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ANALYTICAL SOLUTION OF SPACE-TIME FRACTIONAL FOKKER PLANCK EQUATIONS BY GENERALIZED DIFFERENTIAL TRANSFORM METHOD

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In the present paper, we use generalized differential transform method (GDTM) to derive solutions of some linear and nonlinear space-time fractional Fokker-Planck equations (FPE) in closed form. The space and time fractional derivatives are considered in Caputo sense and the solutions are obtained in terms of Mittag-Leffler functions.

1. Introduction

The Fokker-Planck equation (FPE), first applied to investigate Brownian motion [6] and the diffusion mode of chemical reactions [14], is now largely employed, in various generalized forms, in physics, chemistry, engineering and biology [21]. The FPE arises in kinetic theory [7], where it describes the evolution of the one-particle distribution function of a dilute gas with long-range collisions, such as a Coulomb gas. For some applications of this equation one can refer the works of He and Wu [11], Jumarie [12], Kamitani and Matsuba [13], Xu *et al.* [23], and Zak [27]. The general FPE for the motion of a concentration field

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$u(t,x)$ of one space variable x at time t has the form [21]

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]u(t,x), \tag{1}$$

with the initial condition given by

$$u(0,x) = f(x), \qquad x \in \mathbb{R}, \tag{2}$$

where $B(x) > 0$ is called the diffusion coefficient and $A(x)$ is the drift coefficient. The drift and diffusion coefficients may also depend on time. Mathematically, this equation is a linear second-order partial differential equation of parabolic type. Roughly speaking, it is a diffusion equation with an additional first-order derivative with respect to x.

There is a more general form of Fokker-Planck equation which is called the nonlinear Fokker-Planck equation. The nonlinear Fokker-Planck equation has important applications in various areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing (see [8] and references therein). In the one variable case, the nonlinear FPE is written in the following form

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]u(t,x), \tag{3}$$

with the initial condition given by

$$u(0,x) = f(x), \qquad x \in \mathbb{R}. \tag{4}$$

Due to vast range of applications of the FPE, a lot of work has been done to find numerical solution of this equation. In this context, the works of Buet *et al.* [4], Harrison [10], Palleschi *et al.* [20], Vanaja [22], and Zorzano [31] are worth mentioning.

It has been observed that diffusion processes where the diffusion takes place in a highly nonhomogeneous medium, the traditional FPE may not be adequate [1, 2]. The nonhomogeneities of the medium may alter the laws of Markov diffusion in a fundamental way. In particular, the corresponding probability density of the concentration field may have a heavier tail than the Gaussian density, and its correlation function may decay to zero at a much slower rate than the usual exponential rate of Markov diffusion, resulting in long-range dependence. This phenomenon is known as anomalous diffusion [3]. Fractional derivatives play key role in modeling particle transport in anomalous diffusion including the space fractional Fokker-Planck (advection-dispersion) equation describing

Levy flights, the time fractional Fokker-Planck equation depicting traps, and the space-time fractional equation characterizing the competition between Levy flights and traps [15, 28]. Different assumptions on this probability density function lead to a variety of space-time fractional Fokker-Planck equations.

The non-linear space-time fractional FPE can be written in the following general form

$$D_t^{\alpha} u = \left[-D_x^{\beta} A\left(x,t,u\right) + D_x^{2\beta} B\left(x,t,u\right) \right] u\left(t,x\right), \tag{5}$$

where $t > 0$, $x > 0$, $0 < \alpha \leq 1, 1 < 2\beta \leq 2$. It can be obtained from the general Fokker-Planck equation by replacing the space and time derivatives by Caputo fractional derivatives D_t^{α} and D_x^{β} defined by (6). The function $u\left(t,x\right)$ is assumed to be a causal function of time and space, i.e., vanishing for $t < 0$ and $x < 0$. Particularly for $\alpha = \beta = 1$, the fractional FPE (5) reduces to the classical nonlinear FPE given by (3) in the case $x > 0$.

Recently several numerical methods have been proposed for solutions of space and/or time fractional Fokker-Planck equations [24–26, 30]. In the present paper we obtain closed form solutions of a linear space-time fractional and a non-linear time fractional FPE using generalized differential transform method [17–19]. The differential transform method was proposed by Zhou [29] to solve linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of a series. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions and takes long time in computation whereas the differential transform is an iterative procedure for obtaining analytic Taylor series solution. The method is further developed by Momani, Odibat and Erturk in their papers [17–19] for solving two-dimensional linear and non-linear partial differential equations of fractional order.

