LE MATEMATICHE
Vol. LXVI (2011) – Fasc. II, pp. 103–111
doi: 10.4418/2011.66.2.9

ON DIFFERENTIAL SUBORDINATIONS AND ARGUMENT INEQUALITIES ASSOCIATED WITH THE GENERALIZED HYPERGEOMETRIC FUNCTIONS

ALI MUHAMMAD

The main object of this paper is to derive several interesting argument properties of a linear operator $\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)$ associated with the generalized hypergeometric functions.

1. Introduction

For any integer $m > 1-p$, let $\sum_{p,m}$ denote the class of functions f :

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k, \quad p \in \mathbb{N} \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $E^* = \{z \mid z \in \mathbb{C}$ and $0 < |z| < 1\} = E \backslash \{0\}$.

If f and g are analytic in E, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in E such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in E, then the following equivalence holds (see [7],[8]):

$$f(z) \prec g(z) \quad (z \in E) \Longleftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

Entrato in redazione: 17 ottobre 2011

AMS 2010 Subject Classification: 30C45, 30C50.

Keywords: Analytic functions, Multivalent functions, Differential subordination, Hadamard product (or convolution).

For functions $f, g \in \sum_{p,m}$, where f is given by (1.1) and g is defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k, \quad p \in \mathbb{N}$$

then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z).$$

For real or complex parameters $\alpha_1, \cdots \alpha_q$ and $\beta_1, \cdots, \beta_s$, with $\beta_j \notin Z_0^- = \{0, -1, -2, \cdots\}$, $j = 1, \cdots s$, the generalized hypergeometric function ${}_qF_s$ (see [10]) is given by

$${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k, \cdots, (\beta_s)_k \; k!} z^k, \qquad (1.2)$$

with $q \leq s+1$, $q, s \in N_0 = \mathbb{N} \cup \{0\}$, $z \in E$, where $(v)_k$ is the Pochhammer symbol (or the shifted factorial) defined in (terms of the Gamma function) by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \left\{ \begin{array}{ll} 1 & \text{if } k = 0, \ v \in \mathbb{C} \backslash \{0\} \\ v(v+1), \cdots (v+k-1) & \text{if } k \in N, \ v \in \mathbb{C} \end{array} \right\}$$

Corresponding to the function $\phi_p(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s; z)$ defined by

$$\phi_p(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s; z) = z^{-p} \, {}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z). \qquad (1.3)$$

We introduce a function $\phi_{p,\mu}(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s; z)$ defined by

$$\begin{aligned} \phi_p(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s; z) * \phi_{p,\mu}(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s; z) \\ = \frac{1}{z^p (1-z)^{\mu+p}} \quad (\mu > -p; \; z \in E^*). \quad (1.4) \end{aligned}$$

We now define a linear operator $\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s) f(z) : \sum_{p,m} \longrightarrow \sum_{p,m}$ by

$$\begin{aligned} \mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s) f(z) = \phi_{p,\mu}(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s; z) * f(z) \quad (1.5) \\ (\alpha_i, \; \beta_j \notin Z_0^-; \; i = 1, 2, ...q; \; j = 1, \cdots s; \; \mu > -p; \; f \in \sum_{p,m}; \quad z \in E^*). \end{aligned}$$

For convenience, we write

$$\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1) = \mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s) f(z).$$

If $f \in \sum_{p,m}$ is given by (1.1), then from (1.5), we deduce that

$$\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f(z) = z^{-p} + \sum_{k=m} \frac{(\mu+p)_{p+k}(\beta_1)_{p+k}...(\beta_s)_{p+k}}{(\alpha_1)_{p+k}...(\alpha_q)_{p+k}} a_k z^k, \quad (\mu > -p;\ z \in E^*), \quad (1.6)$$

and it is easily verified from (1.6) that

$$z(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f(z))' = (\mu+p)\mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)f(z) - (\mu+2p)\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f(z). \quad (1.7)$$

and

$$z(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1+1)f(z))' = \alpha_1 \mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f(z) - (p+\alpha_1)\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1+1)f(z). \quad (1.8)$$

