \mathbb{Z}_m$ with an element of order m, so of the form $\mathbb{Z}_2^r \times \mathbb{Z}_m$ with $r \leq 2$.
If $n = m$, all the three possibilities, one for each value of $r \in \{0,1,2\}$, exist, giving the cases (a), (b) and (d) in table 2 (the genera are computed by (1)). We show the existence
- for $r = 0$, $G \cong \mathbb{Z}_{2g}$, take $m_1 = m_2 = g$, $m_3 = 1$, $m_4 = -1$
- for $r = 1$, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{g+1}$, take $m_1 = m_2 = (1,0)$, $m_3 = (0,1)$, $m_4 = (0,-1)$.
- for $r = 2$, $G \cong \mathbb{Z}_2^2 \times \mathbb{Z}_{\frac{g+3}{2}}$, take $m_1 = (1,0,0)$, $m_2 = (0,1,0)$, $m_3 = (0,0,1)$, $m_4 = (-1,-1,-1)$.
Else, if $n = 2m$ and m is odd, by (1) $r = 2$, $m = \frac{g}{2} + 1$ and $G \cong \mathbb{Z}_2^2 \times \mathbb{Z}_{\frac{g}{2}+1} \cong \mathbb{Z}_2 \times \mathbb{Z}_{g+2}$. This is case (c) of table 2.

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_{g+1} \text{ with } \left\{2,2,\frac{g}{2}+1,g+2\right\} \text{ and } \frac{g}{4} \in \mathbb{N}.$$

The following example confirms its existence: $m_1 = (1, \frac{g}{2}+1)$, $m_2 = (1,0)$, $m_3 = (0,2)$, $m_4 = (0, \frac{g}{2}-1)$.

(II,b) Four infinite families: $\{2,3,a,n\}$, $3 \leq a \leq 6$ e $a \leq n$.

If $a = 3$, by Lemma 4.3 $n = 6$ and G is a quotient of $\mathbb{Z}_3 \times \mathbb{Z}_6$, with an element of order 6. Then either $G \cong \mathbb{Z}_3 \times \mathbb{Z}_6$ or $G \cong \mathbb{Z}_6$. Both cases exist: in the first case one can take $m_1 = (0,3)$, $m_2 = (0,2)$, $m_3 = (1,2)$, $m_4 = (2,5)$; in the second case $m_1 = 3$, $m_2 = 2$, $m_3 = 2$, $m_4 = 5$. These are cases (e) and (f) of table 2.

If $a = 4$, the same arguments as in the previous case show $n = 12$ and either $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ or $G \cong \mathbb{Z}_{12}$. Both cases exist: in the first case one can take $m_1 = (0,3)$, $m_2 = (0,2)$, $m_3 = (1,2)$, $m_4 = (1,5)$; in the second case $m_1 = 3$, $m_2 = 2$, $m_3 = 2$, $m_4 = 5$. These are cases (g) and (h) of table 2.

If $a = 5$ the same arguments show $n = 30$ and $G \cong \mathbb{Z}_{30}$. This case exists: *e.g.* $m_1 = 15$, $m_2 = 10$, $m_3 = 6$, $m_4 = 29$. This is case (i) of table 2.

If $a = 6$ we get $n = 6$ and G is a quotient of $\mathbb{Z}_6 \times \mathbb{Z}_6$ with an element of order 6. By (1), $g_C = 1 + 5\frac{|G|}{12}$: 12 divides $|G|$. Then either $G \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ or $G \cong \mathbb{Z}_6 \times \mathbb{Z}_6$. Both cases exist: in the first case one can take $m_1 = (0,3)$, $m_2 = (0,2)$, $m_3 = (1,1)$, $m_4 = (1,0)$; in the second case $m_1 = (0,3)$, $m_2 = (0,2)$, $m_3 = (5,1)$, $m_4 = (1,0)$. These are cases (j) and (k) of table 2.

(II,c-g) All these cases contradict Lemma 4.3.

(II,h) One infinite family: $\{2,4,4,n\}$, $4 \leq n$. By Lemma 4.3 $n = 4$ and G is a quotient of $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ with an element of order 4 and whose order, by (1), is divided by 8. Moreover $\mathbb{Z}_2 \times \mathbb{Z}_4$ is impossible, because the sum of three elements of order 4 in it has order 4. Then either $G \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. Both cases exist: in the first case one can take $m_1 = (2,2)$, $m_2 = (1,0)$, $m_3 = (0,1)$, $m_4 = (1,1)$; in the second case $m_1 = (1,0,0)$, $m_2 = (0,1,0)$, $m_3 = (0,0,1)$, $m_4 = (1,3,3)$. These are cases (l) and (m) of table 2.

(II,i-m) All these cases contradict Lemma 4.3.

(II,n) One infinite family: $\{3,3,3,n\}$, $3 \leq n$. By Lemma 4.3 $n = 3$ and G is a quotient of $\mathbb{Z}_3^3$, so $G \cong \mathbb{Z}_3^r$, with $r \in \{1,2,3\}$, and all cases clearly exist. This gives cases (n), (o) and (p) of table 2.

(II,o) $\{3,3,4,a\}$, $4 \leq a \leq 11$. By Lemma 4.3, $a = 4$, and G is a quotient of $\mathbb{Z}_{12}$ whose order, by (1), is divided by 12. Then $G \cong \mathbb{Z}_{12}$. This case esists: take $m_1 = 4$, $m_2 = 8$, $m_3 = 3$, $m_4 = 9$. This is case (q) of table 2.

(II,p) $\{3,3,5,a\}$, $5 \leq a \leq 7$. Arguing as in the previous case, $a = 5$, and $G \cong \mathbb{Z}_{15}$. This case exists: take $m_1 = 5$, $m_2 = 10$, $m_3 = 3$, $m_4 = 12$. This is case (r) of table 2.

(II,q) It contradicts Lemma 4.3.

The case (III) gives the last row of table 2 by direct application of Proposition 4.6. By (1) $g = 1 + \frac{\alpha}{2}(\alpha\beta\delta_1\delta_2\delta_3 - \delta_1 - \delta_2 - \delta_3)$ and it's immediate to verify that the group of automorphisms is always quasi-large. □

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