2. Preliminaries

Definition 2.1. Caputo fractional derivative of order α is defined as [5]:

$$D_x^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(m-\alpha\right)} \int_{x_0}^{x} \frac{f^{(m)}\left(\xi\right)}{\left(x-\xi\right)^{\alpha-m+1}} d\xi, (m-1 < \alpha \leq m), m \in \mathbb{N}. \tag{6}$$

Definition 2.2. The Mittag-Leffler function which is a generalization of exponential function is defined as [16]:

$$E_{\alpha}\left(z\right) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\alpha n + 1\right)} \left(\alpha \in \mathbb{C}, Re(\alpha) > 0\right). \tag{7}$$

Definition 2.3. Generalized differential transform [17–19] is as given below: Consider a function of two variables $u(t,x)$ which is analytic and differentiated continuously with respect to t and x in the domain of interest, then the generalized differential transform of the function $u(t,x)$ is given by

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[(D_t^{\alpha})^k\left(D_x^{\beta}\right)^h u(t,x)\right]_{(t_0,x_0)}, \qquad (8)$$

where $0<\alpha,\beta\leq 1, (D_t^{\alpha})^k = D_t^{\alpha}.D_t^{\alpha}...D_t^{\alpha}$ (k times) and $U_{\alpha,\beta}(k,h)$ is the transformed function.

The inverse generalized differential transform of $U_{\alpha,\beta}(k,h)$ is given by:

$$u(t,x) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(t-t_0)^{k\alpha}(x-x_0)^{h\beta}. \qquad (9)$$

We now mention a theorem which gives the conditions under which the exponent law holds for Caputo derivative.

Theorem 2.4. [9] Suppose that $f(x) = (x-x_0)^{\lambda} g(x)$, where $\lambda>0$ and $g(x)$ has the generalized power series expansion $g(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n\alpha}$ with radius of convergence $R>0, 0<\alpha\leq 1$. Then

$$D_x^{\gamma} D_x^{\beta} f(x) = D_x^{\gamma+\beta} f(x), \qquad (10)$$

for all $(x-x_0)\in(0,R)$, the coefficients $a_n=0$ for n given by $n\alpha+\lambda-\beta=0$ and either

$\lambda>\mu, \mu = \max(\beta+[\gamma],[\beta+\gamma])$

or

$\lambda\leq\mu, a_k=0$ for $k=0,1,...,\left[\frac{\mu-\lambda}{\alpha}\right]$,

where $[x]$ denotes the greatest integer less than or equal to x.

Lemma 2.5. [17–19] Some basic properties of the generalized differential transform are as given below:
let $U_{\alpha,\beta}(k,h), V_{\alpha,\beta}(k,h)$ and $W_{\alpha,\beta}(k,h)$ be the generalized differential transforms of the functions $u(t,x), v(t,x)$ and $w(t,x)$ respectively, then

(a) if $u(t,x) = v(t,x)\pm w(t,x)$, then $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h)\pm W_{\alpha,\beta}(k,h)$.

(b) If $u(t,x) = av(t,x)$, a is constant, then $U_{\alpha,\beta}(k,h) = aV_{\alpha,\beta}(k,h)$.

(c) If $u(t,x) = v(t,x)w(t,x)$, then

$$U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(k-r,s).$$

(d) If $u(t,x) = (t-t_0)^{n_1\alpha}(x-x_0)^{n_2\beta}, n_1, n_2 \in \mathbb{N}$, then

$$U_{\alpha,\beta}(k,h) = \delta(k-n_1)\,\delta(h-n_2),$$

where δ is defined as

$$\delta(k) = \begin{cases} 1, & \text{when } k=0 \\ 0, & \text{otherwise} \end{cases}.$$

(e) If $u(t,x) = (D_t^{\alpha})^m v(t,x)$ where $0<\alpha\le 1, m\in\mathbb{N}$, then

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+m)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+m,h).$$

(f) If $u(t,x) = \left(D_x^{\beta}\right)^n v(t,x)$ where $0<\beta\le 1, n\in\mathbb{N}$, then

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\beta(h+n)+1)}{\Gamma(\beta h+1)} V_{\alpha,\beta}(k,h+n).$$

3. Applications

In this section, we shall apply this method for solving linear/nonlinear fractional FPE.