We note that the linear operator $\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator [4] for analytic functions and is essentially motivated by the operators defined and studied in [2]. The linear operator $\mathcal{H}_{1,q,s}^{0,\mu}(\alpha_1)$ was investigated recently by Cho and Kim [1], whereas $\mathcal{H}_{p,2,1}^{1-p}(c,1;a;z) = \mathcal{L}_p(a,c)$ $(c \in \mathbb{R}$, $a \in Z_0^-)$ is the operator studied in [6]. In particular, we have the following observations;

i) $\mathcal{H}_{p,s+1,s}^{m,0}(p+1,\beta_1,\cdots,\beta_s;\beta_1,\cdots,\beta_s)f(z) = \frac{p}{z^{2p}}\int_0^z t^{2p-1}f(t)dt$;

ii) $\mathcal{H}_{p,s+1,s}^{m,0}(p,\beta_1,\cdots,\beta_s;\beta_1,\cdots,\beta_s)f(z)$
$= \mathcal{H}_{p,s+1,s}^{m,1}(p+1,\beta_1,\cdots,\beta_s;\ \beta_1,\cdots,\beta_s)f(z) = f(z)$;

iii) $\mathcal{H}_{p,s+1,s}^{m,1}(p,\beta_1,\cdots,\beta_s;\beta_1,\cdots,\beta_s)f(z) = \frac{zf'(z)+2pf(z)}{p}$;

iv) $\mathcal{H}_{p,s+1,s}^{m,2}(p+1,\beta_1,\cdots,\beta_s;\beta_1,\cdots,\beta_s)f(z) = \frac{zf'(z)+(2p+1)f(z)}{p+1}$;

v) $\mathcal{H}_{p,s+1,s}^{1-p,n}(\beta_1,\cdots,\beta_s;1;\beta_1,\cdots,\beta_s)f(z) = \frac{1}{z^p(1-z)^{n+p}} = D^{n+p-1}f(z)$ (n is an integer $> -p$) the operator studied in [5];

vi) $\mathcal{H}_{p,s+1,s}^{m,1-p}(\delta+1,\beta_2,\cdots,\beta_s;1;\delta;\beta_2,\cdots,\beta_s)f(z) = \frac{\delta}{z^{\delta+p}}\int_0^z t^{\delta+p-1}f(t)dt$ $(\delta > 0,\ z \in E^*)$.

Using the operator $\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)$ we now define a function Q by

$$Q(z) = (1-\delta-(\mu+2p)\delta)(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f(z)) + \delta(\mu+p)\mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)f(z),$$
$$f \in \sum_{p,m} \mu > -p; \quad \delta \geq 0;\ z \in E^*. \quad (1.9)$$

We observe that the operator $Q(z) = \mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z)$ when $\delta = 0$. On the other hand for $\delta = 1$, in view of Eq. (1.7), it follows that $Q(z) = z(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z))'$.

In this paper, we investigate some properties of $Q(z)$. The following Lemma will be required in our investigation.

Lemma 1.1. (see [9])

Let a function $h(z) = 1 + c_{m+p}z^{m+p} + c_{m+p+1}z^{m+p+1} + ...$be analytic in E and $h(z) \neq 0$, $z \in E$. If there exists a point $z_0 \in E$ such that

$$|\arg h(z)| < \frac{\pi}{2}\eta \ \ (|z| < |z_0|) \ \text{ and } |\arg h(z_0)| = \frac{\pi}{2}\eta \ \ (0 \leq \alpha < 1), \tag{1.10}$$

then we have,

$$\frac{z_0 h'(z_0)}{h(z_0)} = ik\alpha, \tag{1.11}$$

where

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \ \text{ when } \ \arg h(z_0) = \frac{\pi}{2}\eta, \tag{1.12}$$

$$k \leq -\frac{1}{2}((a + \frac{1}{a}) \ \text{ when } \ \arg h(z_0) = \frac{\pi}{2}\eta, \tag{1.13}$$

and

$$(h(z_0))^{\frac{1}{\eta}} = \pm ia, \ \ (a > 0).$$

2. Main Results

Theorem 2.1. *Let $f \in \sum_p$ and $Q(z)$ be defined by (1.9). If*

$$\left|\arg z^{p-j}(Q(z))^{(j)}\right| < \frac{\pi}{2}\eta \ (z \in E^*), \tag{2.1}$$

then

$$\left|\arg z^{p-j}(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z))^{(j)}\right| < \frac{\pi}{2} \ (z \in E^*),$$

where $0 \leq j \leq p$, $0 < \eta \leq 1$, $\mu > -p$, and $\delta \geq 0$.

Proof. Let $f \in \sum_p$ and set

$$\frac{(p-j)!}{p!} z^{p-j}(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z))^{(j)} = h(z). \tag{2.2}$$

Then h is analytic in E with $h(z) \neq 0$ for all $z \in E$ and $h(0) = 1$. Since

$$(z\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\ f'(z))^{(j)} = j((\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\ f(z))^{(j)} + z(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\ f(z))^{(j+1)}, \tag{2.3}$$

we have from Eqs. $(1.7),(1.9)$ and (2.3) that

$$\begin{aligned}Q^{(j)}(z) &= (1-\delta-(\mu+2p)\delta)(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z))^{(j)} \\ &+\delta(\mu+p)(\mathcal{H}^{m,\mu+1}_{p,q,s}(\alpha_1)f(z))^{(j)} \\ &= (1-\delta-(\mu+2p)\delta)(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z))^{(j)}+\delta(z(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\ f'(z))^{(j)}) \\ &+(\mu+2p)(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z))^{(j)} \\ &= (1-\delta-\delta j)((\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\ f(z))^{(j)}+\delta z((\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\ f(z))^{(j+1)}. \end{aligned} \tag{2.4}$$

It is easy to see from Eqs. (2.4) and (2.2) that

$$\begin{aligned}z^{p-j}Q^{(j)}(z) &= (1-\delta-\delta j)z^{p-j}(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\ f(z))^{(j)} \\ &+\delta z^{p+1-j}(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\ f(z))^{(j+1)} \\ &= \frac{p!(1-\delta-\delta j)}{(p-j)!}h(z)+\frac{p!\delta}{(p-j)!}(p-j)h(z)+zh'(z) \\ &= \frac{p!(1-\delta-\delta p)}{(p-j)!}\left(h(z)+\frac{\delta}{(1-\delta-\delta p)}zh'(z)\right). \end{aligned} \tag{2.5}$$

Suppose there exists a point $z_0\in E$ such that

$$|\arg h(z)|<\frac{\pi}{2}\eta\ (|z|<\in|z_0|)\text{ and }|\arg h(z_0)|=\frac{\pi}{2}\eta.$$

Then, by Lemma 1.1, we can write that

$$\frac{z_0h'(z_0)}{h(z_0)}=ik\eta\text{ and }(h(z_0))^{\frac{1}{\eta}}=\pm ia\ (a>0).$$

Therefore, if $\arg h(z_0)=\frac{\pi\eta}{2}$, then by Eq. (2.5)

$$\begin{aligned}z_0^{p-j}Q^{(j)}(z_0) &= \frac{p!(1-\delta-\delta p)}{(p-j)!}h(z_0)\left(1+\frac{\delta}{(1-\delta-\delta p)}\frac{zh'(z_0)}{h(z_0)}\right) \\ &= \frac{p!(1-\delta-\delta p)}{(p-j)!}a^{\eta}e^{i\frac{\eta\pi}{2}}\left(1+\frac{\delta}{(1-\delta-\delta p)}ik\eta\right).\end{aligned}$$