Example 3.1. Consider the linear space-time fractional FPE:

$$D_t^{\alpha} u(t,x) = \left[-D_x^{\beta}\left(px^{\beta}\right) + \left(D_x^{\beta}\right)^2\left(qx^{2\beta}\right)\right] u(t,x), \tag{11}$$

where $t>0$, $x>0, 0<\alpha\le 1, 1<2\beta\le 2, p,q\in\mathbb{R}, D_t^{\alpha}, D_x^{\beta}$ are Caputo fractional derivatives defined by (6) and initial condition is

$$u(0,x) = x^{a-1},\ a\ge 1, \tag{12}$$

where $(a-1)/\beta$ is a nonnegative integer. Applying generalized differential transform (8) with $x_0 = 0 = t_0$, to both sides of time fractional FPE (11) and making use of properties of generalized differential transform given in Lemma 2.5, equation (11) transforms to

$$\begin{aligned} U_{\alpha,\beta}(k+1,h) = \tfrac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} \Bigg[& -\tfrac{p\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)} \sum_{s=0}^{h+1} \delta(h-s)\, U_{\alpha,\beta}(k,s) \\ & + \tfrac{q\Gamma(\beta(h+2)+1)}{\Gamma(\beta h+1)} \sum_{s=0}^{h+2} \delta(h-s)\, U_{\alpha,\beta}(k,s) \Bigg]. \end{aligned} \tag{13}$$

The generalized differential transform of initial condition (12) is given by

$$U_{\alpha,\beta}(0,h) = \delta\left(h - \frac{a-1}{\beta}\right). \tag{14}$$

Utilizing the recurrence relation (13) and the transformed initial condition (14), for $k = 1, 2, ...$ we obtain

$$U_{\alpha,\beta}(k,h) = \begin{cases} \frac{b^k}{\Gamma(\alpha k+1)} & \text{when } h = \frac{a-1}{\beta} \\ 0 & \text{otherwise} \end{cases}, \tag{15}$$

where $b = q(a)_{2\beta} - p(a)_{\beta}$, with $(a)_{\beta}$ denoting pochhammer symbol. From the inverse transform given by equation (9), we have

$$u(t,x) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)\, t^{\alpha k} x^{\beta h}. \tag{16}$$

Using the values of $U_{\alpha,\beta}(k,h)$ from equation (15) in equation (16) and the definition of Mittag-Leffler function (7), a solution of linear space-time fractional FPE (11) is obtained as

$$u(t,x) = x^{a-1} E_{\alpha}(bt^{\alpha}), b = q(a)_{2\beta} - p(a)_{\beta}. \tag{17}$$

A more general solution of linear space-time fractional FPE (11) is

$$u(t,x) = \sum_{a\in I} x^{a-1} E_{\alpha}(bt^{\alpha}), b = q(a)_{2\beta} - p(a)_{\beta}, I = \left\{a \in \mathbb{R} \middle| \frac{a-1}{\beta} \in \mathbb{Z}^{+} \cup 0\right\}. \tag{18}$$

Further in view of Theorem 2.4 we find for equation (11) and its solution that $\left(D_x^{\beta}\right)^2 x^{2\beta+a-1} = D_x^{2\beta} x^{2\beta+a-1}$, hence the linear space-time fractional FPE (11) can also be written as

$$D_t^{\alpha} u(t,x) = \left[-D_x^{\beta}\left(px^{\beta}\right) + D_x^{2\beta}\left(qx^{2\beta}\right)\right] u(t,x), \tag{19}$$

where $t > 0$, $x > 0, 0 < \alpha \leq 1, 1 < 2\beta \leq 2, p, q \in \mathbb{R}$, which under condition (12) has the solution given by (18).

Remark 3.2. Setting $\alpha = 1$, equation (19) with condition (12) reduces to linear space fractional FPE:

$$\frac{\partial u}{\partial t} = \left[-D_x^{\beta}\left(px^{\beta}\right) + D_x^{2\beta}\left(px^{2\beta}\right)\right] u(t,x), \quad t > 0,\ x > 0, 1 < 2\beta \leq 2, \tag{20}$$

with initial condition

$$u(0,x) = x^{a-1}, a \geq 1, \tag{21}$$

and a solution as

$$u(t,x)=\sum_{a\in I}x^{a-1}e^{bt},b=q(a)_{2\beta}-p(a)_{\beta},I=\left\{a\in\mathbb{R}|\,\frac{a-1}{\beta}\in\mathbb{Z}^{+}\cup 0\right\}. \quad (22)$$

Remark 3.3. Setting $\beta=1$, equation (19) with condition (12) reduces to linear time fractional FPE:

$$D_t^{\alpha}u(t,x)=\left[-\frac{\partial}{\partial x}(px)+\frac{\partial^2}{\partial x^2}\left(qx^2\right)\right]u(t,x),\ \ t>0,\ x>0,0<\alpha\leq 1,p,q\in\mathbb{R}, \quad (23)$$

with initial condition

$$u(0,x)=x^{a-1},a\geq 1, \quad (24)$$

and a solution as

$$u(t,x)=\sum_{a=1}^{\infty}x^{a-1}E_{\alpha}(bt^{\alpha}),b=qa^2+a(q-p). \quad (25)$$