This shows that

$$\begin{aligned}\arg\left(z_0^{p-j}Q^{(j)}(z_0)\right) &= \frac{\pi}{2}\eta+\arg\left(1+\frac{\delta}{(1-\delta-\delta p)}ik\eta\right) \\ &= \frac{\pi}{2}\eta+\tan^{-1}\left(\frac{\delta}{(1-\delta-\delta p)}k\eta\right) \\ &\geq \frac{\pi}{2}\eta\quad\left(\text{where } k\geq\frac{1}{2}(a+\frac{1}{a})\geq 1\right),\end{aligned}$$

which contradicts the condition Eq. (2.1). Similarly, if $\arg(h(z_0)) = -\frac{\pi\eta}{2}$, then we obtain

$$\arg\left(z_0^{p-j}Q^{(j)}(z_0)\right) \leq \frac{-\pi}{2}\eta,$$

which also contradicts the Eq. (2.1). Thus, the function h satisfies

$$|\arg(h(z_0))| < \frac{\eta\pi}{2}\ (z \in E^*).$$

This shows that

$$\left|\arg\left(z^{p-j}\right)(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)\ f(z))^{(j)}\right| < \frac{\pi}{2}\eta\ \ (z \in E^*).$$

This completes the proof. □

Taking $\alpha_1 = p$, $\mu = 0$, $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ...s)$ in Theorem 2.1, we immediately have the following result.

Corollary 2.2. *Let* $Q(z) = (1-\delta)f(z) + \delta z f'(z)$ *for* $f \in \sum_p$. *If*

$$\left|\arg\left(z^{p-j}Q^{(j)}(z)\right)\right| < \frac{\pi}{2}\eta\ \ (z \in E^*),$$

then

$$\left|\arg\left(\left(z^{p-j}f^{(j)}(z)\right)\right)\right| < \frac{\pi}{2}\eta\ \ (z \in E^*),$$

where $0 \leq j \leq p$, $0 < \eta \leq 1$ *and* $\delta \geq 0$.

Theorem 2.3. *Let* $f(z) \in \sum_p$ *and let* Q *be defined by Eq.* (1.9). *If*

$$z^{p-j}(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)\ f(z))^{(j)} \prec \frac{p!}{(p-j)!}\frac{1+(1-2\sigma)z}{1-z}\ \ (z \in E), \tag{2.6}$$

then

$$\left(z^{p-j}Q^{(j)}(z)\right) \prec \frac{p!(1-\delta-p\delta)}{(p-j)!}\frac{1+(1-2\sigma)z}{1-z}\ \ (|z| < \rho), \tag{2.7}$$

where $0 \leq j \leq p$, $0 \leq \sigma < 1$, *and*

$$\rho = \left[1 + \left(\frac{\delta}{(1-\delta-p\delta)}\right)^2\right]^{\frac{1}{2}} - \frac{\delta}{(1-\delta-p\delta)}. \tag{2.8}$$

The bound $\rho \in (0,1)$ *is best possible.*

Proof. Set

$$\Psi(z) = (1-\gamma)\frac{1}{z}\frac{1}{1-z} + \gamma\frac{1}{z}\frac{1}{(1-z)^2}, \ \ z \in E^*,$$

where $\gamma = \frac{\delta}{(1-\delta-p\delta)} > 0$. We need to verify that

$$\Re\{\rho z\Psi(z)\} > \frac{1}{2}, \quad z \in E^*, \tag{2.9}$$

where $\rho = (1+\gamma^2)^{\frac{1}{2}} - \gamma$ and $\rho \in (0,1)$. Let $\frac{1}{1-z} = R\,e^{i\theta}$ and $|z| = r < 1$. In view of

$$\cos\theta = \frac{1+R^2(1-r^2)}{2R}, \ \ R \geq \frac{1}{1+r},$$

we have

$$\begin{aligned}2\Re\left\{z\Psi(z) - \frac{1}{2}\right\} &= 2(1-\gamma)R\cos\theta + 2\gamma R^2\cos 2\theta - 1\\ &= R^4\gamma(1-r^2)^2 + R^2((1-\gamma)(1-r^2) - 2\gamma r^2)\\ &\geq R^2(\gamma(1-r)^2 + (1-\gamma)(1-r^2) - 2\gamma r^2)\\ &= R^2(1-2\gamma r - r^2) > 0,\end{aligned}$$

for $|z| = r < \rho$, which gives Eq. (2.9). Thus the function $\Psi(z)$ has the integral representation

$$\rho z\Psi(z) = \int\limits_{|x|=1} \frac{d\mu(x)}{1-xz}, \ \ z \in E^*, \tag{2.10}$$

where $\mu(x)$ is a probability measure on $|x| = 1$.

Now, setting

$$\frac{(p-j)!}{p!}\left(z^{p-j}(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\ f(z)\right)^{(j)} = h(z),$$

where $h(z)$ is analytic in in E with $h(0) = 1$. Then it follows from Eq. (2.6) that $\Re h(z) > \sigma$, $0 \leq \sigma < 1$, $z \in E$.

Since we can write

$$\{z\Psi(z) * h(z)\} = h(z) + \gamma z h'(z),$$

it follows from Eq. (2.10) that

$$\begin{aligned}\Re\{h(\rho z) + \gamma\rho z h'(z)\} &= \Re\{\rho z\Psi(\rho z) * h(z)\}\\ &= \Re\left\{\int\limits_{|x|=1} h(xz)d\mu(x)\right\} > \sigma, \quad z \in E^*.\end{aligned} \tag{2.11}$$

Thus, from Eqs. (2.3) and (2.11), we conclude that Eq. (2.7) holds.

To show that the bound ρ is best possible we take $f \in \sum_p$ defined by

$$\frac{(p-j)!}{p!} z^{p-j} \left(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)\, f(z)\right)^{(j)} = (1-\sigma)\frac{1+z}{1-z} + \sigma.$$

Since

$$\begin{aligned}\frac{(p-j)! z^{p-j} \left((\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)\, f(z)\right)^{(j)}}{p!(1-\delta-\delta p)} &= (1-\sigma)\frac{1+z}{1-z} + \sigma + \gamma(1-\sigma)z\left(\frac{1+z}{1-z}\right)' \\ &= (1-\sigma)\frac{1+2\gamma z - z^2}{(1-z)^2} + \sigma = \sigma,\end{aligned}$$

for $|z| = -\rho$, it follows that ρ is the best possible. □

REFERENCES

[1] N. E. Cho - I. H. Kim, *Inclusion properties of certain classes of meromorphic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. 187 (2007), 115–121.

[2] N. E. Cho - K. I. Noor, *Inclusion properties for certain classes of meromorphic functions associated with Choi-Saigo-Srivastava operator*, J. Math. Anal. Appl. 320 (2006), 779–786.

[3] N. E. Cho - O. S. Kwon - H. M. Srivastava, *Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations*, Integral Transforms Spec. Funct. 16 (2005), 647–659.

[4] J. H. Choi - M. Saigo - H. M. Srivastava, *Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl. 276 (2002), 432–445.

[5] M. R. Ganigi - B. A. Uralegaddi, *New criteria for meromorphic univalent functions*, Bull. Math. Soc. Sci. Math. Roumanie (N.S) 33 (81) (1989), 9–13.

[6] J. L. Liu - H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl. 259 (2000), 566–581.

[7] S. S. Miller - P. T. Mocanu, *Differential subordination Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics 225, Marcel Dekker Inc., New York, Basel, 2000.

[8] S. S. Miller - P. T. Mocanu, *Subordinations of differential superordinations*, Complex Variables 48 (10) (2003), 815–826.

[9] M. Nunokawa, *On the order of strongly starlikness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci. 69 (1993), 234–237.

[10] H. M. Srivastava - P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1985.

ALI MUHAMMAD
Department of Basic Sciences
University of Engineering and Technology
Peshawar, Pakistan
e-mail: ali7887@gmail.com