Remark 3.4. Setting $\alpha=\beta=1$, equation (19) with condition (12) reduces to linear FPE:

$$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}(px)+\frac{\partial^2}{\partial x^2}\left(qx^2\right)\right]u(t,x),\ \ t>0,\ x>0,p,q\in\mathbb{R}, \quad (26)$$

with initial condition

$$u(0,x)=x^{a-1},a\geq 1, \quad (27)$$

and a solution as

$$u(t,x)=\sum_{a=1}^{\infty}x^{a-1}e^{bt},b=qa^2+a(q-p). \quad (28)$$

Remark 3.5. Setting $\alpha=\beta=1,a=2,p=1,q=1/2$, equation (19) with condition (12) reduces to linear FPE [26]:

$$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}(x)+\frac{\partial^2}{\partial x^2}\left(\frac{x^2}{2}\right)\right]u(t,x),\ \ t>0,\ x>0, \quad (29)$$

with initial condition

$$u(0,x)=x, \quad (30)$$

and a solution as

$$u(t,x)=xe^{t}. \quad (31)$$

Remark 3.6. Setting $\alpha=\beta=1, a=3, p=1/6, q=1/12$, equation (19) with condition (12) reduces to linear FPE [26]:

$$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}\left(\frac{x}{6}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{x^2}{12}\right)\right]u(t,x),\quad t>0,\ x>0, \tag{32}$$

with initial condition

$$u(0,x)=x^2, \tag{33}$$

and a solution as

$$u(t,x)=x^2e^{t/2}. \tag{34}$$

Example 3.7. Consider the nonlinear time fractional FPE:

$$D_t^{\alpha}u(t,x)=\left[-\frac{\partial}{\partial x}\left(3u-\frac{x}{2}\right)+\frac{\partial^2}{\partial x^2}(xu)\right]u(t,x), \tag{35}$$

where $t>0,\ x>0, 0<\alpha\leq 1, D_t^{\alpha}$ is Caputo fractional derivative defined by (6) and initial condition is

$$u(0,x)=x. \tag{36}$$

Applying generalized differential transform (8) with $x_0=0=t_0, \beta=1$, to both sides of time fractional FPE (35) and making use of properties of generalized differential transform given in Lemma 2.5, equation (35) transforms to

$$\begin{aligned}U_{\alpha,1}(k+1,h)&=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\\ &\cdot\left[-(h+1)\left\{3\sum_{r=0}^{k}\sum_{s=0}^{h+1}U_{\alpha,1}(r,h+1-s)U_{\alpha,1}(k-r,s)-\tfrac{1}{2}U_{\alpha,1}(k,h)\right\}\right.\\ &\left.+(h+2)(h+1)\sum_{r=0}^{k}\sum_{s=0}^{h+1}U_{\alpha,1}(r,h+1-s)U_{\alpha,1}(k-r,s)\right].\end{aligned} \tag{37}$$

The generalized differential transform of initial condition (36) is given by

$$U_{\alpha,1}(0,h)=\delta(h-1). \tag{38}$$

Utilizing the recurrence relation (37) and the transformed initial condition (38), for $k=1,2,\ldots$ we obtain

$$U_{\alpha,1}(k,h)=\begin{cases}\frac{1}{\Gamma(\alpha k+1)} & \text{when}\, h=1\\ 0 & \text{otherwise}\end{cases}. \tag{39}$$

From the inverse transform given by equation (9), we have

$$u(t,x)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,1}(k,h)t^{\alpha k}x^h. \tag{40}$$

Using the values of $U_{\alpha,1}(k,h)$ from equation (39) in equation (40) and the definition of Mittag-Leffler function (7), we obtain a solution of problem (35)-(36) as

$$u(t,x) = xE_\alpha(t^\alpha). \tag{41}$$

Remark 3.8. Setting $\alpha = 1$, problem (35)-(36) reduces to non-linear FPE:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}\left(3u - \frac{x}{2}\right) + \frac{\partial^2}{\partial x^2}(xu)\right] u(t,x), \quad t > 0,\ x > 0, \tag{42}$$

with initial condition

$$u(0,x) = x, \tag{43}$$

and a solution as

$$u(t,x) = xe^t. \tag{44}$$

4. Conclusions

Analytical exact solutions of fractional FPE, with both space and time fractional derivatives in Caputo sense, are obtained using generalized differential transform method. The solutions are given in terms of Mittag-Leffler function. It may be concluded that generalized differential transform method is a powerful and efficient technique that provides closed form solutions of fractional differential equations.

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